0.1 Preface

This textbook is designed for a one-semester undergraduate course in ordinary differential equations and linear algebra. We have had such a course at Northeastern University since our conversion from the quarter to semester system required us to offer one course instead of two. Many other institutions have a similarly combined course, perhaps for a similar reason; consequently, there are many other textbooks available that cover both differential equations and linear algebra. Let me describe some of the features of my book and draw some contrasts with the other texts on this subject.

Because many students taking the course at Northeastern are electrical engineering majors who concurrently take a course in circuits, we always include the Laplace transform in the first half of the course. For this reason, in my textbook I cover first and second-order differential equations as well as the Laplace transform in the first three chapters, then I turn to linear algebra in Chapters 4-6, and finally draw on both in the analysis of systems of differential equations in Chapter 7. This ordering of material is unusual (perhaps unique) amongst other textbooks for this course, which generally alternate more between differential equations and linear algebra, and put Laplace transform near the end of the book.

Another feature of my textbook is a fairly concise writing style and selection of topics. I find that many textbooks on this subject are excessively long: they use a verbose writing style, include too many sections, and many of the sections contain too much material. As an instructor using such a book for a one-semester course, I am constantly deciding what to not cover: not only what sections to skip, but what topics in each section to leave out. I think that students using such a textbook also find it difficult to know what has been covered and what has not. On the other hand, I think it is good to have some additional or optional material, to provide some flexibility for the instructor, and to make the book more appropriate for advanced or honors students. Consequently, in my book I have tried to make judicious choices about what material to include, and to arrange it in such a way as to conveniently allow the instructor to omit certain topics. For example, an instructor can cover separable first-order differential equations with applications to unlimited population growth and Newton's law of cooling, and then decide whether or not to include the subsections on resistive force models and on the logistic model for population growth.

The careful selection and arrangement of material is also reflected in the exercises for the student. At the end of each section I have provided exercises that are designed to develop fairly basic skills, and I grouped problems together according to the skill that they are intended to develop. For example, Exercise #1 may address a certain skill, and there are six specific problems (a-f) to do this. Moreover, the answers to all of these exercises (not just the odd-numbered ones) are provided in the Appendix. In fact, some exercises have solution videos on YouTube, which is indicated by *Solution*; a full list of the solution videos can be found at

http://www.centerofmath.org/textbooks/diff_eq/supplements.html

In addition to the exercises at the end of each section, I have provided at the end of each chapter a list of Additional Exercises. These include exercises involving additional applications and some more challenging problems. Only the odd-numbered problems from the Additional Exercises sections are given answers in the Appendix.

Let me add that, in order to keep this book at a length that is convenient for a single semester course, I have had to leave out some important topics. For example, I have not tried to cover numerical methods in this book. While I believe that numerical methods (including the use of computational software) should be taught along with theoretical techniques, there are so many of the latter in a course that covers both differential equations and linear algebra, that it seemed inadvisable to try to also include numerical methods. Consequently, I made the difficult decision to leave numerical methods out of this textbook.

On the other hand, I have taken some advantage of the fact that this book is being primarily distributed in electronic form to expand the coverage. For example, I have included links to online resources (especially Wikipedia articles) that provide more information about topics that are only briefly mentioned in this book. Again, I have tried to make judicious choices about this: if a Wikipedia article on a certain topic exists but does not provide significantly more information than is given in the text, then I chose not to include it.

I hope that the choices that I have made in writing this book make it a valuable learning tool for the students and instructors alike.

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Chapter 1

First-Order Differential Equations

1.1 Differential Equations and Mathematical Models

A **differential equation** is an equation that involves an unknown function and its derivatives. These arise naturally in the physical sciences. For example, Newton's second law of motion $F = ma$ concerns the acceleration a of an object of mass m under a force F. But if we denote the object's velocity by v and assume that F could depend on both v and t, then this can be written as a first-order differential equation for v

$$m\frac{dv}{dt} = F(t, v). \tag{1.1}$$

The simplest example of (1.1) is when F is a constant, such as the gravitational force F_g near the surface of the earth. In this case, $F_g = mg$ where g is the constant acceleration due to gravity, which is given approximately by $g \approx 9.8$ m/sec$^2 \approx 32$ ft/sec^2. If we use this in (1.1), we can easily integrate to find $v(t)$:

$$m\frac{dv}{dt} = mg \quad \Rightarrow \quad \frac{dv}{dt} = g \quad \Rightarrow \quad v(t) - v_0 = \int_0^t g\,dt \quad \Rightarrow \quad v(t) = gt + v_0,$$

where v_0 is the initial velocity. Notice that we need to know the initial velocity in order to determine the velocity at time t.

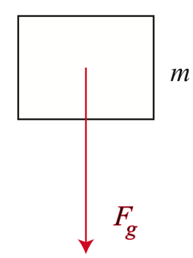


Fig.1. Gravitational force

While equations in which time is the independent variable occur frequently in applications, it is often more convenient to consider x as the independent variable. Let us use this notation and consider a **first-order differential equation** in which we can solve for dy/dx in terms of x and y:

$$\frac{dy}{dx} = f(x, y). \tag{1.2}$$

We are frequently interested in finding a solution of (1.2) that also satisfies an **initial condition**:

$$y(x_0) = y_0. \tag{1.3}$$

The combination of (1.2) and (1.3) is called an **initial-value problem**. The class of **all** solutions of (1.2) is called the **general solution** and it usually depends upon a constant that can be evaluated to find the **particular solution** satisfying a given initial condition. Most of this chapter is devoted to finding general solutions for (1.2), as well as particular solutions of initial-value problems, when $f(x,y)$ takes various forms.

A very easy case of (1.2) occurs when f is independent of y, i.e.

$$\frac{dy}{dx} = f(x),$$

since we can simply integrate to obtain the general solution as

$$y(x) = \int f(x)\,dx + C, \quad \text{where } C \text{ is an arbitrary constant.}$$

On the other hand, if we also require y to satisfy the initial condition (1.3), then we can evaluate C to find the particular solution. This technique was used to solve the gravitational force problem in the first paragraph and should be familiar from calculus, but further examples are given in the Exercises.

In (1.1), if we replace v by dx/dt where x denotes the position of the object and we assume that F could also depend on x, then we obtain an example of a second-order differential equation for x:

$$m\frac{d^2x}{dt^2} = F\left(t, x, \frac{dx}{dt}\right). \tag{1.4}$$

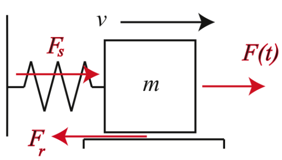


Fig.2. Spring-mass system

An instance of (1.4) is the damped, forced spring-mass system considered in Chapter 2:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t). \tag{1.5}$$

But now initial conditions at t_0 must specify the values of both x and dx/dt:

$$x(t_0) = x_0, \qquad \frac{dx}{dt}(t_0) = v_0. \tag{1.6}$$

In general, the **order** of the differential equation is determined by the highest-order derivative of the unknown function appearing in the equation. Moreover, an initial-value problem for an n-th-order differential equation should specify the first $n-1$ derivatives of the unknown function at some initial point.

An important concept for differential equations is linearity: a differential equation is **linear** if the unknown function and all of its derivatives occur linearly. For example, (1.1) is linear if $F(t,v) = f(t) + g(t)v$, but not if $F(t,v) = v^2$. Similarly, (1.4) is linear if $F(t,x,v) = f(t) + g(t)x + h(t)v$, but not if $F(t,x,v) = \sin x$ or $F(t,x,v) = e^v$. For example, (1.5) is linear. (Note that the coefficient functions $f(t)$, etc do *not* have to be linear in t.)

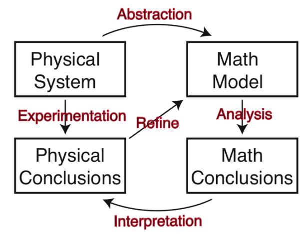
In the above examples, the independent variable is sometimes x and sometimes t. However, when it is clear what is the independent variable, we may use $'$ to denote derivatives; for example, we can write (1.5) as

$$m\,x'' + c\,x' + k\,x = F(t).$$

Notation: for simplicity, we generally do not show the dependence of the unknown function on the independent variable, so we will not write $mx''(t)+cx'(t)+kx(t) = F(t)$.

Mathematical Models

Before we begin the analysis of differential equations, let us consider a little more carefully how they arise in **mathematical models**. Mathematical models are used to reach conclusions and make predictions about the physical world. Suppose there is a particular physical system that we want to study. Let us describe the modeling process for the system in several steps:



1. **Abstraction:** Describe the physical system using mathematical terms and relationships; this provides the model itself.

2. **Analysis:** Apply mathematical analysis of the model to obtain mathematical conclusions.

3. **Interpretation:** Use the mathematical conclusions to obtain conclusions about the physical system.

4. **Refinement (if necessary):** If the conclusions of the model do not agree with experiments, it may be necessary to replace or at least refine the model to make it more accurate.

In this textbook, of course, we consider models involving differential equations. Perhaps the simplest case is **population growth**. It is based upon the observation that populations of all kinds grow (or shrink) at a rate proportional to the size of the population. If we let $P(t)$ denote the size of the population at time t, then this translates to the mathematical statement $dP/dt = kP$, where k is the proportionality constant: if $k > 0$ then the population is growing, and if $k < 0$ then it is shrinking. If we know the population is P_0 at time $t = 0$, then we have an initial-value problem for a first-order linear differential equation:

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0. \tag{1.7}$$

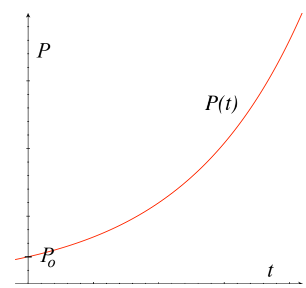


Fig.3. Population growth

This is our mathematical model for population growth. It can easily be analyzed by separation of variables (see Section 1.3), and the solution is found to be $P(t) = P_0\, e^{kt}$. When $k > 0$, the interpretation of this analysis is that the population grows exponentially, and without any upper bound. While this may be true for a while, growing populations eventually begin to slow down due to additional factors like overcrowding or limited food supply. This means that the model must be refined to account for these additional factors; we will discuss one such refinement of (1.7) in Section 1.3. We should also mention that the case $k < 0$ in (1.7) provides a model for **radioactive decay**.

Similar reasoning lies behind Newton's **law of cooling** in **heat transfer**: it is observed that a body with temperature T that is higher than the ambient temperature A will cool at a rate proportional to the temperature differential $T - A$. Consequently, if the initial temperature T_0 is greater than A, then the body will cool: rapidly at first,

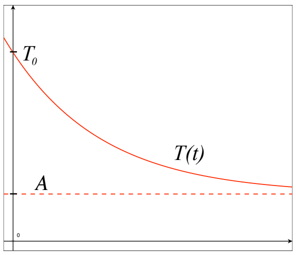


Fig.4. Newton's law of cooling

but then gradually as it decreases to A. Our mathematical model for cooling is the initial-value problem for a first-order linear differential equation:

$$\frac{dT}{dt} = -k(T - A), \quad T(0) = T_0, \tag{1.8}$$

where $k > 0$. In fact, if T_0 is less than A, then (1.8) also governs the **warming** of the body; cf. Example 4 in Section 1.3 where we shall solve (1.8) by separation of variables. But for now, observe that (1.8) implies that $T = A$ is an **equilibrium**, i.e. $dT/dt = 0$ and the object remains at the ambient temperature. We shall have more to say about equilibria in the next section. Also, note that we have assumed that the proportionality constant k is independent of T; of course, this may not be strictly true for some materials, which means that a refinement of the model may be necessary.

We can also use mathematical models to study resistive forces that depend on the velocity of a moving body; these are of the form (1.1) and will be discussed in Section 1.3. Other mathematical models discussed in this textbook include the damped, forced spring-mass system (1.5) and other mechanical vibrations, electrical circuits, and mixture problems.

Remark. *Differential equations involving unknown functions of a single variable, e.g. $x(t)$ or $y(x)$, are often called* **ordinary differential equations**. *On the other hand, differential equations involving unknown functions of several variables, such as $u(x, y)$, are called* **partial differential equations** *since the derivatives are partial derivatives, u_x and u_y. We shall not consider partial differential equations in this textbook.*

Exercises

1. For each differential equation, determine (i) the order, and (ii) whether it is linear:

   (a) $y' + xy^2 = \cos x$ (c) $y''' + y = x^2$
   (b) $x'' + 2x' + x = \sin t$ (d) $x''' + t\,x = x^2$

2. For the given differential equation, use integration to (i) find the general solution, and (ii) find the particular solution satisfying the initial condition $y(0) = 1$.

   (a) $\frac{dy}{dt} = \sin t$ (c) $\frac{dy}{dx} = x \cos x$
   (b) $\frac{dy}{dx} = xe^{x^2}$ (d) $\frac{dy}{dx} = \frac{1}{\sqrt{1-x}}$

3. A rock is hurled into the air with an initial velocity of 64 ft/sec. Assuming only gravitational force with $g = 32$ ft/sec^2 applies, when does the rock reach its maximum height? What is the maximum height that the rock achieves?

4. A ball is dropped from a tower that is 100 m tall. Assuming only gravitational force $g = 9.8$ m/sec^2, how long does it take to reach the ground? What is the speed upon impact?