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MOTIVIC MILNOR CLASSES

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ABSTRACT. The Milnor class is a generalization of the Milnor number, defined as the difference (up to sign) of Chern–Schwartz–MacPherson's class and Fulton–Johnson's canonical Chern class of a local complete intersection variety in a smooth variety. In this paper we introduce a "motivic" Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ and natural transformations from this Grothendieck group to the homology theory. We capture the Milnor class, more generally Milnor–Hirzebruch class, as a special value of a distinguished element under these natural transformations. We also show a Verdier-type Riemann–Roch formula for our motivic Milnor–Hirzebruch class. We use Fulton–MacPherson's bivariant theory and the motivic Hirzebruch class.

1. INTRODUCTION

The Milnor class is defined for a local complete intersection variety X in a non-singular variety M as follows. The local complete intersection variety X defines a normal bundle N_X in M, from which we can define the virtual tangent bundle T_X of X by

$$T_X := TM|_X - N_X M$$

which is a well-defined element of the Grothendieck group $K^0(X)$. Then Fulton-Johnson's or Fulton's canonical (Chern) class of X (see [FJ] and [Fu]) is defined by

$$c_*^{FJ}(X) := c(T_X) \cap [X].$$

Here $c(T_X)$ is the total Chern class of the virtual bundle T_X.

In general, Fulton-Johnson's and Fulton's canonical (Chern) classes are defined for any scheme X embedded as a closed subscheme of a non-singular variety M(see [Fu, Example 4.2.6]): Fulton–Johnson's canonical class $c_*^{FJ}(X)$ ([Fu, Example 4.2.6 (c)]) is defined by

$$c(TM|_X) \cap s(\mathcal{N}_X M),$$

where TM is the tangent bundle of M and $s(\mathcal{N}_X M)$ is the Segre class of the conormal sheaf $\mathcal{N}_X M$ of X in M [Fu, §4.2]. Fulton's canonical class $c_*^F(X)$ ([Fu, Example 4.2.6 (a)]) is defined by

$$c(TM|_X) \cap s(X, M),$$

where $s(X, M)$ is the relative Segre class [Fu, §4.2]. As shown in [Fu, Example 4.2.6], for a local complete intersection variety X in a non-singular variety M these two classes are both equal to $c(T_X) \cap [X]$.

On the other hand there is another well-known notion of Chern class for possibly singular varieties. That is Chern–Schwartz–MacPherson's class $c_*(X)$ [Mac1, Schw1, Schw2,

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Schw3, BrSc]. Then the Milnor class of the local complete intersection variety X, denoted by $\mathcal{M}(X)$, is defined by, up to sign, the difference of Fulton–Johnson's class and Chern–Schwartz–MacPherson's class $c_*(X)$; more precisely

$$\mathcal{M}(X) := (-1)^{\dim X}\left(c_*^{FJ}(X) - c_*(X)\right).$$

Since Chern–Schwartz–MacPherson's class $c_*(X)$ and Fulton–Johnson's class $c_*^{FJ}(X)$ are identical for a nonsingular variety, the Milnor class is certainly supported on the singular locus of the given variety, thus is an invariant of singularities. Prototypes of the Milnor class were studied by P. Aluffi [Alu1, Alu2], A. Parusiński [Pa1, Pa2], A. Parusiński and P. Pragacz [PP2] and T. Suwa [Su3]. Many people have been investigating on the Milnor class from their own viewpoints or interests, and many papers are now available [Alu2, Alu3, Br, BLSS1, BLSS2, Max, Pa3, PP1, PP3, Sea1, SeSu, Su2, Yo2, Yo3]. A category-functorial aspect of the Milnor class is its connection to the so-called Verdier–Riemann–Roch theorem for MacPherson's Chern class [Yo4, Sch1].

Some functoriality of the Milnor class was investigated in [Yo4], but so far it has never been captured as a natural transformation from a certain covariant functor to the homology theory. In this paper we try to capture the Milnor class from a motivic viewpoint and we show that in fact we can capture it as a natural transformation from a pre-motivic covariant functor to the homology theory. For this we need to use the motivic Hirzebruch class [BSY1, BSY2] and a key idea comes from the construction of a universal bivariant theory given in [Yo5].

In §2 we make a quick review of the motivic Hirzebruch class, following [BSY1] (also see [Yo6] and [Sch4]). In §3 we construct the motivic Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$, motivated by the construction of an oriented bivariant theory [Yo5]. The main results are given in §4 and §5. In §4 we construct a motivic Milnor–Hirzebruch class as a natural transformation from the above motivic Grothendieck group to Fulton–MacPherson's bivariant homology theory, a special case of which captures the Milnor class as a natural transformation from the motivic Grothendieck group to the Borel–Moore homology theory. In §5 we show a Verdier-type Riemann–Roch theorem for the motivic Milnor–Hirzebruch class.

In [CMSS] (also see [CLMS1, CLMS2, CMS1, CMS2, CS2, CS3]) Sylvain Cappell et al. independently consider the motivic Hirzebruch–Milnor class and they describe it in terms of other invariants of singularities, thus dealing more with singularities. Our present work is more category-functorial, compared with [CMSS]. A more general work is done in [Yo8].

2. Motivic Hirzebruch classes

In the following sections we use the motivic Hirzebruch class [BSY1, BSY2], thus we very quickly recall some ingredients which are needed later.

Let $\mathcal{V}$ denote the category of complex algebraic varieties. The relative Grothendieck group $K_0(\mathcal{V}/X)$ of a variety X is the quotient of the free abelian group $\mathrm{Iso}^{\mathcal{P}rop}(\mathcal{V}/X)$ of isomorphism classes $[V \xrightarrow{h} X]$ of proper morphisms to X , modulo the following *additivity* relation:

$$[V \xrightarrow{h} X] = [Z \hookrightarrow V \xrightarrow{h} X] + [V \setminus Z \hookrightarrow Y \xrightarrow{h} X]$$

for $Z \subset Y$ a closed subvariety of Y. We set the quotient homomorphism by

$$\Theta : \mathrm{Iso}^{\mathcal{P}rop}(\mathcal{V}/X) \to K_0(\mathcal{V}/X).$$

From now on the equivalence class $\Theta([V \xrightarrow{h} X])$ of the isomorphism class $[V \xrightarrow{h} X]$ is denoted by the same symbol $[V \xrightarrow{h} X]$ unless some possible confusion occurs.

Remark 2.1. Furthermore it follows from Hironaka's resolution of singularities that the restriction $\Theta^{sm} := \Theta|_{\mathrm{Iso}^{\mathcal{P}rop}(Sm/X)}$ of Θ to the subgroup $\mathrm{Iso}^{\mathcal{P}rop}(Sm/X)$ of isomorphism classes $[V \xrightarrow{h} X]$ of proper morphisms from <u>smooth varieties</u> V to X is surjective:

$$\Theta^{sm} : \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \to K_0(\mathcal{V}/X).$$

Here we just remark that F. Bittner [Bit] identified the kernel of the above map $\Theta^{sm} : \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \to K_0(\mathcal{V}/X)$ by some "blow-up relation", for the details of which see [Bit]. This "blow-up relation" plays an important role for constructing a bivariant analogue of the motivic Hirzebruch classes. Since we do not deal with this bivariant analogue, we do not go further into details of this "blow-up relation".

If we use the above "pre-motivic" group $\mathrm{Iso}^{\mathcal{P}rop}(Sm/X)$ we can get the following *"pre-motivic" characteristic classes of singular varieties for an arbitrary characteristic class $c\ell$ of complex vector bundles*.

For a proper morphism $f : X \to Y$ we have the obvious pushforward

$$f_* : \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \to \mathrm{Iso}^{\mathcal{P}rop}(Sm/Y)$$

defined by $f_*([V \xrightarrow{h} X]) := [V \xrightarrow{f \circ h} Y]$. Let $c\ell$ be any characteristic class of complex vector bundles with values in the cohomology theory $H^*(\quad) \otimes R$, where R is a coefficient ring. Then we define

$$\gamma_{c\ell} : \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \to H_*^{BM}(X) \otimes R$$

by

$$\gamma_{c\ell}([V \xrightarrow{h} X]) := h_*(c\ell(TV) \cap [V]).$$

Then it is clear that

$$\gamma_{c\ell} : \mathrm{Iso}^{\mathcal{P}rop}(Sm/\quad) \to H_*^{BM}(\quad) \otimes R$$

is a unique natural transformation satisfying the normalization condition that for a smooth variety X the homomorphism $\gamma_{c\ell} : \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \to H_*^{BM}(X) \otimes R$ satisfies that

$$\gamma_{c\ell}([X \xrightarrow{\mathrm{id}_X} X]) := c\ell(TX) \cap [X].$$

A naïve question is whether $\gamma_{c\ell}$ can be pushed down to the relative Grothendieck group $K_0(\mathcal{V}/X)$, i.e., for some natural transformation $? : K_0(\mathcal{V}/X) \to H_*^{BM}(\quad) \otimes R$ so that the following diagram commutes:

$$\begin{tikzcd}
& \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \arrow[dl, "\Theta^{sm}"'] \arrow[dr, "\gamma_{c\ell}"] & \\
K_0(\mathcal{V}/X) \arrow[rr, "?"'] & & H_*^{BM}(X) \otimes R
\end{tikzcd}$$

If we require that $c\ell$ is a multiplicative characteristic class, the above normalization condition and another extra condition that the degree of the 0-dimensional component of the class $\gamma_{c\ell}(\mathbb{CP}^n)$ equals $1 - y + y^2 + \cdots (-y)^n$, then the characteristic class $c\ell$ can be identified as the Hirzebruch class. Namely, let α_i's be the Chern roots of a complex vector bundle E over X. Then

$$td(E) = \prod_{i=1}^{\operatorname{rank} E} \frac{\alpha_i}{1 - e^{-\alpha_i}} \in H^{2*}(X; \mathbb{Q})$$

is the *Todd class* of E, and its modified version of it

$$td_{(y)}(V) := \prod_{i=1}^{\operatorname{rank} E} \left(\frac{\alpha_i(1+y)}{1 - e^{-\alpha_i(1+y)}} - \alpha_i y \right) \in H^*(X) \otimes \mathbb{Q}[y]$$

is called the *Hirzebruch class* (see [Hir] and [HBJ]. In fact, the Hirzebruch class unifies Chern class, Todd class and Thom–Hirzebruch L-class:

(1) $\underline{y = -1}$: $td_{(-1)}(E) = c(E)$ Chern class,
(2) $\underline{y = 0}$: $td_{(0)}(E) = td(E)$ Todd class,
(3) $\underline{y = 1}$: $td_{(1)}(E) = L(E)$ Thom–Hirzebruch L-class.

Our previous paper [BSY1] (also see [BSY2] and [SY]) showed the following theorem (originally using Saito's theory of mixed Hodge modules [Sai]):

Theorem 2.2. *(Motivic Hirzebruch class of singular varieties) There exists a unique natural transformation*

$$T_{y_*} : K_0(\mathcal{V}/\quad) \to H_*^{BM}(\quad) \otimes \mathbb{Q}[y]$$

satisfying the normalization condition that for a smooth variety X

$$T_{y_*}([X \xrightarrow{\mathrm{id}_X} X]) = td_{(y)}(TX) \cap [X].$$

This motivic Hirzebruch class $T_{y_*} : K_0(\mathcal{V}/\quad) \to H_*^{BM}(\quad) \otimes \mathbb{Q}[y]$ in a sense "unifies" the following three well-known characteristic classes of singular varieties:

Theorem 2.3. *(A "unification" of three characteristic classes)*

(1) *$\underline{c = \text{Chern class}}$: There exists a unique natural transformation*

$$\gamma_F : K_0(\mathcal{V}/\quad) \to F(\quad)$$

such that for X nonsingular $\gamma_F([X \xrightarrow{\mathrm{id}} X]) = 1\!\!1_X$. And the following diagram commutes

$$\begin{tikzcd} & K_0(\mathcal{V}/X) \arrow[dl, "\gamma_F"'] \arrow[dr, "T_{-1_*}"] & \\ F(X) \arrow[rr, "c_* \otimes \mathbb{Q}"'] & & H_*^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

Here $c_* : F(X) \to H_*^{BM}(X)$ is MacPherson's Chern class transformation [Mac1] *defined on the group $F(X)$ of complex algebraically constructible functions.*

(2) *td = Todd class: There exists a unique natural transformation*

$$\gamma_{G_0} : K_0(\mathcal{V}/\quad) \to G_0(\quad)$$

such that for X nonsingular $\gamma([X \xrightarrow{\mathrm{id}} X]) = [\mathcal{O}_X]$. And the following diagram commutes

$$\begin{tikzcd} & K_0(\mathcal{V}/X) \arrow[dl, "\gamma_{G_0}"'] \arrow[dr, "T_{0*}"] & \\ G_0(X) \arrow[rr, "td_*"'] & & H_*^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

Here $td_* : G_0(X) \to H_*^{BM}(X) \otimes \mathbb{Q}$ is Baum–Fulton–MacPherson's Todd class (or Riemann–Roch) transformation [BFM1] *defined on the Grothendieck group $G_0(X)$ of coherent algebraic $\mathcal{O}_X$-sheaves.*

(3) *L = Thom-Hirzebruch L-class:There exists a unique natural transformation*

$$\gamma_\Omega : K_0(\mathcal{V}/\quad) \to \Omega(\quad)$$

such that for X nonsingular $\gamma_\Omega([X \xrightarrow{\mathrm{id}} X]) = [\mathbb{Q}_X[\dim X]]$. And the following diagram commutes

$$\begin{tikzcd} & K_0(\mathcal{V}/X) \arrow[dl, "\gamma_\Omega"'] \arrow[dr, "T_{1*}"] & \\ \Omega(X) \arrow[rr, "L_*"'] & & H_*^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

Here $\Omega(X)$ is the Cappell–Shaneson–Youssin's cobordism group of self-dual constructible sheaves (see [CS1] *and* [You]*) and $L_* : \Omega(X) \to H_*^{BM}(X) \otimes \mathbb{Q}$ is Cappell–Shaneson's homology L-class transformation* [CS1] *(also see* [GM]*).*

We also have the following

Corollary 2.4. *The following diagram commutes:*

$$\begin{tikzcd} & \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \arrow[dl, "\Theta^{sm}"'] \arrow[dr, "\gamma_{td_{(y)}}"] & \\ K_0(\mathcal{V}/X) \arrow[rr, "T_{y*}"'] & & H_*^{BM}(X) \otimes R \end{tikzcd}$$

Definition 2.5. For a complex algebraic variety X

$$T_{y*}(X) := T_{y*}([X \xrightarrow{\mathrm{id}} X]) \in H_*^{BM}(X) \otimes \mathbb{Q}[y]$$

is called *the motivic Hirzebruch class of X*.

Remark 2.6. As to the homomorphism $\gamma_F : K_0(\mathcal{V}/X) \to F(X)$ we have that for *any* variety X

$$\gamma_F([X \xrightarrow{\mathrm{id}} X]) = 1\!\!1_X, \text{therefore} \quad T_{-1*}(X) = c_*(X) \otimes \mathbb{Q},$$

whether X is singular or non-singular. However, as to the other two homomorphisms $\gamma_{G_0} : K_0(\mathcal{V}/X) \to G_0(X)$ and $\gamma_\Omega : K_0(\mathcal{V}/X) \to \Omega(X)$, if X is singular, in general we have that

$$\gamma_{G_0}([X \xrightarrow{\mathrm{id}} X]) \neq [\mathcal{O}_X], \quad \gamma_\Omega([X \xrightarrow{\mathrm{id}} X]) \neq [\mathcal{IC}_X],$$

where $\mathcal{IC}_X$ is the middle intersection homology complex of Goresky–MacPherson [GM]. Hence, if X is singular, in general we have that

$$T_{0*}(X) \neq td_*(X), \quad T_{1*}(X) \neq L_*(X).$$

If X is a Du Bois variety, i.e., a variety with Du Bois singularities, then we have that

$$\gamma_{G_0}([X \xrightarrow{\mathrm{id}} X]) = [\mathcal{O}_X], \text{therefore} \quad T_{0*}(X) = td_*(X).$$

If X is a rational homology manifold, then conjecturally

$$\gamma_\Omega([X \xrightarrow{\mathrm{id}} X]) = [\mathcal{IC}_X], \text{therefore} \quad T_{1*}(X) = L_*(X).$$

For more details, see [BSY1] and also [CMSS, Theorem 4.3], where the conjecture is proved in some special cases.

3. The Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$

Let S be a complex algebraic variety and fixed. Let $\mathcal{V}_S$ be the category of S-varieties, i.e., an object is a morphism $h : X \to S$ and a morphism from $h : X \to S$ to $k : Y \to S$ is a morphism $f : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\ & S & \end{tikzcd} .$$

A morphism $f : X \to Y$ is called *a local complete intersection ($\ell.c.i.$) morphism* if f admits a factorization into a closed regular embedding followed by a smooth morphism (e.g., see [Fu] or [FM]). In particular, regular embeddings and smooth morphisms are $\ell.c.i.$ morphisms. The composite of $\ell.c.i.$ morphisms are again an $\ell.c.i.$ morphism.

Definition 3.1. Let $M^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ be the monoid consisting of isomorphism classes $[V \xrightarrow{p} X]$ of proper morphisms $p : V \to X$ such that the composite $h \circ p : V \to S$ is an $\ell.c.i.$ morphism, with the addition $(+)$ and zero (0) defined by

- $[V \xrightarrow{h} X] + [V' \xrightarrow{h'} X] := [V \sqcup V' \xrightarrow{h+h'} X]$,
- $0 := [\phi \to X]$.

Then we define

$$K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$$

to be the Grothendieck group of the monoid $M^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$.

Remark 3.2. In other words, $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ is the free abelian group generated by the set of all isomorphism classes of $[V \xrightarrow{p} X]$ of proper morphisms $p : V \to X$ such that the composite $h \circ p : V \to S$ is an $\ell.c.i.$ morphism, modulo the subgroup generated by the elements of the following form

$$[V \xrightarrow{h} X] + [V' \xrightarrow{h'} X] - [V \sqcup V' \xrightarrow{h+h'} X].$$

Lemma 3.3. (1) *The Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ is a covariant functor with pushforwards for proper morphisms, i.e., for a proper morhism $f : X \to Y \in \mathcal{V}_S$*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\ & S & \end{tikzcd}$$

the pushforward

$$f_* : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S)$$

defined by

$$f_*([V \xrightarrow{p} X]) := [V \xrightarrow{f\circ p} Y]$$

is covariantly functorial.

(2) *The Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ is a contravariant functor with pullbacks for smooth morphisms, i.e., for a smooth morhism $f : X \to Y \in \mathcal{V}_S$ the pullback*

$$f^* : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S) \to K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$$

defined by

$$f^*([W \xrightarrow{p} Y]) := [W' \xrightarrow{p'} X]$$

is contravariantly functorial. Here we consider the following commutative diagrams whose top square is a fiber square:

$$\begin{tikzcd} W' \arrow[rr, "f'"] \arrow[d, "p'"'] & & W \arrow[d, "p"] \\ X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\ & S. & \end{tikzcd}$$

Proof. (1) The well-definedness of the pushforward homomorphism f_* is clear.

(2) In the diagram of Lemma 3.3 (2) , by the definition $k \circ p : W \to S$ is an $\ell.c.i.$ morphism, and $f' : W' \to W$ is smooth since it is a base change of a smooth morphism $f : X \to Y$. The composite $h \circ p' : W' \to S$ is equal to the composite $k \circ p \circ f'$, thus it is an $\ell.c.i.$ morphism because it is the composite of two $\ell.c.i.$ morphisms. Thus the pullback homomorphism f^* is well-defined. □

Remark 3.4. (1) As to the contravariance of the Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$, one might be tempted to consider the pullback for a local complete intersection morphism $f : X \to Y$ instead of a smooth morphism. But a crucial problem for this is that the pullback of a local complete intersection morphism is not necessarily a local complete intersection morphism, thus in the diagram of Lemma 3.3 (2) $f' : W' \to W$ is not necessarily a local complete intersection morphism and hence we do not know whether or not the composite $k \circ p \circ f' = h \circ p'$ is a local complete intersection morphism.

(2) If we consider the finer class $\mathcal{S}m$ of smooth morphisms instead of the class $\mathcal{L}.c.i$ of local complete intersection morphisms, we do have a bivariant theory, from which we can construct a motivic bivariant characteristic class [Yo7].

4. MOTIVIC MILNOR–HIRZEBRUCH CLASSES

For a morphism $f : X \to Y$, $\mathbb{H}(X \to Y)$ is the Fulton–MacPherson bivariant homology theory [FM]. Since the main theme of the present paper is not a bivariant theoretic, we do not recall a general bivariant theory, thus see [FM] for details. In the paper $\bullet$ denotes the bivariant product, i.e., for morphisms $f : X \to Y$, $g : Y \to Z$ the bivariant product $\bullet$ is

$$\bullet : \mathbb{H}(X \xrightarrow{f} Y) \times \mathbb{H}(Y \xrightarrow{g} Z) \to \mathbb{H}(X \xrightarrow{g\circ f} Z).$$

Then $\mathbb{H}(X \xrightarrow{\mathrm{id}_X} X)$ is the usual cohomology theory $H^*(X)$ and $\mathbb{H}(X \to pt)$ (for a mapping to a point) is the Borel–Moore homology theory $H_*^{BM}(X)$.

Proposition 4.1. *Let $c\ell : K^0 \to H^*(\quad) \otimes R$ be a characteristic class of complex vector bundles with a suitable coefficients R. Then on the category $\mathcal{V}_S$ we have that*

(1) *There exists a unique natural transformation (not a Grotendieck transformation)*

$$\widetilde{\gamma_{c\ell *}} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}(X \xrightarrow{h} S) \otimes R$$

such that for a local complete intersection morphism $h : X \to S$

$$\widetilde{\gamma_{c\ell *}}([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(T_h) \bullet U_h.$$

Here T_h is the (virtual) relative tangent bundle of h and $U_h \in \mathbb{H}(X \xrightarrow{h} S)$ is the canonical orientation.

(2) *There exists a unique natural transformation*

$$\gamma_{c\ell *} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to H_*^{BM}(X) \otimes R$$

such that for a local complete intersection morphism $h : X \to S$

$$\gamma_{c\ell *}([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(T_h) \cap [X].$$

Proof. (1) We define $\widetilde{\gamma_{c\ell *}} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}(X \xrightarrow{h} S) \otimes R$ by

$$\widetilde{\gamma_{c\ell *}}([V \xrightarrow{p} X]) := p_*(c\ell(T_{h\circ p}) \bullet U_{h\circ p}).$$

First we observe that $\widetilde{\gamma_{c\ell *}}$ is well-defined. Let $p' : V' \to X$ be another representative of $[V \xrightarrow{p} X]$, i.e., the composite $h \circ p'$ is an $\ell.c.i.$ morphism and there is an isomorphism $g : V' \cong V$ such that the following diagram commutes:

$$\begin{tikzcd} V' \arrow[rr, "g", "\cong"'] \arrow[dr, "p'"'] & & V \arrow[dl, "p"] \\ & X. & \end{tikzcd}$$

Then we have

$$
\begin{aligned}
\widetilde{\gamma_{c\ell *}}([V' \xrightarrow{p'} X]) &= p'_*(c\ell(T_{h\circ p'}) \bullet U_{h\circ p'}) \\
&= p_* g_*(c\ell(g^* T_{h\circ p}) \bullet U_{h\circ p'}) \\
&= p_* g_*(c\ell(g^* T_{h\circ p}) \bullet U_{h\circ p'}) \\
&= p_* g_*(g^* c\ell(T_{h\circ p}) \bullet U_{h\circ p'}) \\
&= p_*(c\ell(T_{h\circ p}) \bullet g_* U_{h\circ p'}) \quad \text{(projection formula)} \\
&= p_*(c\ell(T_{h\circ p}) \bullet U_{h\circ p}) \quad \text{(since } g \text{ is an isomorphism)} \\
&= \widetilde{\gamma_{c\ell *}}([V \xrightarrow{p} X]).
\end{aligned}
$$

The equality $g_* U_{h\circ p'} = U_{h\circ p}$ is due to the following observation. By the definition or the construction of Fulton–MacPherson's bivariant homology theory $\mathbb{H}$ (see [FM]), for the isomorphism $g : V' \xrightarrow{\cong} V$ we have

- $\mathbb{H}^i(V' \xrightarrow{g} V) = H^i(V)$
- $g_* : \mathbb{H}^i(V' \xrightarrow{g} V) \to \mathbb{H}^i(V \xrightarrow{\mathrm{id}_V} V)$ is the identity map,
- $U_g = 1_V \in H^0(V)$.

Since $h \circ p' = (h \circ p) \circ g$ and g is also an $\ell.c.i.$ morphism, it follows from [FM, Part II, §1.3] that we have

$$
U_{h\circ p'} = U_{(h\circ p)\circ g} = U_g \bullet U_{h\circ p}.
$$

Then we have

$$
\begin{aligned}
g_* U_{h\circ p'} &= g_*\big(U_g \bullet U_{h\circ p}\big) \\
&= g_* U_g \bullet U_{h\circ p} \quad \text{([FM, } A_{12}\text{, p.20])} \\
&= U_{h\circ p} \quad \text{(since } g_* U_g = 1_V)
\end{aligned}
$$

Thus $\widetilde{\gamma_{c\ell *}}$ is well-defined.

Now, for a morphism $f : X \to Y$, i.e., for the following commutative diagram

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\
& S &
\end{tikzcd}
$$

the following diagram commutes:

$$
\begin{tikzcd}
K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \arrow[r, "\widetilde{\gamma_{c\ell *}}"] \arrow[d, "f_*"'] & \mathbb{H}(X \xrightarrow{h} S) \otimes R \arrow[d, "f_*"] \\
K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S) \arrow[r, "\widetilde{\gamma_{c\ell *}}"'] & \mathbb{H}(Y \xrightarrow{k} S) \otimes R,
\end{tikzcd}
$$

Indeed, for $[V \xrightarrow{p} X] \in K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ we have that

$$\begin{aligned}
f_*\left(\widetilde{\gamma_{c\ell *}}([V \xrightarrow{p} X])\right) &= f_*\left(p_*(c\ell(T_{h\circ p}) \bullet U_{h\circ p})\right) \\
&= (f\circ p)_*\left(c\ell(T_{h\circ p}) \bullet U_{h\circ p}\right) \\
&= (f\circ p)_*\left(c\ell(T_{k\circ f\circ p}) \bullet U_{k\circ f\circ p}\right) \\
&= (f\circ p)_*\left(c\ell(T_{k\circ f\circ p}) \bullet U_{k\circ f\circ p}\right) \\
&= \gamma_{c\ell *}([V \xrightarrow{f\circ p} Y]) \\
&= \widetilde{\gamma_{c\ell *}}\left(f_*([V \xrightarrow{p} X])\right).
\end{aligned}$$

Since, for a local complete intersection morphism $h : X \to S$, by definition of $\gamma_{c\ell *}$ we have $\gamma_{c\ell *}([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(T_h) \bullet U_h$, the uniqueness of $\gamma_{c\ell *}$ follows.

(2) We define $\gamma_{c\ell *} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to H^{BM}_*(X) \otimes R$ by

$$\gamma_{c\ell *}([V \xrightarrow{p} X]) := p_*(c\ell(T_{h\circ p}) \cap [V]).$$

The well-definedness of $\gamma_{c\ell *}$ is similar to the above, but more straightforward. Indeed, we have

$$\begin{aligned}
\gamma_{c\ell *}([V' \xrightarrow{p} X]) &= p'_*(c\ell(T_{h\circ p'}) \cap [V']) \\
&= p_*g_*(c\ell(g^*T_{h\circ p}) \cap [V']) \\
&= p_*g_*(c\ell(g^*T_{h\circ p}) \cap [V']) \\
&= p_*g_*(g^*c\ell(T_{h\circ p}) \cap [V']) \\
&= p_*(c\ell(T_{h\circ p}) \cap g_*[V']) \\
&= p_*(c\ell(T_{h\circ p}) \cap [V]) \\
&= \gamma_{c\ell *}([V \xrightarrow{p} X]).
\end{aligned}$$

Then the following diagram commutes:

$$\begin{array}{ccc}
K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) & \xrightarrow{\gamma_{c\ell *}} & H^{BM}_*(X) \otimes R \\
{\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\
K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S) & \xrightarrow[\gamma_{c\ell *}]{} & H^{BM}_*(Y) \otimes R,
\end{array}$$

which follows from replacing $\bullet U_{h\circ p}$ and $\bullet U_{k\circ f\circ p}$ by $\cap[V]$ in the proof of (1). $\square$

Remark 4.2. For a local complete intersection morphism $f : X \to S$, we have

$$\bullet U_h \bullet [S] = \cap [X].$$

Here $[W]$ is the fundamental class of W and $[W] \in \mathbb{H}(W \to pt) = H^{BM}_*(W)$. Thus the relation between the above two natural transformations $\widetilde{\gamma_{c\ell *}}$ and $\gamma_{c\ell *}$ is that

$$\gamma_{c\ell *} = \widetilde{\gamma_{c\ell *}} \bullet [S].$$

Remark 4.3. When the fixed variety S is a point, the above two natural transformations $\widetilde{\gamma_{c\ell *}}$ and $\gamma_{c\ell *}$ are the same: $\gamma_{c\ell *} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H^{BM}_*(X) \otimes R$.

If S is a point and $c\ell = c$ the Chern class, then for a local complete intersection variety X in a smooth manifold, we have that

$$\gamma_{c*}([X \xrightarrow{\mathrm{id}_X} X]) = c(T_X) \cap [X]$$

which is Fulton–Johnson's class $c_*^{FJ}(X)$. Thus the above natural transformations

$$\widetilde{\gamma_{c\ell *}} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}(X \xrightarrow{h} S) \otimes R$$

$$\gamma_{c\ell *} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to H_*^{BM}(X) \otimes R$$

are both generalizations of Fulton–Johnson's class as natural transformations. They are respecively called *a motivic "bivariant" FJ-$c\ell$ class*, denoted by $\widetilde{c\ell_*^{FJ}}$, and *a motivic FJ-$c\ell$ class*, denoted by $c\ell_*^{FJ}$, since it is modelled after Fulton–Johnson's class c_*^{FJ}.

From here on we consider the Hirzebruch class $td_{(y)}$, instead of an arbitrary characteristic class $c\ell$, because we use the motivic Hirzebruch class $T_{y_*} : K_0(\mathcal{V}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}[y]$ below. We use the above natural transformations

$$\widetilde{\gamma_{td_{(y)}*}} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y],$$

$$\gamma_{td_{(y)}*} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to H_*^{BM}(X) \otimes \mathbb{Q}[y],$$

which are respectively called *the motivic "bivariant" FJ-Hirzebruch class* and *the motivic FJ-Hirzebruch class* and denoted by $\widetilde{T_{y_*}^{FJ}}$ and $T_{y_*}^{FJ}$.

We define the twisted pushforward for homology as follows: for a morphism $f : X \to Y$, the relative dimension of f and the co-relative dimension of f are respectively defined by

$$\dim(f) := \dim X - \dim Y \quad \operatorname{codim}(f) := \dim Y - \dim X.$$

For the Borel–Moore homology theory H_*, the twisted pushforward for a proper morphism $f : X \to Y$ is define by

$$f_{**} := (-1)^{\operatorname{codim}(f)} f_* : H_*^{BM}(X) \to H_*^{BM}(Y).$$

With this twisted pushforward the Borel–Moore homology theory is still a covariant functor. To avoid a possible confusion we denote $H_{**}^{BM}(X)$ for the Borel–Moore homology theory with the twisted pushforward.

Corollary 4.4. *On the category $\mathcal{V}_S$ there exists a unique natural transformation*

$$\mathcal{M}T_{y_*} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to H_{**}^{BM}(X) \otimes \mathbb{Q}[y]$$

such that for a local complete intersection morphism $h : X \to S$ the homomorphism $\mathcal{M}T_{y_*} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to H_{**}^{BM}(X) \otimes \mathbb{Q}[y]$ satisfies that

$$\mathcal{M}T_{y_*}([X \xrightarrow{\mathrm{id}_X} X]) = (-1)^{\dim X} \left(T_{y_*}^{FJ} - T_{y_*} \circ \Theta\right)([X \xrightarrow{\mathrm{id}_X} X]).$$

Proof. We define $\mathcal{M}T_{y_*} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to H_{**}^{BM}(X) \otimes \mathbb{Q}[y]$ by

$$\mathcal{M}T_{y_*}([V \xrightarrow{p} X]) := (-1)^{\dim V} \left(T_{y_*}^{FJ} - T_{y_*} \circ \Theta\right)([V \xrightarrow{p} X]).$$

This is equal to

$$(-1)^{\dim X} p_* \left(td_{(y)}(T_{p \circ h}) \cap [V] - T_{y_*}(V)\right).$$

□

From here on we denote $T_{y_*} \circ \Theta$ simply by T_{y_*}. When S is a point, the above motivic natural transformation

$$\mathcal{M}T_{y_*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H^{BM}_{**}(X) \otimes \mathbb{Q}[y]$$

shall be called *a motivic Milnor–Hirzebruch class*, even though $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X)$ is not (a subgroup of) the motivic group $K_0(\mathcal{V}/X)$, but because it is defined by using the motivic Hirzebruch class $T_{y_*} : K_0(\mathcal{V}/X) \to H^{BM}_*(X) \otimes \mathbb{Q}[y]$ and because, if we specialize $\mathcal{M}T_{y_*}$ to the case when $y = -1$ and X is a local complete intersection variety in a smooth manifold, we have

$$\begin{aligned}
&\mathcal{M}T_{-1_*}([X \xrightarrow{\mathrm{id}} X]) \\
&= (-1)^{\dim X} \left\{ td_{(-1)}(T_X) \cap [X] - T_{-1_*}\left(\Theta([X \xrightarrow{\mathrm{id}} X])\right)\right\} \\
&= (-1)^{\dim X} \left(c^{FJ}_*(X) - c_*(X)\right),
\end{aligned}$$

which is the Milnor class $\mathcal{M}(X)$ of X. Thus $\mathcal{M}T_{-1_*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H^{BM}_{**}(X) \otimes \mathbb{Q}[y]$ is called *the motivic Milnor class (or Milnor–Chern class)*. The more general one

$$\mathcal{M}T_{y_*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to H^{BM}_{**}(X) \otimes \mathbb{Q}[y]$$

is called *a generalized motivic Milnor–Hirzebruch class*.

In fact, if the base variety S is a $\mathbb{Q}$-homology manifold or a rational homology manifold, the fundamental class $[S] \in \mathbb{H}(S \to pt) = H^{BM}_*(S)$ is a strong orientation (see [FM, Part I, §2.6]), namely we have the following isomorphism (see [BSY3])

$$\bullet[S] : \mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q} \xrightarrow{\cong} \mathbb{H}(X \to pt) \otimes \mathbb{Q} = H^{BM}_*(X) \otimes \mathbb{Q}.$$

Which is a generalized Poincaré duality isomorphism, hence denoted by $\mathcal{PD}_h$. Indeed, when X is a rational homology compact manifold, for the identity $id_X : X \to X$, the above isomorphism is nothing but the classical Poincaré duality isomorphism

$$\cap[X] : H^*(X) \otimes \mathbb{Q} \to H_*(X) \otimes \mathbb{Q}.$$

Examples of a $\mathbb{Q}$-homology manifold (e.g., see [BM, §1.4 Rational homology manifolds]) are surfaces with Kleinian singularities, the moduli space of curves of a given genus, Satake's V-manifolds or orbifolds, in particular, the quotient of a nonsingular variety by a finite group action on.

Thus we can get the following corollary:

Corollary 4.5. *Let the base variety S be a $\mathbb{Q}$-homology manifold. On the category $\mathcal{V}_S$ there exists a unique natural transformation*

$$\widetilde{\mathcal{M}T_{y_*}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}_{**}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y]$$

such that for a local complete intersection morphism $h : X \to S$ the homomorphism $\widetilde{\mathcal{M}T_{y_*}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y]$ satisfies that

$$\widetilde{\mathcal{M}T_{y_*}}([X \xrightarrow{\mathrm{id}_X} X]) = (-1)^{\dim X} \left(\widetilde{T^{FJ}_{y_*}} - \mathcal{PD}_h^{-1} \circ T_{y_*} \right) ([X \xrightarrow{\mathrm{id}_X} X]).$$

Here $\mathbb{H}_{**}(X \xrightarrow{h} S)$ is the twisted bivariant homology theory with the twisted pushforward $f_{**} := (-1)^{\mathrm{codim}(f)} f_*$.

Remark 4.6. (1) $\widetilde{\mathcal{M}T_{y_*}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}_{**}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y]$ shall be called *a motivic "bivariant" Milnor–Hirzebruch class*, even thought the source group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ is not a bivariant theory, but the target group $\mathbb{H}_{**}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y]$ is a bivariant theory.

(2) Note that when the base variety S is a point, $\widetilde{\mathcal{M}T_{y_*}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}_{**}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y]$ is the same as $\mathcal{M}T_{y_*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H^{BM}_{**}(X) \otimes \mathbb{Q}[y]$.

Proposition 4.7. *In the case when $y = 0$, the Milnor–Todd class $\mathcal{M}T_{0*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H^{BM}_*(X) \otimes \mathbb{Q}$ vanishes on the subgroup generated by $[V \xrightarrow{p} X]$ with V being Du Bois varieties:*

$$\mathcal{M}T_{0*}([V \xrightarrow{p} X]) = 0 \quad \textit{if } V \textit{ is a Du Bois variety.}$$

Proof. For a local complete intersection variety V in a smooth variety M, we have that

$$\begin{aligned}
&\mathcal{M}T_{0*}([V \xrightarrow{p} X]) \\
&= p_{**}\mathcal{M}T_{0*}([V \xrightarrow{\mathrm{id}} V]) \\
&= (-1)^{\dim X} p_* \Big(td(T_V) \cap [V] - T_{0*}([V \xrightarrow{\mathrm{id}} V]) \Big) \\
&= (-1)^{\dim X} p_* \big(td(T_V) \cap [V] - T_{0*}(V) \big).
\end{aligned}$$

If V is a Du Bois variety, it follows from Remark 2.6 that $T_{0*}(V) = td_*(\mathcal{O}_V)$. On the other hand we observe that it follows from the properties of the Baum–Fulton–MacPherson's Riemann–Roch $td_* : G_0(X) \to H^{BM}_*(X) \otimes \mathbb{Q}$ (see [Fu, Corollary 18.3.1 (b)], or more generally [FM, PART II, §0.2 Summary of results]) that for a local complete intersection variety V in a smooth variety M we have

$$td_*(\mathcal{O}_V) = td(T_V) \cap [V],$$

for T_V the virtual tangent bundle of V in M. Therefore, if V is a local complete intersection variety V in a smooth variety M and V is also a Du Bois variety, then we have

$$\mathcal{M}T_{0*}([V \xrightarrow{p} X]) = 0.$$

□

Corollary 4.8. *If the base variety S is a $\mathbb{Q}$-homology manifold, then the motivic bivariant Milnor–Todd class $\widetilde{\mathcal{M}T_{0*}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}_{**}(X \xrightarrow{h} S) \otimes \mathbb{Q}$ vanishes on the subgroup generated by $[V \xrightarrow{p} X]$ with V being Du Bois varieties.*

Proof. This follows from the fact that for an element $[V \xrightarrow{p} X]$ with V a Du Bois variety $\widetilde{\mathcal{M}T_{0*}}([V \xrightarrow{p} X]) \bullet [S] = \mathcal{M}T_{0*}([V \xrightarrow{p} X]) = 0$ and $\bullet[S] : \mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q} \xrightarrow{\cong} \mathbb{H}(X \to pt) \otimes \mathbb{Q} = H^{BM}_*(X) \otimes \mathbb{Q}$ is an isomorphism when S is a $\mathbb{Q}$-homology manifold. □

Remark 4.9. Let us compare with the results in Theorem 2.3. Neither of the following three diagrams commutes in general:

- $\underline{y = -1}$:

$$\begin{tikzcd}
 & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_F"'] \arrow[dr, "{T_{-1}}_*^{FJ}"] & \\
F(X) \arrow[rr, "c_*"'] & & H^{BM}_*(X) \otimes \mathbb{Q}.
\end{tikzcd}$$

- $\underline{y=0}$:

$$\begin{tikzcd} & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_{G_0}"'] \arrow[dr, "{T_0}_*^{FJ}"] & \\ G_0(X) \arrow[rr, "td_*"'] & & H_*^{BM}(X)\otimes\mathbb{Q}. \end{tikzcd}$$

- $\underline{y=1}$:

$$\begin{tikzcd} & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_{\Omega}"'] \arrow[dr, "{T_1}_*^{FJ}"] & \\ \Omega(X) \arrow[rr, "L_*"'] & & H_*^{BM}(X)\otimes\mathbb{Q}. \end{tikzcd}$$

Hence it is natural or reasonable to consider the following commutative diagrams with the corresponding Milnor classes and the corresponding looked-for natural transformations

$$\mathcal{M}T_{-1_*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H_{**}^{BM}(X)\otimes\mathbb{Q}, \quad \mathcal{M}c_* : F(X) \to H_{**}^{BM}(X)\otimes\mathbb{Q}$$
$$\mathcal{M}T_{0*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H_{**}^{BM}(X)\otimes\mathbb{Q}, \quad \mathcal{M}td_* : G_0(X) \to H_{**}^{BM}(X)\otimes\mathbb{Q},$$
$$\mathcal{M}T_{1*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H_{**}^{BM}(X)\otimes\mathbb{Q}, \quad \mathcal{M}L_* : \Omega(X) \to H_{**}^{BM}(X)\otimes\mathbb{Q} :$$

- $\underline{y=-1}$:

$$\begin{tikzcd} & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_{F}"'] \arrow[dr, "\mathcal{M}T_{-1_*}"] & \\ F(X) \arrow[rr, "\mathcal{M}c_*"'] & & H_{**}^{BM}(X)\otimes\mathbb{Q}. \end{tikzcd}$$

- $\underline{y=0}$:

$$\begin{tikzcd} & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_{G_0}"'] \arrow[dr, "\mathcal{M}T_{0*}"] & \\ G_0(X) \arrow[rr, "\mathcal{M}td_*"'] & & H_{**}^{BM}(X)\otimes\mathbb{Q}. \end{tikzcd}$$

- $\underline{y=1}$:

$$\begin{tikzcd} & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_{\Omega}"'] \arrow[dr, "\mathcal{M}T_{1*}"] & \\ \Omega(X) \arrow[rr, "\mathcal{M}L_*"'] & & H_{**}^{BM}(X)\otimes\mathbb{Q}. \end{tikzcd}$$

5. VERDIER-TYPE RIEMANN–ROCH FORMULAS

In this section we show *Verdier-type Riemann–Roch formulas*.

First we show *a Verdier-type Riemann–Roch formula for the motivic canonical $c\ell$ class for a smooth morphism*. Here we emphasize that we need a smooth morphism instead of a local complete intersection morphism:

Proposition 5.1. *Let $f : X \to Y$ be a smooth morphism in the category $\mathcal{V}_S$:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\ & S. & \end{tikzcd}$$

Then the following diagram commutes:

$$\begin{array}{ccc} K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S) & \xrightarrow{c\ell^{FJ}_*} & H^{BM}_*(Y) \otimes R \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle c\ell(T_f)\cap f^*} \\ K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) & \xrightarrow[c\ell^{FJ}_*]{} & H^{BM}_*(X) \otimes R, \end{array}$$

Here $f^* : H^{BM}_*(Y) \to H^{BM}_*(X)$ is the Gysin pullback homomorphism.

Proof. Let $[W \xrightarrow{p} Y] \in K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S)$ and consider the following diagram whose top square is a fiber square:

(5.2)
$$\begin{tikzcd} W' \arrow[rr, "f'"] \arrow[d, "p'"'] & & W \arrow[d, "p"] \\ X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\ & S. & \end{tikzcd}$$

We want to show that

$$c\ell^{FJ}_* f^*([W \xrightarrow{p} Y]) = c\ell(T_f) \cap f^*\left(c\ell^{FJ}_*([W \xrightarrow{p} Y])\right).$$

$$\begin{aligned} c\ell^{FJ}_* f^*([W \xrightarrow{p} Y]) &= c\ell^{FJ}_*([W' \xrightarrow{p'} X]) \\ &= p'_*(c\ell(T_{h\circ p'}) \cap [W']) \quad \text{(by definition of } c\ell^{FJ}_*) \end{aligned}$$

$$c\ell(T_f) \cap f^*\left(c\ell^{FJ}_*([W \xrightarrow{p} Y])\right) = c\ell(T_f) \cap f^*\left(p_*(c\ell(T_{k\circ p}) \cap [W])\right).$$

Since $p : W \to Y$ is proper and $f : X \to Y$ is smooth, hence flat, it follows from [Fu, Proposition 1.7] that we have the *base change formula*: $f^*p_* = p'_*{f'}^*$. The above equality continues as follows:

$$\begin{aligned} &= c\ell(T_f) \cap p'_*{f'}^*(c\ell(T_{k\circ p}) \cap [W]) \\ &= p'_*\left({p'}^*c\ell(T_f) \cap {f'}^*(c\ell(T_{k\circ p}) \cap [W])\right) \ \text{(projection formula)} \\ &= p'_*\left(c\ell({p'}^*T_f) \cap (c\ell({f'}^*T_{k\circ p}) \cap {f'}^*[W])\right) \ \text{(by [Fu, Theorem 3.2])} \\ &= p'_*\left((c\ell(T_{f'}) \cup c\ell({f'}^*T_{k\circ p})) \cap [{f'}^{-1}(W)]\right) \ \text{(by [Fu, Lemma1.7.1])} \\ &= p'_*\left(c\ell(T_{f'} + {f'}^*T_{k\circ p}) \cap [W']\right) \\ &= p'_*\left(c\ell(T_{k\circ p\circ f'}) \cap [W']\right) \quad (T_{k\circ p\circ f'} = T_{f'} + {f'}^*T_{k\circ p} \in K^0(W')\ ) \\ &= p'_*\left(c\ell(T_{h\circ p'}) \cap [W']\right). \end{aligned}$$

Therefore we get that $c\ell_*^{FJ} f^*([W \xrightarrow{p} Y]) = c\ell(T_f) \cap f^* \left(c\ell_*^{FJ}([W \xrightarrow{p} Y]) \right)$. $\square$

By the definition $K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{k} S)$ is the Grothendieck group of the monoid consisting of some elements of $\mathrm{Iso}^{Prop}(\mathcal{V}/X)$, hence a homomorphism

$$\Psi : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{k} S) \to H_*^{BM}(X) \otimes \mathbb{Q}[y]$$

satisfying

$$\Psi([V \xrightarrow{p} X]) = T_{y*}([V \xrightarrow{p} X]) \, (= T_{y*} \circ \Theta([V \xrightarrow{p} X]))$$

is uniquely determined. So we denote Ψ by the same symbol T_{y*}:

$$T_{y*} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{k} S) \to H_*^{BM}(X) \otimes \mathbb{Q}[y],$$

which is also called a motivic Hirzebruch class in the present set-up.

Secondly we show *a Verdier-type Riemann–Roch formula for the motivic Hirzebruch class for a smooth morphism*:

Proposition 5.3. *Let $f : X \to Y$ be a smooth morphism in the category $\mathcal{V}_S$ as in Proposition 5.1. Then the following diagram commutes:*

$$\begin{array}{ccc}
K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/Y \xrightarrow{k} S) & \xrightarrow{T_{y*}} & H_*^{BM}(Y) \otimes \mathbb{Q}[y] \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle td_{(y)}(T_f)\cap f^*} \\
K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) & \xrightarrow[T_{y*}]{} & H_*^{BM}(X) \otimes \mathbb{Q}[y].
\end{array}$$

Proof. For the above diagram (5.2) we want to show that

$$T_{y*} f^*([W \xrightarrow{p} Y]) = td_{(y)}(T_f) \cap f^* \left(T_{y*}([W \xrightarrow{p} Y]) \right).$$

Since it follows from Hironaka's resolution of singularities that any $[W \xrightarrow{p} Y]$ can be expressed as a linear combination

$$\sum_V a_V [V \xrightarrow{p_V} Y]$$

where $a_V \in \mathbb{Z}$, V is a smooth variety , and $p_V : V \to Y$ is proper, it suffices to show that

$$T_{y*} f^*([V \xrightarrow{p_V} Y]) = td_{(y)}(T_f) \cap f^* \left(T_{y*}([V \xrightarrow{p_V} Y]) \right).$$

Hence, from the beginning we can assume that in the above diagram 5.2 W is smooth and $p : W \to Y$ is proper, but here note that we DO NOT need the requirement that the composite $k \circ p : W \to S$ is a local complete intersection morphism. Here it should be noted that since W is smooth and $f' : W' \to W$ is smooth (because f' is the pullback of the smooth morphism $f : X \to Y$), W' is also smooth, which is crucial below.

$$\begin{aligned}
T_{y*} f^*([W \xrightarrow{p} Y]) &= T_{y*}([W' \xrightarrow{p'} X]) \\
&= T_{y*}(p'_*[W' \xrightarrow{\mathrm{id}_{W'}} W']) \\
&= p'_* T_{y*}([W' \xrightarrow{\mathrm{id}_{W'}} W']) \\
&= p'_*(td_{(y)}(TW') \cap [W']) \quad \text{(since } W' \text{ is smooth).}
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
& td_{(y)}(T_f) \cap f^*T_{y*}([W \xrightarrow{p} Y]) \\
&= td_{(y)}(T_f) \cap f^*T_{y*}(p_*[W \xrightarrow{\mathrm{id}_W} W]) \\
&= td_{(y)}(T_f) \cap f^*p_*(T_{y*}([W \xrightarrow{\mathrm{id}_W} W])) \\
&= td_{(y)}(T_f) \cap f^*p_*(td_{(y)}(TW) \cap [W])) \quad \text{(since } W \text{ is smooth)} \\
&= td_{(y)}(T_f) \cap p'_*{f'}^*(td_{(y)}(TW) \cap [W])) \\
&= p'_*\left({p'}^* td_{(y)}(T_f) \cap {f'}^*(td_{(y)}(TW) \cap [W])\right) \\
&= p'_*\left(td_{(y)}({p'}^*T_f) \cap \left({f'}^* td_{(y)}(TW) \cap {f'}^*[W]\right)\right) \\
&= p'_*\Big(\left(td_{(y)}(T_{f'}) \cup td_{(y)}({f'}^*TW)\right) \cap [{f'}^{-1}W])\Big) \\
&= p'_*\left(td_{(y)}(T_{f'} + {f'}^*TW) \cap [W']\right) \\
&= p'_*\left(td_{(y)}(TW') \cap [W']\right) \quad \text{(since } T_{f'} = TW' - {f'}^*TW).
\end{aligned}$$

Therefore we get that $T_{y*}f^*([W \xrightarrow{p} Y]) = td_{(y)}(T_f) \cap f^*\Big(T_{y*}([W \xrightarrow{p} Y])\Big)$. $\square$

Remark 5.4. The above proof of course implies that the following Verdier-type Riemann–Roch formula holds for the motivic Hirzebruch class $T_{y_*} : K_0(\mathcal{V}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}[y]$: for a smooth morphism $f : X \to Y$ in the category $\mathcal{V}$ the following diagram commutes:

$$\begin{array}{ccc}
K_0(\mathcal{V}/Y) & \xrightarrow{T_{y_*}} & H_*^{BM}(Y) \otimes \mathbb{Q}[y] \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle td_{(y)}(T_f) \cap f^*} \\
K_0(\mathcal{V}/X) & \xrightarrow[T_{y_*}]{} & H_*^{BM}(X) \otimes \mathbb{Q}[y].
\end{array}$$

Definition 5.5. For a smooth morphism $f : X \to Y$, the twisted Gysin pullback homomophism $f^{**} : H_*^{BM}(Y) \to H_*^{BM}(X)$ is defined by

$$f^{**} = (-)^{\dim(f)} f^* = (-1)^{\dim X - \dim Y} f^*.$$

(In other words, $(-)^{\mathrm{codim}(f)} f^{**} = (-1)^{\dim Y - \dim X} f^{**} = f^*$.) The contravariant Borel–Moore homology theory with this twisted pullback homomotphism for smoth morphisms is denoted by H_{**}^{BM}.

In [Yo4, Theorem 2.2] we obtained a Verdier-type Riemann–Roch formula of the Milnor class in a special case. The following Verdier-type Riemann–Roch formula of the motivc Milnor–Hirzebruch class is a generalization of this result:

Theorem 5.6. *For a smooth morphism $f : X \to Y$ in the category $\mathcal{V}_S$ as in Proposition 5.1, the following diagram commutes:*

$$\begin{array}{ccc}
K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/Y \xrightarrow{k} S) & \xrightarrow{\mathcal{M}T_{y_*}} & H_{**}^{BM}(Y) \otimes \mathbb{Q}[y] \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle td_{(y)}(T_f) \cap f^{**}} \\
K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) & \xrightarrow[\mathcal{M}T_{y_*}]{} & H_{**}^{BM}(X) \otimes \mathbb{Q}[y].
\end{array}$$

Proof. Let $[W \xrightarrow{p} Y] \in K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S)$. Then we have that

$$\begin{aligned}
&\mathcal{M}T_{y_*} f^*([W \xrightarrow{p} Y]) \\
&= \mathcal{M}T_{y_*}[W' \xrightarrow{p'} X]) \\
&= (-1)^{\dim W'} \left(T^{FJ}_{y_*} - T_{y_*}\right)([W' \xrightarrow{p'} X]) \\
&= (-1)^{\dim W'} \left(T^{FJ}_{y_*} - T_{y_*}\right)(f^*[W \xrightarrow{p} Y]) \\
&= (-1)^{\dim W'} \left(T^{FJ}_{y_*} f^* - T_{y_*} f^*\right)([W \xrightarrow{p} Y]) \\
&= (-1)^{\dim W'} \left(td_{(y)}(T_f) \cap f^* T^{FJ}_{y_*} - td_{(y)}(T_f) \cap f^* T_{y_*}\right)([W \xrightarrow{p} Y]) \\
&= (-1)^{\dim W'} td_{(y)}(T_f) \cap f^* \left(T^{FJ}_{y_*} - T_{y_*}\right)([W \xrightarrow{p} Y]) \\
&= (-1)^{\dim W'} (-)^{\operatorname{codim}(f)} td_{(y)}(T_f) \cap f^{**} \left(T^{FJ}_{y_*} - T_{y_*}\right)([W \xrightarrow{p} Y]) \\
&= (-1)^{\dim W' + \dim Y - \dim X} td_{(y)}(T_f) \cap f^{**} \left(T^{FJ}_{y_*} - T_{y_*}\right)([W \xrightarrow{p} Y]) \\
&= (-1)^{\dim W} td_{(y)}(T_f) \cap f^{**} \left(T^{FJ}_{y_*} - T_{y_*}\right)([W \xrightarrow{p} Y]) \\
&= td_{(y)}(T_f) \cap f^{**} \left((-1)^{\dim W} \left(T^{FJ}_{y_*} - T_{y_*}\right)([W \xrightarrow{p} Y])\right) \\
&= td_{(y)}(T_f) \cap f^{**} \left(\mathcal{M}T_{y_*}([W \xrightarrow{p} Y])\right).
\end{aligned}$$

□

Finally we give a "bivariant version" of Theorem 5.6:

Corollary 5.7. *For a smooth morphism $f : X \to Y$ in the category $\mathcal{V}_S$ as in Proposition 5.1, the following diagram commutes:*

$$\begin{array}{ccc}
K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S) & \xrightarrow{\widetilde{\mathcal{M}T_{y_*}}} & \mathbb{H}(Y \xrightarrow{h} S) \otimes \mathbb{Q}[y] \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (-1)^{\dim(f)} td_{(y)}(T_f) \bullet U_f \bullet} \\
K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) & \xrightarrow[\widetilde{\mathcal{M}T_{y_*}}]{} & \mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y],
\end{array}$$

Proof. The commutativity of the above diagram follows from Theorem 5.6, the following commutative diagram

$$\begin{array}{ccc}
\mathbb{H}(Y \xrightarrow{k} S) \otimes \mathbb{Q}[y] & \xrightarrow{\bullet[S]} & \mathbb{H}^{BM}_*(Y) \otimes \mathbb{Q}[y] \\
{\scriptstyle (-1)^{\dim(f)} td_{(y)}(T_f) \bullet U_f \bullet}\downarrow & & \downarrow{\scriptstyle td_{(y)}(T_f) \cap f^{**}} \\
\mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y] & \xrightarrow[\bullet[S]]{} & \mathbb{H}^{BM}_*(X) \otimes \mathbb{Q}[y],
\end{array}$$

and the fact (see [FM]) that for any $\beta \in \mathbb{H}(Y \to pt) = H^{BM}_*(Y)$

$$U_f \bullet \beta = f^* \beta$$

and also using the fact that $\bullet[S] : \mathbb{H}(X \xrightarrow{h} S) \xrightarrow{\cong} H^{BM}_*(X)$ is an isomorphism.

□

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