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ON REGULARITY CONDITIONS AT INFINITY

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Abstract. Let $f\colon X \to \mathbb{K}^p$ be a restriction of a polynomial mapping on X, where $X \subset \mathbb{K}^n$ is a smooth affine variety. We prove the equivalence of regularity conditions at infinity, which are useful to control the bifurcation set of f.

1. Introduction

Let $f\colon X \to \mathbb{K}^p$ be a differentiable mapping, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, X is a smooth affine variety and $\dim X \geq p$. The *bifurcation set of* f, denoted by $B(f)$, is the smallest subset of $\mathbb{K}^p$ such that f is a locally trivial topological fibration on $\mathbb{K}^p \setminus B(f)$.

The elements of $B(f)$ may come from critical values but also from regular values of f, i.e., $B(f) \setminus (B(f) \cap f(\text{Sing} f))$ can be not empty. In the example $f\colon \mathbb{K}^2 \to \mathbb{K}$, $f(x, y) = x + x^2 y$, the value $0 \in \mathbb{K}$ is not critical but there is no trivial fibration on any neighborhood of 0.

The study of bifurcation set $B(f)$ has connections with many other topics such as problems of optimization of polynomial functions $f\colon \mathbb{R}^n \to \mathbb{R}$ (see e.g. [HP]), generalizations of Ehresmann's Theorem (see e.g. [Ga, Je3, Ra]), Jacobian Conjecture (see e.g. [LW, ST]), global Łojasiewicz exponents (see e.g. [PZ, DG]), equisingularity and Milnor numbers (see e.g. [Ga, Pa1, ST, Ti2, Ti3]), stratification theory (see e.g. [KOS, Ti1]), etc...

A complete characterization of $B(f) \setminus (B(f) \cap f(\text{Sing} f))$ is yet an open problem. In fact, a characterization of $B(f) \setminus (B(f) \cap f(\text{Sing} f))$ is available only for polynomial functions $f\colon \mathbb{K}^2 \to \mathbb{K}$, see [Su, HL] for $\mathbb{K} = \mathbb{C}$ and [TZ] for $\mathbb{K} = \mathbb{R}$.

Through the use of *regularity conditions at infinity*, one has obtained some ways to approximate $B(f)$. For polynomial functions $f\colon \mathbb{K}^n \to \mathbb{K}$, see for instance [Br, CT, NZ, Pa1, Pa2, PZ, ST, Ti2, Ti3, Ti4].

For mappings, i.e., $p \geq 1$, Rabier [Ra] considered a regularity condition, which we call here *Rabier condition*. From this condition, Rabier defined the set of *asymptotic critical values* $K_\infty(f)$ and proved that $B(f) \subset (f(\text{Sing} f) \cup K_\infty(f))$. In fact, Rabier's results apply to C^2 maps $f\colon M \to N$, where M, N are Finsler manifolds.

For polynomial mappings $f\colon \mathbb{C}^n \to \mathbb{C}^p$, Gaffney [Ga] defined the *generalized Malgrange condition*, which we call here *Gaffney condition*. This condition yields the set $A_{G_\infty}(f)$ of non-regular values at infinity and, under additional hypothesis on f, Gaffney obtained

$$B(f) \subset (f(\text{Sing} f) \cup A_{G_\infty}(f)).$$

Kurdyka, Orro and Simon [KOS] also considered Rabier condition. They obtained an equivalence between Rabier condition and another condition which depends on *Kuo function*([Kuo]) (we call this last of *Kuo-KOS condition*). They showed that, for C^2 semi-algebraic mappings $f\colon \mathbb{R}^n \to \mathbb{R}^p$ (respectively, polynomial mappings $f\colon \mathbb{C}^n \to \mathbb{C}^p$), the set $K_\infty(f)$ is a closed semi-algebraic set (respectively, a closed algebraic set) of dimension at most $p - 1$.

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Jelonek [Je3] used another condition, which turns out to be equivalent to Rabier condition and to Gaffney condition. We call that condition *Jelonek condition*. Then, Jelonek [Je3] gave a more direct proof of the inclusion $B(f) \subset (f(\mathrm{Sing} f) \cup K_\infty(f))$.

The above four conditions are asymptotic conditions, which depend on the behaviours of the fibres of f and Jacobian matrix of f.

Another regularity condition at infinity is the *t-regularity*, a geometric grounded condition at infinity. The t-regularity has been introduced in [ST] for polynomial functions $f\colon \mathbb{C}^n \to \mathbb{C}$ and in [Ti3] for polynomial functions $f\colon \mathbb{R}^n \to \mathbb{R}$.

In [DRT], we considered the t-regularity for C^1 semi-algebraic mappings $f\colon \mathbb{R}^n \to \mathbb{R}^p$ and we proved that t-regularity is equivalent to the conditions of [Ra, KOS] (consequently, equivalent to the conditions of [Ga, Je3]).

In this paper, we extend the use of t-regularity to algebraic mappings $f\colon X \to \mathbb{K}^p$ and we replace $\mathbb{K}^n$ in the above results by a smooth affine variety X.

In section 4, we prove that t-regularity is equivalent to Rabier condition for $f\colon X \to \mathbb{K}^p$ (Theorem 4.1). This extends for mappings defined on X the equivalence proved in [DRT, Theorem 3.2] and the equivalence proved for $p = 1$ in [Pa2, ST].

It follows from Jelonek [Je4] that Rabier, Gaffney, Kuo-KOS and Jelonek conditions are also equivalent for mappings defined on X. Therefore, our Theorem 4.1 completes for these mappings the equivalences above mentioned in the case of mappings $f\colon \mathbb{K}^n \to \mathbb{K}^p$.

Another important set in the study of polynomial mappings is the set J_f of points at which f is not proper (see e.g. [Je1, Je2]). It was proved in [KOS, Proposition 3.1] that in the case of semi-algebraic maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$, the set J_f coincides with $K_\infty(f)$. This equality is crucial in the proof of the injectivity criterion of [CDTT, CDT].

In section 5, we consider $f\colon X \to \mathbb{R}^p$, where $\dim X = p$. We prove (Proposition 5.3) that $K_\infty(f) = J_f$, which extends for mappings defined on X the equality proved in [KOS, Proposition 3.1].

2. Basic Definitions

The goal of this section is to present Lemma 2.1, which will be useful to compute the Rabier function. We also introduce here some notations.

Let V, W be normed finite dimensional vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$. We denote by $\mathcal{L}(V, W)$ the set of linear mappings from V to W. For simplicity, we denote $\mathcal{L}(V, \mathbb{K})$ by V^*. Given $A \in \mathcal{L}(V, W)$, we denote by $A^* \in \mathcal{L}(W^*, V^*)$ the adjoint operator induced by A. For any linear subspace V of $\mathbb{K}^n$, we set

$$V^\perp := \{w \in \mathbb{K}^n \mid \langle w, v\rangle = 0, \forall v \in V\}.$$

We consider the following norm on $\mathcal{L}(V, W)$:

$$\|A\| := \max\{\|A(x)\|;\ x \in V \text{ and } \|x\| = 1\}, \text{ where } A \in \mathcal{L}(V, W). \tag{1}$$

We denote by e_i the vector of $\mathbb{K}^n$ with 1 in the i-th coordinate and zeros elsewhere. Let $A \in \mathcal{L}(\mathbb{K}^n, \mathbb{K})$, we denote by $\|(A(e_1), \dots, A(e_n))\|$ the Euclidean norm of the vector

$$(A(e_1), \dots, A(e_n)) \in \mathbb{K}^n.$$

Another norm on $\mathcal{L}(\mathbb{K}^n, \mathbb{K})$ can be defined as follows:

$$\|A\|_1 := \|(A(e_1), \dots, A(e_n))\|. \tag{2}$$

It is well known that norms (1) and (2) of $\mathcal{L}(\mathbb{K}^n, \mathbb{K})$ are equivalents (see e.g. [Yo, Theorem 6.8]). The next lemma will be useful in the sequel:

Lemma 2.1. *Let $V \subset \mathbb{K}^n$ be a linear subspace of $\mathbb{K}^n$. Given $A \in \mathcal{L}(\mathbb{K}^n, \mathbb{K})$, we denote by $A_{|V}$ the restriction of A to V and we set:*

$$\|A_{|V}\|_3 := \min\{\|(A(e_1), \dots, A(e_n)) + w\|; w \in V^{\perp}\}. \tag{3}$$

Then, the norms (1) and (3) of $A_{|V}$ are equivalent (indeed, one has $\|A_{|V}\|_3 = \|A_{|V}\|$).

Proof. Let $A \in \mathcal{L}(\mathbb{K}^n, \mathbb{K})$. For any vector $w \in V^{\perp}$ and $v = (v_1, \dots, v_n) \in V$, we may write $A(v) = \sum_{i=1}^{n} v_i A(e_i) = \langle v, (A(e_1), \dots, A(e_n))\rangle = \langle v, (A(e_1), \dots, A(e_n)) + w\rangle$, where the last equality follows from the fact that $w \in V^{\perp}$. These equalities and Cauchy-Schwarz inequality imply:

$$\|A(v)\| = \|\langle v, (A(e_1), \dots, A(e_n)) + w\rangle\| \leq \|v\|\|(A(e_1), \dots, A(e_n)) + w\|, \tag{4}$$

If $\|v\| = 1$, the inequality (4) gives $\|A(v)\| \leq \|(A(e_1), \dots, A(e_n)) + w\|$. Since v, w are arbitrary elements, this last inequality implies:

$$\|A_{|V}\| \leq \|A_{|V}\|_3. \tag{5}$$

To show $\|A_{|V}\|_3 \leq \|A_{|V}\|$, we write $(A(e_1), \dots, A(e_n)) = \mathrm{v}_1 + \mathrm{w}_1$, with $\mathrm{v}_1 \in V$ and $\mathrm{w}_1 \in V^{\perp}$ (this is possible since $\mathbb{K}^n = V \oplus V^{\perp}$). Then, for any $v \in V$, one obtains

$$A(v) = \langle v, (A(e_1), \dots, A(e_n))\rangle = \langle v, \mathrm{v}_1 + \mathrm{w}_1\rangle = \langle v, \mathrm{v}_1\rangle,$$

where the last equality follows from the fact that $\mathrm{w}_1 \in V^{\perp}$.

If $\mathrm{v}_1 = 0$ then $A_{|V} \equiv 0$ and $(A(e_1), \dots, A(e_n)) = \mathrm{w}_1$, which implies $\|A_{|V}\| = 0$ and $\|A_{|V}\|_1 = 0$. Therefore, the inequality $\|A_{|V}\|_3 \leq \|A_{|V}\|$ holds if $\mathrm{v}_1 = 0$.

If $\mathrm{v}_1 \neq 0$, we set $z := \frac{\mathrm{v}_1}{\|\mathrm{v}_1\|}$. Thus, $z \in V$, $\|z\| = 1$ and $A(z) = \langle z, \mathrm{v}_1\rangle = \|\mathrm{v}_1\|$, where the last equality follows from definition of z. Since $\|z\| = 1$, one has $\|A(z)\| = \|\mathrm{v}_1\| \leq \|A_{|V}\|$.

To finish, we observe that $(A(e_1), \dots, A(e_n)) - \mathrm{w}_1 = \mathrm{v}_1$, with $\mathrm{w}_1 \in V^{\perp}$. By definition of $\|A_{|V}\|_3$, this last equality implies $\|A_{|V}\|_3 \leq \|\mathrm{v}_1\|$. Thus, we conclude $\|A_{|V}\|_3 \leq \|\mathrm{v}_1\| \leq \|A_{|V}\|$, which follows $\|A_{|V}\|_3 \leq \|A_{|V}\|$. Therefore, from this last inequality and inequality (5), we obtain $\|A_{|V}\| = \|A_{|V}\|_3$, which finishes the proof. □

3. Regularity conditions for mappings

We introduce the main definitions leading to the notion of t-regularity and we define Rabier condition in §3.3.

3.1. t-regularity.

Let $\mathcal{X} \subset \mathbb{K}^m$ be a $\mathbb{K}$-analytic variety, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote the set of regular points of $\mathcal{X}$ by $\mathcal{X}_{\text{reg}}$ and the set of singular points of $\mathcal{X}$ by $\mathcal{X}_{\text{sing}}$. We assume that $\mathcal{X}$ contains at least a regular point.

Definition 3.1. Let $g : \mathcal{X} \to \mathbb{K}$ be an analytic function defined in some neighbourhood of $\mathcal{X}$ in $\mathbb{K}^m$. Let $\mathcal{X}_0$ denote the subset of $\mathcal{X}_{\text{reg}}$ where g is a submersion. The *relative conormal space of* g is defined as follows:

$$C_g(\mathcal{X}) := \text{closure}\{(x, H) \in \mathcal{X}_0 \times \check{\mathbb{P}}^{m-1} \mid T_x(g^{-1}(g(x))) \subset H\} \subset \overline{\mathcal{X}} \times \check{\mathbb{P}}^{m-1}.$$

We denote by $\pi : C_g(\mathcal{X}) \to \overline{\mathcal{X}}$ the projection $\pi(x, H) = x$.

For any $y \in \overline{\mathcal{X}}$ such that $g(y) = 0$, we define $C_{g,y}(\mathcal{X}) := \pi^{-1}(y)$. The following result shows that $C_{g,y}(\mathcal{X})$ depends on the germ of g at y only up to multiplication by some invertible analytic function germ γ.

Lemma 3.2 ([Ti4, Lemma 1.2.7])**.** *Let $\gamma : (\mathbb{K}^m, y) \to \mathbb{K}$ be an analytic function such that $\gamma(y) \neq 0$. Then $C_{\gamma g,y}(\mathcal{X}) = C_{g,y}(\mathcal{X})$.* □

We use coordinates $(x_1, \ldots, x_n)$ for $\mathbb{K}^n$ and coordinates $[x_0 : x_1 : \ldots : x_n]$ for the projective space $\mathbb{P}^n$. We denote by $\mathbb{H}^\infty = \{[x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n \mid x_0 = 0\}$ the hyperplane at infinity.

Let $f : X \to \mathbb{K}^p$ be the restriction of a polynomial mapping to a smooth affine variety $X \subset \mathbb{K}^n$, where $\dim X \geq p$. We set $\mathbb{X} := \overline{\mathrm{graph} f}$ as the closure of the graph of f in $\mathbb{P}^n \times \mathbb{K}^p$ and we set $\mathbb{X}^\infty := \mathbb{X} \cap (\mathbb{H}^\infty \times \mathbb{K}^p)$.

We consider the affine charts $U_j \times \mathbb{K}^p$ of $\mathbb{P}^n \times \mathbb{K}^p$, where $U_j = \{x_j \neq 0\}$ and $j = 0, 1, \ldots, n$. We identify the chart U_0 with the affine space $\mathbb{K}^n$. Thus, we have $\mathbb{X} \cap (U_0 \times \mathbb{K}^p) = \mathbb{X} \setminus \mathbb{X}^\infty = \mathrm{graph} f$ and $\mathbb{X}^\infty$ is covered by the charts $U_1 \times \mathbb{K}^p, \ldots, U_n \times \mathbb{K}^p$.

If g denotes the projection to the variable x_0 in some affine chart $U_j \times \mathbb{K}^p$, then the relative conormal $C_g(\mathbb{X} \backslash \mathbb{X}^\infty \cap U_j \times \mathbb{K}^p) \subset \mathbb{X} \times \check{\mathbb{P}}^{n+p-1}$ and the projection $\pi : C_g(\mathbb{X} \backslash \mathbb{X}^\infty \cap U_j \times \mathbb{K}^p) \to \mathbb{X}$, $\pi(y, H) = y$, are well-defined.

Let us then consider the space $\pi^{-1}(\mathbb{X}^\infty)$, which is well-defined for every chart $U_j \times \mathbb{K}^p$ as a subset of $C_g(\mathbb{X} \backslash \mathbb{X}^\infty \cap U_j \times \mathbb{K}^p)$. By Lemma 3.2, the definitions coincide at the intersections of the charts and one has:

Definition 3.3. The *space of characteristic covectors at infinity* is the well-defined set

$$\mathcal{C}^\infty := \pi^{-1}(\mathbb{X}^\infty).$$

For any $z_0 \in \mathbb{X}^\infty$, we denote $\mathcal{C}^\infty_{z_0} := \pi^{-1}(z_0)$.

We denote by $\tau : \mathbb{P}^n \times \mathbb{K}^p \to \mathbb{K}^p$ the second projection. The relative conormal space $C_\tau(\mathbb{P}^n \times \mathbb{K}^p)$ is defined as in Definition 3.1, where the function g is replaced by the application τ.

Definition 3.4 (t-regularity)**.** We say that f is *t-regular* at $z_0 \in \mathbb{X}^\infty$ if $C_\tau(\mathbb{P}^n \times \mathbb{K}^p) \cap \mathcal{C}^\infty_{z_0} = \emptyset$.

3.2. ***t*-regularity interpretation.** Let $X \subset \mathbb{K}^n$ be a smooth affine variety over $\mathbb{K}$. We suppose that X is a global complete intersection. In other words,

$$X = \{x \in \mathbb{K}^n \mid h_1(x) = h_2(x) = \ldots = h_r(x) = 0\}$$

and $\mathrm{rank}\, \mathrm{D}h(x) = r$, where $h = (h_1, \ldots, h_r) : \mathbb{K}^n \to \mathbb{K}^r$ and $\mathrm{D}h(x)$ denotes the Jacobian matrix of h at x.

Let $f = (f_1, \ldots, f_p) \colon X \to \mathbb{K}^p$ be the restriction of a polynomial mapping to X, where $\dim X \geq p$. Given $z_0 \in \mathbb{X}^\infty$, up to some linear change of coordinate, we may assume that $z_0 \in \mathbb{X}^\infty \cap (U_n \times \mathbb{K}^p)$. In the intersection of charts $(U_0 \cap U_n) \times \mathbb{K}^p$, we consider the change of coordinates $x_1 = y_1/y_0, \ldots, x_{n-1} = y_{n-1}/y_0$, $x_n = 1/y_0$, where $(x_1, \ldots, x_n)$ are the coordinates in U_0 and $(y_0, \ldots, y_{n-1})$ are those in U_n. Then for $i = 1, \ldots, p$ and $j = 1, \ldots, r$, we define:

$$F_i(y, t) = F_i(y_0, y_1, \ldots, y_{n-1}, t_1, \ldots, t_p) := f_i\left(y_1/y_0, \ldots, y_{n-1}/y_0, 1/y_0\right) - t_i, \tag{6}$$

$$H_j(y, t) = H_j(y_0, y_1, \ldots, y_{n-1}, t_1, \ldots, t_p) := h_j(y_1/y_0, \ldots, y_{n-1}/y_0, 1/y_0). \tag{7}$$

Define $H(y, t) := (H_1(y, t), \ldots, H_r(y, t))$ and $F(y, t) := (F_1(y, t), \ldots, F_p(y, t))$. Then

$$(X \times \mathbb{K}^p) \cap ((U_0 \cap U_n) \times \mathbb{K}^p) = H^{-1}(0)$$

and $\mathbb{X} \cap ((U_0 \cap U_n) \times \mathbb{K}^p) = F^{-1}(0) \cap H^{-1}(0)$.

We denote the normal vector to the hypersurface $\{y_0 = \text{constant}\}$ by

$$\vec{n_0} = (1, 0, \ldots, 0) \in \mathbb{K}^n \times \mathbb{K}^p.$$

Let us define $p + r$ normal vectors to $F^{-1}(0)$ at $(y, t) \in \mathbb{X} \cap ((U_0 \cap U_n) \times \mathbb{K}^p)$, as follows:
For $i = 1, \ldots, p$, define:

$$\vec{n_i}(y, t) = \nabla F_i(y, t) = (\nabla_n F_i(y, t), \nabla_p F_i(y, t)), \tag{8}$$

where

$$\nabla_n F_i(y,t) := \left(\frac{\partial F_i}{\partial y_0}(y,t), \cdots, \frac{\partial F_i}{\partial y_{n-1}}(y,t)\right), \quad \nabla_p F_i(y,t) := \left(\frac{\partial F_i}{\partial t_1}(y,t), \cdots, \frac{\partial F_i}{\partial t_p}(y,t)\right).$$

For $j = 1, \ldots, r$, define:

$$\vec{m}_j(y,t) = \nabla H_j(y,t) = \left(\frac{\partial H_j}{\partial y_0}(y,t), \ldots, \frac{\partial H_j}{\partial y_{n-1}}(y,t), 0, \ldots, 0\right). \tag{9}$$

By Definition 3.4, f is not t-regular at $z_0 \in \mathbb{X}^\infty$ if and only if there exists a sequence $\{(y_k, t_k)\}_{k\in\mathbb{N}} \subset \mathbb{X} \cap ((U_0 \cap U_n) \times \mathbb{K}^p)$ such that $(y_k, t_k) \to z_0$ and the tangent hyperplanes to the fibres of $g_{|\mathbb{X}}$ at (y_k, t_k) tend to a hyperplane W such that its normal line has a direction of the form $[0 : \cdots : 0 : b_1 : \cdots : b_p]$ in $\mathbb{P}^{n+p-1}$. More explicitly, there exists a sequence $\{(\psi_{0k}, \psi_{1k}, \ldots, \psi_{p_k}, \varphi_{1k}, \ldots, \varphi_{r_k})\}_{k\in\mathbb{N}} \subset \mathbb{K}^{p+r+1}$ such that

$$\lim_{k\to\infty}\left(\sum_{i=0}^{p} \psi_{ik}\vec{n}_i(y_k, t_k) + \sum_{j=1}^{r} \varphi_{jk}\vec{m}_j(y_k, t_k)\right)$$

of the linear combination of normal vectors $\vec{n}_i, \vec{m}_j$ has the direction

$$\vec{n}_W = [0 : 0 : \cdots : 0 : b_1 : \cdots : b_p] \in \mathbb{P}^{n+p-1}.$$

3.3. **Rabier function and Rabier condition.**

Definition 3.5 ([Ra, p. 651])**.** Given $A \in \mathcal{L}(V, W)$. The *Rabier function at* A is defined as follows:

$$\nu(A) := \inf\{\|A^*(\varphi)\|; \varphi \in W^* \text{ and } \|\varphi\| = 1\}. \tag{10}$$

For any vector $w = (w_1, \ldots, w_m) \in \mathbb{K}^m$, we denote the line matrix associated to w by $[w]$, i.e., $[w] = \left[\begin{array}{ccc} w_1 & \ldots & w_m \end{array}\right]$. Given $A \in \mathcal{L}(\mathbb{K}^n, \mathbb{K}^p)$, we denote by $[A]$ the matrix of A with respect to the canonical basis of $\mathbb{K}^n$ and $\mathbb{K}^p$. Thus, one has:

Lemma 3.6. *Let V be a linear subspace of $\mathbb{K}^n$. For any $A \in \mathcal{L}(\mathbb{K}^n, \mathbb{K}^p)$, if we set*

$$\nu_1(A_{|V}) := \inf\{\, \|\, [u][A] + [w]\,\|\,; w \in V^\perp, u \in \mathbb{K}^p \text{ and } \|u\| = 1\}, \tag{11}$$

then there are positive constants C_1 and C_2 such that $C_1\nu_1(A_{|V}) \leq \nu(A_{|V}) \leq C_2\nu_1(A_{|V})$.

Proof. The proof follows from Lemma 2.1 and Definition 3.5. □

Now, let $X \subset \mathbb{K}^n$ be a smooth affine variety over $\mathbb{K}$ and let $f : X \to \mathbb{K}^p$ be the restriction of a polynomial mapping to X, where $\dim X \geq p$. We have:

Definition 3.7 ([Ra])**.** The set of *asymptotic critical values of* f is defined as follows:

$$\begin{aligned} K_\infty(f) \;\; := \;\; & \{t \in \mathbb{K}^p \mid \exists\{x_j\}_{j\in\mathbb{N}} \subset X, \lim_{j\to\infty}\|x_j\| = \infty, \\ & \lim_{j\to\infty} f(x_j) = t \text{ and } \lim_{j\to\infty}\|x_j\|\nu(\mathrm{D}f(x_j)_{|T_{x_j}X}) = 0\}, \end{aligned} \tag{12}$$

where $\nu(-)$ is defined as in Definition 3.5.

We reformulate the above condition in a localized version, at some point at infinity $z_0 \in \mathbb{X}^\infty$, as follows:

Definition 3.8 (Rabier condition)**.** We say that $z_0 \in \mathbb{X}^\infty$ is an *asymptotic critical point of* f if and only if there exists $\{x_j\}_{j\in\mathbb{N}} \subset X \simeq \text{graph}f$ such that $\lim_{j\to\infty}(x_j, f(x_j)) = z_0$ and $\tau(z_0) \in K_\infty(f)$, where $\tau\colon \mathbb{P}^n \times \mathbb{K}^p \to \mathbb{K}^p$ denotes the second projection.

We say that $z_0 \in \mathbb{X}^\infty$ satisfies *Rabier condition* if z_0 is not an asymptotic critical point of f.

Remark 3.9. From Lemma 3.6, we obtain the same set of Definition 3.7 if we replace ν by the function ν_1 defined in (11).

4. Equivalence of regularity conditions

The goal of this section is to prove an equivalence between t-regularity and Rabier condition.

Let $X \subset \mathbb{K}^n$ be a smooth affine variety over $\mathbb{K}$. We suppose that X is a global complete intersection. In other words, $X = \{x \in \mathbb{K}^n \mid h_1(x) = h_2(x) = \ldots = h_r(x) = 0\}$ and $\operatorname{rank} \mathrm{D}h(x) = r$, for any $x \in X$, where $h = (h_1, \ldots, h_r) : \mathbb{K}^n \to \mathbb{K}^r$ and $\mathrm{D}h(x)$ denotes the Jacobian matrix of h at x (see Remark 4.2). With above definitions and statements, we have:

Theorem 4.1. *Let $f : X \to \mathbb{K}^p$ be a non-constant polynomial mapping, with $\dim X \geq p$. Let $z_0 \in \mathbb{X}^\infty$. Then f is t-regular at z_0 if and only if z_0 is not an asymptotic critical point of f.*

Proof. We may assume (eventually after some linear change of coordinates) that

$$z_0 \in \mathbb{X}^\infty \cap (U_n \times \mathbb{R}^p)$$

and that $|x_n| \geq |x_i|$, $i = 1, \ldots, n-1$, for x in some neighbourhood of z_0.

"$\Rightarrow$". Let z_0 be an asymptotic critical point of f. By Definition 3.8 and Remark 3.9, this means that there exist sequences $\{(\psi_k, \varphi_k) = ((\psi_{1k}, \ldots, \psi_{pk}), (\varphi_{1k}, \ldots, \varphi_{r_k}))\}_{k \in \mathbb{N}} \subset \mathbb{K}^{p+r}$ and $\{x_k := (x_{1k}, \ldots, x_{nk})\}_{k \in \mathbb{N}} \subset X$, where $\|\psi_k\| = 1$ and $\lim_{k\to\infty}(\psi_k, \varphi_k) = (\psi, \varphi)$, such that $\lim_{k\to\infty} \psi_k = \psi = (\psi_1, \ldots, \psi_p) \neq (0, \ldots, 0)$, $\lim_{k\to\infty}(x_k, f(x_k)) = z_0$ and:

(13)
$$\|x_k\| \left\| \left(\sum_{i=1}^{p} \psi_{ik} \frac{\partial f_i}{\partial x_1}(x_k) + \sum_{j=1}^{r} \varphi_{jk} \frac{\partial h_j}{\partial x_1}(x_k), \ldots, \sum_{i=1}^{p} \psi_{ik} \frac{\partial f_i}{\partial x_n}(x_k) + \sum_{j=1}^{r} \psi_{jk} \frac{\partial h_j}{\partial x_n}(x_k) \right) \right\| \to 0.$$

Since for large enough k we have $|x_{nk}| \geq |x_{ik}|$, $i = 1, \ldots, n-1$, we may replace in (13) $\|x_k\|$ by $|x_{nk}|$ and then multiply the sums of (13) by x_{nk}.

In the notations of §3.2, by changing coordinates within $U_0 \cap U_n$, one has $y_0 = 1/x_n, y_i = x_i/x_n$ and the relations:

$$(14) \qquad \begin{cases} \frac{\partial F_j}{\partial y_i}(y,t) = x_n \frac{\partial f_j}{\partial x_i}(x), & 1 \leq i \leq n-1,\ 1 \leq j \leq p, \\ \frac{\partial F_j}{\partial t_l}(y,t) = -\delta_{l,j}, & 1 \leq j, l \leq p, \\ \frac{\partial F_j}{\partial y_0}(y,t) = -x_n(x_1 \frac{\partial f_j}{\partial x_1}(x) + \ldots + x_n \frac{\partial f_j}{\partial x_n}(x)), & 1 \leq j \leq p. \end{cases}$$

$$(15) \qquad \begin{cases} \frac{\partial H_j}{\partial y_i}(y,t) = x_n \frac{\partial h_j}{\partial x_i}(x), & 1 \leq i \leq n-1,\ 1 \leq j \leq r, \\ \frac{\partial H_j}{\partial t_l}(y,t) = 0, & 1 \leq j \leq r,\ 1 \leq l \leq p, \\ \frac{\partial H_j}{\partial y_0}(y,t) = -x_n(x_1 \frac{\partial h_j}{\partial x_1}(x) + \ldots + x_n \frac{\partial h_j}{\partial x_n}(x)), & 1 \leq j \leq r. \end{cases}$$

The condition (13) yields:

(16)
$$\left\| \left(\left(\sum_{i=1}^{p} \psi_{ik} \frac{\partial F_i}{\partial y_1} + \sum_{j=1}^{r} \varphi_{jk} \frac{\partial H_j}{\partial y_1} \right)(y_k, t_k), \ldots, \left(\sum_{i=1}^{p} \psi_{ik} \frac{\partial F_i}{\partial y_{n-1}} + \sum_{j=1}^{r} \varphi_{jk} \frac{\partial H_j}{\partial y_{n-1}} \right)(y_k, t_k) \right) \right\| \to 0.$$

We set $\vec{n}_{W_k} := (0, \omega_k, -\psi_{1k}, \ldots, -\psi_{pk})$, where ω_k is the vector of equation (16). Let W_k be the hyperplane defined by $\vec{n}_{W_k}$. Let $\vec{n}_i$ and $\vec{m}_j$ be the vectors defined in §3.2. Then, the vectors

$\{\vec{n}_{W_k}\}$ are linear combinations of $\vec{n}_i$ and $\vec{m}_j$ with coefficients $\{\psi_{ik}, \varphi_{jk}\}$, and the hyperplanes W_k are tangent to the levels of the function $g_{|\mathbb{X}}$. Since we have supposed

$$\lim_{k\to\infty}(\psi_{1k}, \dots, \psi_{pk}) = (\psi_1, \dots, \psi_p) \neq (0, \dots, 0),$$

it follows from definition of $\vec{n}_{W_k}$ and equation (16) that:

$$\lim_{k\to\infty} \vec{n}_{W_k} = [0 : 0 : \dots : 0 : \psi_1 : \dots : \psi_p].$$

Denote by W the hyperplane defined by $[0 : 0 : \dots : 0 : \psi_1 : \dots : \psi_p]$. Then $W = \lim_{k\to\infty} W_k$, which implies that W belongs to $\mathcal{C}^\infty_{z_0}$ and consequently f is not t-regular at z_0 (see §3.2).

"$\Leftarrow$". Let $z_0 \in \mathbb{X}^\infty$ be not t-regular. By Definition 3.4, this means that there exist a sequence of points $\{(\mathrm{y}_k, \mathrm{t}_k)\}_{k\in\mathbb{N}} \subset \mathbb{X} \cap ((U_0 \cap U_n) \times \mathbb{K}^p)$ tending to z_0, and a sequence of hyperplanes W_k tangent to the levels of g at $(\mathrm{y}_k, \mathrm{t}_k)$, such that $W_k \to W \in \mathcal{C}^\infty_{z_0}$.

Let $\vec{n}_i$ and $\vec{m}_j$ be the vectors defined in §3.2. From §3.2, if f is not t-regular at z_0 then there exist sequences $\{\tilde{\psi}_k = (\tilde{\psi}_{1k}, \dots, \tilde{\psi}_{pk})\}_{k\in\mathbb{N}} \subset \mathbb{K}^p$, $\{\tilde{\varphi}_k = (\tilde{\varphi}_{1k}, \dots, \tilde{\varphi}_{rk})\}_{k\in\mathbb{N}} \subset \mathbb{K}^r$ and $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{K}$ such that $\vec{n}_{W_k} = \lambda_k \vec{n}_0(\mathrm{y}_k, \mathrm{t}_k) + \sum_i \tilde{\psi}_{ik}\vec{n}_i(\mathrm{y}_k, \mathrm{t}_k) + \sum_j \tilde{\varphi}_{jk}\vec{m}_j(\mathrm{y}_k, \mathrm{t}_k)$ and that $\lim_{k\to\infty} \vec{n}_{W_k} = \left[0 : 0 : \dots : 0 : \tilde{\psi}_1 : \dots : \tilde{\psi}_p\right]$, where $\left(\tilde{\psi}_1, \dots, \tilde{\psi}_p\right) \neq (0, \dots, 0)$. By assumption, the vector $\vec{n}_{W_k}$ has the following expression:

(a) In the first coordinate of $\vec{n}_{W_k}$ one has: $\lambda_k + \left(\sum_{i=1}^{p} \tilde{\psi}_{ik} \frac{\partial F_i}{\partial y_0} + \sum_{j=1}^{r} \tilde{\varphi}_{jk} \frac{\partial H_i}{\partial y_0}\right)(\mathrm{y}_k, \mathrm{t}_k)$.

(b) In the l-th coordinate, with $2 \leq l \leq n$, one has: $\left(\sum_{i=1}^{p} \tilde{\psi}_{ik} \frac{\partial F_i}{\partial y_l} + \sum_{j=1}^{r} \tilde{\varphi}_{jk} \frac{\partial H_j}{\partial y_l}\right)(\mathrm{y}_k, \mathrm{t}_k)$.

(c) In the q-th coordinate, with $n+1 \leq q \leq n+p$, one has: $-\tilde{\psi}_{qk}$.

We may take $\lambda_k := -\sum_{i=1}^{p} \tilde{\psi}_{ik} \frac{\partial F_i}{\partial y_0}(\mathrm{y}_k, \mathrm{t}_k) - \sum_{j=1}^{r} \tilde{\varphi}_{jk} \frac{\partial H_i}{\partial y_0}(\mathrm{y}_k, \mathrm{t}_k)$. After, we divide out by $\mu_k := \|(\tilde{\psi}_{1k}, \dots, \tilde{\psi}_{pk})\|$. Then, we replace $\tilde{\psi}_{ik}$ and $\tilde{\varphi}_{jk}$ by $\psi_{ik} := \frac{\tilde{\psi}_{ik}}{\mu_k}$ and $\varphi_{jk} := \frac{\tilde{\varphi}_{jk}}{\mu_k}$, respectively. This implies that $\|(\psi_{1k}, \dots, \psi_{pk})\| = 1$ and $\lim_{k\to\infty} \vec{n}_{W_k} = [0 : \dots : 0 : \psi_1 : \dots : \psi_p]$, where $(\psi_1, \dots, \psi_p) \neq (0, \dots, 0)$. Therefore,

$$\lim_{k\to\infty} \sum_{i=1}^{p} \psi_{ik} \frac{\partial F_i}{\partial y_l}(\mathrm{y}_k, \mathrm{t}_k) + \sum_{j=1}^{r} \varphi_{jk} \frac{\partial H_j}{\partial y_l}(\mathrm{y}_k, \mathrm{t}_k) = 0, \text{ for any } 1 \leq l \leq n-1. \tag{17}$$

By using (14) and (15), this is equivalent to:

$$\lim_{k\to\infty} x_{nk} \left(\sum_{i=1}^{p} \psi_{ik} \frac{\partial f_i}{\partial x_l}(x_k) + \sum_{j=1}^{r} \varphi_{jk} \frac{\partial h_j}{\partial x_l}(x_k)\right) = 0, \tag{18}$$

for $1 \leq l \leq n-1$, and one has $|x_{nk}| \geq \frac{1}{\sqrt{n}}\|x_k\|$ for large enough k. Therefore, in order to get the limit (13) it remains to prove that (18) is true for $l = n$. The rest of our argument is devoted to this proof.

From relations (14) and (15), we obtain $x_n \frac{\partial f_i}{\partial x_n}(x) = -\sum_{j=0}^{n-1} y_j \frac{\partial F_i}{\partial y_j}(y, t)$ and

$$x_n \frac{\partial h_i}{\partial x_n}(x) = -\sum_{j=0}^{n-1} y_j \frac{\partial H_i}{\partial y_j}(y, t).$$

Therefore:

$$x_{nk}\sum_{i=1}^{p}\psi_{ik}\frac{\partial f_i}{\partial x_n}(x_k)=-\sum_{j=1}^{n-1}\sum_{i=1}^{p}y_{jk}\psi_{ik}\frac{\partial F_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k)-\sum_{i=1}^{p}\psi_{ik}y_{0k}\frac{\partial F_i}{\partial y_0}(\mathrm{y}_k,\mathrm{t}_k). \tag{19}$$

$$x_{nk}\sum_{i=1}^{r}\varphi_{ik}\frac{\partial h_i}{\partial x_n}(x_k)=-\sum_{j=1}^{n-1}\sum_{i=1}^{r}y_{jk}\varphi_{ik}\frac{\partial H_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k)-\sum_{i=1}^{r}\varphi_{ik}y_{0k}\frac{\partial H_i}{\partial y_0}(\mathrm{y}_k,\mathrm{t}_k). \tag{20}$$

We will show that the following two terms tend to zero:

$$\sum_{j=1}^{n-1}\sum_{i=1}^{p}y_{jk}\psi_{ik}\frac{\partial F_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k)+\sum_{j=1}^{n-1}\sum_{i=1}^{r}y_{jk}\varphi_{ik}\frac{\partial H_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k),\ \text{and} \tag{21}$$

$$\sum_{i=1}^{p}\psi_{ik}y_{0k}\frac{\partial F_i}{\partial y_0}(\mathrm{y}_k,\mathrm{t}_k)+\sum_{i=1}^{r}\varphi_{ik}y_{0k}\frac{\partial H_i}{\partial y_0}(\mathrm{y}_k,\mathrm{t}_k). \tag{22}$$

First, we have:

$$\left\|\sum_{j=1}^{n-1}\sum_{i=1}^{p}y_{jk}\psi_{ik}\frac{\partial F_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k)+\sum_{j=1}^{n-1}\sum_{i=1}^{r}y_{jk}\varphi_{ik}\frac{\partial H_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k)\right\|\le \tag{23}$$

$$\left\|\frac{x_k}{x_{nk}}\right\|\left\|((\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial y_1}+\sum_{i=1}^{r}\varphi_{ik}\frac{\partial H_i}{\partial y_1})(\mathrm{y}_k,\mathrm{t}_k),\dots,(\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial y_{n-1}}+\sum_{i=1}^{r}\varphi_{ik}\frac{\partial H_i}{\partial y_{n-1}})(\mathrm{y}_k,\mathrm{t}_k))\right\|,$$

since by hypothesis $|y_{jk}|=|\frac{x_{jk}}{x_{nk}}|\le 1$ for large enough k. Then we obtain from (17) that the right hand side of (23) tends to zero as $k\to\infty$, which shows that (21) tends to zero.

To show that (22) tends to zero, let us assume that the following inequality holds for large enough $k\gg 1$, the proof of which will be given below:

$$\left\|\sum_{i=1}^{p}\psi_{ik}y_{0k}\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\varphi_{jk}y_{0k}\frac{\partial H_j}{\partial y_0}\right\|\ll \tag{24}$$

$$\left\|(\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial y_1}+\sum_{j=1}^{r}\varphi_{jk}\frac{\partial H_j}{\partial y_1},\dots,\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial y_{n-1}}+\sum_{j=1}^{r}\varphi_{jk}\frac{\partial H_j}{\partial y_{n-1}},\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial t_1},\dots,\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial t_p})\right\|.$$

Then, by using (17), (24) and the equality $\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial t_l}=-\psi_{lk}$ for any $1\le l\le p$ (implied by (14)), we have:

$$\left\|\sum_{i=1}^{p}\psi_{ik}y_{0k}\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\varphi_{jk}y_{0k}\frac{\partial H_j}{\partial y_0}\right\|\ll\|\psi_k\|=1.$$

This implies $\lim_{k\to\infty}\|(\sum_{i=1}^{p}\psi_{ik}y_{0k}\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\varphi_{jk}y_{0k}\frac{\partial H_j}{\partial y_0})(\mathrm{y}_k,\mathrm{t}_k)\|=0$, which shows that (22) tends to zero as $k\to\infty$.

We have shown that (21) and (22) tend to zero as $k\to\infty$. From the equations (19) and (20), we have that the sum (21) + (22) is equal to equation of (18) with $l=n$. These imply that (18) is also true for $l=n$. This completes our proof of relation (13) showing that z_0 is an asymptotic critical point of f.

Let us now give the proof of (24). Suppose not; this means that there exists $\delta > 0$ such that for $k \gg 1$ we have:
(25)

$$\frac{\left\|\sum_{i=1}^{p} \psi_{ik} y_{0k} \frac{\partial F_i}{\partial y_0} + \sum_{j=1}^{r} \varphi_{jk} y_{0k} \frac{\partial H_j}{\partial y_0}\right\|}{\left\|(\sum_{i=1}^{p} \psi_{ik} \frac{\partial F_i}{\partial y_1} + \sum_{j=1}^{r} \varphi_{jk} \frac{\partial H_j}{\partial y_1}, \dots, \sum_{i=1}^{p} \psi_{ik} \frac{\partial F_i}{\partial y_{n-1}} + \sum_{j=1}^{r} \varphi_{jk} \frac{\partial H_j}{\partial y_{n-1}}, -\psi_{1k}, \dots, -\psi_{pk})\right\|} > \delta,$$

where, by relations (14), we have $-\psi_{lk} = \sum_{i=1}^{p} \psi_{ik} \frac{\partial F_i}{\partial t_l}$, for $1 \leq l \leq p$. The set:

$$\mathcal{W} = \{((y,t),\psi,\varphi) \in ((U_n \cap U_0) \times \mathbb{K}^p \times \mathbb{K}^p \times \mathbb{K}^r) \cap (\mathbb{X} \times S_1^{p-1} \times \mathbb{K}^r) \mid (25) \text{ holds for } ((y,t),\psi,\varphi)\}$$

is a semi-algebraic set and we have $((\mathrm{y}_k, \mathrm{t}_k), \psi_k, \varphi_k) \in \mathcal{W}$ for $k \gg 1$. We observe that if $((y,t),\psi,\varphi) \in \mathcal{W}$ then $((y,t),\gamma\psi,\gamma\varphi) \in \mathcal{W}$, for any $\gamma \in \mathbb{K}^*$. This last observation implies that $((\mathrm{y}_k, \mathrm{t}_k), \tilde{\psi_k}, \tilde{\varphi_k}) \in \mathcal{W}$, where $\tilde{\psi_k} := \frac{\psi_k}{\|(\psi_k,\varphi_k)\|}$ and $\tilde{\varphi_k} := \frac{\varphi_k}{\|(\psi_k,\varphi_k)\|}$.

Since $\lim_{k\to\infty} \psi_k \to \psi \neq 0$, one may suppose that $\lim_{k\to\infty}(\tilde{\psi_k}, \tilde{\varphi_k}) \to (\tilde{\psi}, \tilde{\varphi})$, with $(\tilde{\psi}, \tilde{\varphi}) \neq 0$. Then $\lim_{k\to\infty}((\mathrm{y}_k, \mathrm{t}_k), \tilde{\psi_k}, \tilde{\varphi_k}) = (z_0, \tilde{\psi}, \tilde{\varphi})$ and by the curve selection lemma [Mi] there exists an analytic curve $\lambda = (\phi, \psi, \varphi) : [0, \varepsilon[\to \overline{\mathcal{W}}$ such that $\lambda(]0,\varepsilon[) \subset \mathcal{W}$ and $\lambda(0) = (z_0, \psi, \varphi)$. We denote

$$\phi(s) = (y_0(s), y_1(s), \dots, y_{n-1}(s), t_1(s), \dots, t_p(s)), \qquad \psi(s) = (\psi_1(s), \dots, \psi_p(s)), \text{ and}$$

$$\varphi(s) = (\varphi_1(s), \dots, \varphi_r(s)).$$

Since $(F,H)(\phi(s)) \equiv 0$, we have:

$$0 = \frac{d}{ds}(F,H)(\phi(s)) = y_0'(s)\frac{\partial(F,H)}{\partial y_0}(\phi(s)) + \sum_{i=1}^{n-1} y_i'(s)\frac{\partial(F,H)}{\partial y_i}(\phi(s)) + \sum_{i=1}^{p} t_i'(s)\frac{\partial(F,H)}{\partial t_i}(\phi(s)),$$

where $\frac{\partial(F,H)}{\partial y_i} = (\frac{\partial F_1}{\partial y_i}, \dots, \frac{\partial F_p}{\partial y_i}, \frac{\partial H_1}{\partial y_i}, \dots, \frac{\partial H_r}{\partial y_i})$.

Multiplying by $(\psi(s), \varphi(s))$ we obtain:

$$(26) \quad -y_0'(s)\left(\left(\sum_{i=1}^{p} \psi_i(s)\frac{\partial F_i}{\partial y_0} + \sum_{j=1}^{r} \varphi_j \frac{\partial H_j}{\partial y_0}\right)(\phi(s))\right) =$$
$$\sum_{l=1}^{n-1} y_l'(s)\left(\left(\sum_{i=1}^{p} \psi_i(s)\frac{\partial F_i}{\partial y_l} + \sum_{j=1}^{r} \varphi_j \frac{\partial H_j}{\partial y_l}\right)(\phi(s))\right) + \sum_{l=1}^{p} t_l'(s) \sum_{i=1}^{p} \psi_i(s) \frac{\partial F_i}{\partial t_l}(\phi(s)).$$

Since ϕ is analytic, thus bounded at $s = 0$, by applying the Cauchy-Schwarz inequality one finds a constant $C > 0$ such that:

$$(27) \quad \left| y_0'(s)\left(\sum_{i=1}^{p} \psi_i(s)\frac{\partial F_i}{\partial y_0} + \sum_{j=1}^{r} \varphi_j \frac{\partial H_j}{\partial y_0}\right)(\phi(s)) \right| \leq$$
$$C \left\| \left((\sum_{i=1}^{p} \psi_i \frac{\partial F_i}{\partial y_1} + \sum_{j=1}^{r} \varphi_j \frac{\partial H_j}{\partial y_1})(\phi), \dots, (\sum_{i=1}^{p} \psi_i \frac{\partial F_i}{\partial y_{n-1}} + \sum_{j=1}^{r} \varphi_j \frac{\partial H_j}{\partial y_{n-1}})(\phi), \psi_1, \dots, \psi_p \right)(s) \right\|.$$

We have $l := \mathrm{ord}_s y_0'(s) \geq 0$ and $\mathrm{ord}_s y_0(s) = l + 1 \geq 1$ since $y_0(0) = 0$. Thus $\left| y_0(s)(\sum_{i=1}^{p} \psi_i(s) \frac{\partial F_i}{\partial y_0} + \sum_{j=1}^{r} \psi_j \frac{\partial H_j}{\partial y_0})(\phi(s)) \right| \ll \left| y_0'(s)(\sum_{i=1}^{p} \psi_i(s) \frac{\partial F_i}{\partial y_0} + \sum_{j=1}^{r} \psi_j \frac{\partial H_j}{\partial y_0})(\phi(s)) \right|$.

This and (27) give:

$$\left\| y_0(s)(\sum_{i=1}^{p}\psi_i(s)\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\psi_j\frac{\partial H_j}{\partial y_0})(\phi(s))\right\| \ll$$
$$\left\| \left((\sum_{i=1}^{p}\psi_i\frac{\partial F_i}{\partial y_1}+\sum_{j=1}^{r}\varphi_j\frac{\partial H_j}{\partial y_1})(\phi),\ldots,(\sum_{i=1}^{p}\psi_i\frac{\partial F_i}{\partial y_{n-1}}+\sum_{j=1}^{r}\varphi_j\frac{\partial H_j}{\partial y_{n-1}})(\phi),\psi_1,\ldots,\psi_p\right)(s)\right\|,$$

which contradicts our assumption that $(\phi(s),\psi(s),\varphi(s))\in\mathcal{W}$, for $s\in]0,\varepsilon[$. Therefore, we conclude that (24) holds, which completes the proof of Theorem 4.1. □

The above theorem extends for mappings defined on X the equivalence proved in [DRT, Theorem 3.2]. It also extends an equivalence proved for $p=1$ in [Pa2, ST].

Remark 4.2. In Theorem 4.1 we suppose that $X\subset\mathbb{K}^n$ is a complete intersection. It is well known that any manifold is a locally complete intersection (see e.g [GP, p. 18]). So, in the general case of a smooth affine variety X, one may take a locally finite cover $\{U_i\}$ of $\mathbb{K}^n$ such that the manifold $X_i:=X\cap U_i$ is a complete intersection. Then we consider the normal vector fields on each X_i as in §3.2 and we use a partition of unity subordinate to the cover $\{U_i\}$ to obtain normal vector fields defined on X. Then the proof of Theorem 4.1 in the general case is the same as above.

5. t-regularity and Jelonek set

In this section, we consider $f\colon X\to\mathbb{R}^p$, where $\dim X=p$. We prove that, in this case, t-regularity is related with the Jelonek set J_f ([Je1]). We begin with:

Definition 5.1 ([Je1, Definition 3.3])**.** Let $f\colon M\to N$ be a continuous mapping, where M,N are manifolds. We say that f is proper at a point $t_0\in N$ if there exists an open neighbourhood U of t_0 such that the restriction $f_{|f^{-1}(U)}\colon f^{-1}(U)\to U$ is a proper mapping. We denote by J_f the set of points at which f is not proper.

See for instance [Je1, Je2] for applications and related problems with J_f.

Definition 5.2. Let $f:X\to\mathbb{K}^p$ be the restriction of a polynomial mapping to a smooth variety X, where $\dim X\geq p$. We set

$$\mathcal{NT}_\infty(f):=\{t_0=\tau(z_0)\in\mathbb{K}^p\mid z_0\in\mathbb{X}^\infty\text{ and }z_0\text{ is not }t\text{-regular}\}. \tag{28}$$

When $\dim X=p$, we have:

Proposition 5.3. *Let $X\subset\mathbb{R}^n$ be a smooth affine variety over $\mathbb{R}$. We suppose that X is a global complete intersection. In other words $X=\{x\in\mathbb{R}^n\mid h_1(x)=h_2(x)=\ldots=h_r(x)=0\}$ and $\operatorname{rank}\mathrm{D}h(x)=r$, for any $x\in X$, where $h=(h_1,\ldots,h_r):\mathbb{R}^n\to\mathbb{R}^r$ and $\mathrm{D}h(x)$ denotes the Jacobian matrix of h at x.*

Let $f=(f_1,\ldots,f_p):X\to\mathbb{R}^p$ be the restriction of a polynomial mapping to X, where $\dim X=n-r=p$. Then $\mathcal{NT}_\infty(f)=K_\infty(f)=J_f$.

Proof. The equality $\mathcal{NT}_\infty(f)=K_\infty(f)$ follows directly from Theorem 4.1. Thus, we need only show the equality $K_\infty(f)=J_f$.

The inclusion $K_\infty(f)\subset J_f$ follows directly from Definitions 3.7 and 5.1. On the other hand, let $t_0\in J_f$. By the curve selection lemma [Mi], there exists an analytic path

$$\phi=(\phi_1,\ldots,\phi_n):]0,\epsilon[\to X\subset\mathbb{R}^n$$

such that $\lim_{s\to 0}\|\phi(s)\| = \infty$ and $\lim_{s\to 0} f(\phi(s)) = t_0$.

Consider

$$\frac{\partial f_i}{\partial x}(x) := \left(\frac{\partial f_i}{\partial x_1}(x), \dots, \frac{\partial f_i}{\partial x_n}(x)\right), \text{ for } i = 1, \dots, p, \tag{29}$$

$$\frac{\partial h_j}{\partial x}(x) := \left(\frac{\partial h_j}{\partial x_1}(x), \dots, \frac{\partial h_j}{\partial x_n}(x)\right), \text{ for } j = 1, \dots, r. \tag{30}$$

Since $n = h + r$, there exist analytic curves $\tilde{\lambda}(s), \tilde{\varphi}_1(s), \dots, \tilde{\varphi}_p(s), \tilde{\psi}_1(s), \dots, \tilde{\psi}_r(s)$, from $]0, \epsilon[$ to $\mathbb{R}$, such that $(\tilde{\lambda}(s), \tilde{\varphi}_1(s), \dots, \tilde{\varphi}_p(s), \tilde{\psi}_1(s), \dots, \tilde{\psi}_r(s)) \neq (0, \dots, 0)$, for any $s \in]0, \epsilon[$, and the following equality holds:

$$\tilde{\lambda}(s)(\phi_1(s), \dots, \phi_n(s)) = \sum_{i=1}^{p} \tilde{\varphi}_i(s)\frac{\partial f_i}{\partial x}(\phi(s)) + \sum_{j=1}^{r} \tilde{\psi}_j(s)\frac{\partial h_j}{\partial x}(\phi(s)). \tag{31}$$

Let $\tilde{\varphi}(s) := (\tilde{\varphi}_1(s), \dots, \tilde{\varphi}_p(s))$. Let us assume that there exists $0 < \epsilon_1 \leq \epsilon$ such that $\tilde{\varphi}(s) \neq 0$, for any $s \in]0, \epsilon_1[$, the proof of which will be given below.

We consider the curves $\lambda(s)$, $\varphi(s) := (\varphi_1(s), \dots, \varphi_p(s))$ and $\psi(s) := (\psi_1(s), \dots, \psi_r(s))$, where $\lambda(s) := \frac{\tilde{\lambda}(s)}{\|\tilde{\varphi}(s)\|}$, $\varphi_i(s) := \frac{\tilde{\varphi}_i(s)}{\|\tilde{\varphi}(s)\|}$, $i = 1, \dots, p$, and $\psi_j(s) = \frac{\tilde{\psi}_j(s)}{\|\tilde{\varphi}(s)\|}$, $j = 1, \dots, r$.

Then $\|\varphi(s)\| = 1$ and we can rewrite equation (31) as follows:

$$\lambda(s)(\phi_1(s), \dots, \phi_n(s)) = \sum_{i=1}^{p} \varphi_i(s)\frac{\partial f_i}{\partial x}(\phi(s)) + \sum_{j=1}^{r} \psi_j(s)\frac{\partial h_j}{\partial x}(\phi(s)). \tag{32}$$

By chain rule and from (32), we obtain the following equalities:

$$\begin{aligned}\sum_{i=1}^{p} \varphi_i(s)\frac{d}{ds}f_i(\phi(s)) + \sum_{j=1}^{r} \psi_j(s)\frac{d}{ds}h_j(\phi(s)) = \\ \left\langle \sum_{i=1}^{p} \varphi_i(s)\frac{\partial f_i}{\partial x}(\phi(s)) + \sum_{j=1}^{r} \psi_j(s)\frac{\partial h_j}{\partial x}(\phi(s)); \frac{d}{ds}\phi(s) \right\rangle = \frac{1}{2}\lambda(s)\left(\frac{d}{ds}\|\phi(s)\|^2\right).\end{aligned} \tag{33}$$

Since $\lim_{s\to 0} f(\phi(s)) = t_0$ and $h(\phi(s)) \equiv 0$, we have that $\mathrm{ord_s}\left(\frac{d}{ds}f_i(\phi(s))\right) \geq 0$, for $i = 1, \dots, p$, and $\frac{d}{ds}h_j(\phi(s)) \equiv 0$, for $j = 1, \dots, r$. These and (33) imply:

$$0 \leq \mathrm{ord_s}\left(\lambda(s)\left(\frac{d}{ds}\|\phi(s)\|^2\right)\right) < \mathrm{ord_s}\left(\lambda(s)\|\phi(s)\|^2\right). \tag{34}$$

On the other hand, the equality (32) yields:

$$\mathrm{ord_s}\left(|\lambda(s)|\|\phi(t)\|^2\right) = \mathrm{ord_s}\left(\|\phi(s)\|\left\|\sum_{i=1}^{p} \varphi_i(s)\frac{\partial f_i}{\partial x}(\phi(s)) + \sum_{j=1}^{r} \psi_j(s)\frac{\partial h_j}{\partial x}(\phi(s))\right\|\right). \tag{35}$$

From (34), we conclude that (35) is positive, which implies:

$$\lim_{s\to 0}\|\phi(s)\|\left\|\sum_{i=1}^{p} \varphi_i(s)\frac{\partial f_i}{\partial x}(\phi(s)) + \sum_{j=1}^{r} \psi_j(s)\frac{\partial h_j}{\partial x}(\phi(s))\right\| = 0. \tag{36}$$

Therefore, since $\lim_{s\to 0} f(\phi(s)) = t_0$, $\|\varphi(s)\| = 1$, $\sum_{j=1}^{r} \psi_j(s)\frac{\partial h_j}{\partial x}(\phi(s)) \in (T_{\phi(s)}X)^{\perp}$, we conclude from (36), Definition 3.7 and Lemma 3.6 that $t_0 \in K_\infty(f)$.

Let us now show that there exists $0 < \epsilon_1 \le \epsilon$ such that $\tilde{\varphi}(s) \neq 0$, for any $s \in]0, \epsilon_1[$. Suppose not; this means that there exists a sequence $\{s_k\}_{k\in\mathbb{N}} \subset]0, \epsilon[$ such that $\lim_{k\to\infty} s_k = 0$ and $\tilde{\varphi}(s_k) = (0, \ldots, 0)$. This and (31) yield the following equality:

$$\tilde{\lambda}(s_k)(\phi_1(s_k), \ldots, \phi_n(s_k)) = \sum_{j=1}^{r} \tilde{\psi}_j(s_k) \frac{\partial h_j}{\partial x}(\phi(s_k)), \text{ for any } k \in \mathbb{N}. \tag{37}$$

We remember that $(\tilde{\lambda}(s), \tilde{\varphi}_1(s), \ldots, \tilde{\varphi}_p(s), \tilde{\psi}_1(s), \ldots, \tilde{\psi}_r(s)) \neq (0, \ldots, 0)$, for any $s \in]0, \epsilon[$. Consequently, the condition on $\tilde{\varphi}$ implies $(\tilde{\lambda}(s_k), \tilde{\psi}_1(s_k), \ldots, \tilde{\psi}_r(s_k)) \neq (0, \ldots, 0)$, for any $k \in \mathbb{N}$. Moreover, since $\lim_{k\to\infty} s_k = 0$, we have $\lim_{k\to\infty} \|\phi(s_k)\| = \infty$ and $\lim_{k\to\infty} f(\phi(s_k)) = t_0$. From these conditions, equality (37) and curve selection lemma, we can obtain new analytic curves $\lambda(s), \psi_1(s), \ldots, \psi_r(s)$ and an analytic curve $\alpha = (\alpha_1, \ldots, \alpha_n) :]0, \epsilon[\to X \subset \mathbb{R}^n$ such that $\lim_{s\to 0} \|\alpha(s)\| = \infty$, $\lim_{s\to 0} f(\alpha(s)) = t_0$, $(\lambda(s), \psi_1(s), \ldots, \psi_r(s)) \neq (0, \ldots, 0)$, for any s, and the following equality holds:

$$\lambda(s)(\alpha_1(s), \ldots, \alpha_n(s)) = \sum_{j=1}^{r} \psi_j(s) \frac{\partial h_j}{\partial x}(\phi(s)). \tag{38}$$

Since $\alpha(s) \in X$, we have $h_j(\alpha(s)) \equiv 0$, which implies $\frac{d}{ds} h_j(\alpha(s)) \equiv 0$, for $j = 1, \ldots, r$. These and chain rule give:

$$0 \equiv \sum_{j=1}^{r} \psi_j(s) \frac{d}{ds} h_j(\alpha(s)) = \left\langle \sum_{j=1}^{r} \psi_j(s) \frac{\partial h_j}{\partial x}(\alpha(s)), \frac{d}{ds}\alpha(s) \right\rangle = \frac{1}{2}\lambda(s) \left(\frac{d}{ds}\|\alpha(s)\|^2 \right). \tag{39}$$

Since λ and α are analytic curves, equality (39) gives $\lambda(s) \equiv 0$ or $\frac{d}{ds}\|\alpha(s)\|^2 \equiv 0$. If $\lambda(s) \equiv 0$ then, from (38) and statements on $\lambda, \psi_1, \ldots, \psi_r$, we obtain that $\sum_{j=1}^{r} \psi_j(s) \frac{\partial h_j}{\partial x}(\phi(s)) \equiv 0$, with $(\psi_1(s), \ldots, \psi_r(s)) \neq (0, \ldots, 0)$. But this contradicts the hypothesis that X is a global intersection. If $\frac{d}{ds}\|\alpha(s)\|^2 \equiv 0$ then $\|\alpha(s)\|^2$ is constant, which contradicts the assumption $\lim_{s\to 0} \|\alpha(s)\| = \infty$. Therefore, we have shown by contradiction that the assertion "there exists $0 < \epsilon_1 \le \epsilon$ such that $\tilde{\varphi}(s) \neq 0$, for any $s \in]0, \epsilon_1[$," is true, which completes the proof of Proposition 5.3. □

The above proposition extends for mappings defined on X the equality proved in [KOS, Proposition 3.1].

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