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ON THE EULER CHARACTERISTIC OF REAL MILNOR FIBRES

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Abstract. We study the Milnor fibres of a real analytic mapping defined on a real analytic space which has an isolated critical point. In particular we look at the Euler characteristic. We discuss the global case, too.

0. Introduction

Mappings $f : \mathbb{R}^n \to \mathbb{R}^k$ with an isolated singularity have been already studied by J. Milnor [M]. It is not important whether one works in the real algebraic or real analytic category, here we prefer the real analytic one. We replace $\mathbb{R}^n$ by a germ of a real analytic space with an isolated singularity, introduce a kind of Milnor fibration and study the Euler characteristic of its fibres. Finally we pass shortly to the global case.

Part of the results has been announced in [H].

1. The real Milnor fibration

Let $f : (X, 0) \to (\mathbb{R}^k, 0)$ be a real analytic mapping between real analytic space germs with an isolated singularity, which means that $f : X \setminus \{0\} \to \mathbb{R}^k$ is a submersion between manifolds. Let X be purely n-dimensional. We may suppose that $(X, 0)$ is embedded in $(\mathbb{R}^N, 0)$. Let $D_\epsilon := \{x \in \mathbb{R}^N \mid \|x\| \leq \epsilon\}$, $S_\epsilon := \partial D_\epsilon$. Let $L := X \cap S_\epsilon$ and $K := f^{-1}(\{0\}) \cap S_\epsilon$, $0 < \epsilon \ll 1$, be the links of $(X, 0)$ and $(f^{-1}(\{0\}), 0)$. Note that $X \setminus \{0\}, L$ and K are manifolds which are not necessarily orientable!

Similarly, let $B_\alpha := \{t \in \mathbb{R}^k \mid \|t\| \leq \alpha\}$.

Theorem 1.1:

a) Let $0 < \alpha \ll \epsilon \ll 1$. Then $f : X \cap D_\epsilon \cap f^{-1}(B_\alpha \setminus \{0\}) \to B_\alpha \setminus \{0\}$ is a locally trivial fibration ("Milnor fibration").

b) The mapping $f : X \cap S_\epsilon \cap f^{-1}(B_\alpha) \to B_\alpha$ is a locally trivial, hence a trivial fibration, so ∂F_t is diffeomorphic to K for every "Milnor fibre" $F_t = f^{-1}(\{t\}) \cap D_\epsilon$.

Proof. Note that we have supposed that 0 is an isolated singularity of f. In particular $f^{-1}(0)$ has an isolated singularity at 0, and S_ϵ is transversal to $f^{-1}(0)$, $0 < \epsilon \ll 1$. Hence S_ϵ is transversal to $f^{-1}(t)$ for $\|t\| \leq \alpha, 0 < \alpha \ll \epsilon \ll 1$. □

The base space in a) is connected if $k \geq 2$ but not if $k = 1$, so we treat these cases separately.

Note that we have a lemma which goes back to Milnor ([M] Lemma 11.3) in the case $X = \mathbb{R}^n$:

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Lemma 1.2: For $0 < \alpha \ll \epsilon \ll 1$ we have a homeomorphism

$$X \cap \big((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha \setminus \{0\}))\big) \approx L \setminus K\,,$$

hence a homotopy equivalence $X \cap D_\epsilon \cap f^{-1}(\partial B_\alpha) \sim L \setminus K$.

So we use the symbol $\approx$ in the case of a homeomorphism and $\sim$ in the case of a homotopy equivalence.

Proof. We have assumed $X \subset \mathbb{R}^N$. Put $\phi, \psi : X \to \mathbb{R} : \phi(x) := \|f(x)\|^2, \psi(x) := \|x\|^2$. By the Curve Selection Lemma we know that there are no $x \in D_\epsilon \cap X \setminus f^{-1}(0)$ such that there is a $\lambda \leq 0$ with $d\psi_x = \lambda\, d\phi_x$ if $0 < \epsilon \ll 1$. Therefore we can find on $X \setminus f^{-1}(0)$ a vector field v such that $d\psi_x(v(x)) > 0$, $d\phi_x(v(x)) = 1$ for $\|x\| \leq \epsilon$. Using the flow we can construct the desired homeomorphism. Furthermore $X \cap D_\epsilon \cap f^{-1}(\partial B_\alpha)$ is a deformation retract of $X \cap ((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha \setminus \{0\})))$. □

According to Milnor [M], p. 99, the homotopy equivalence can in general not be chosen as to be fibre preserving with respect to $x \mapsto \frac{f(x)}{\|f(x)\|}$.

2. The Milnor fibre of a real analytic mapping ($k \geq 2$)

First we suppose $k \geq 2$. Then we can speak of the typical Milnor fibre F because all Milnor fibres are diffeomorphic.

Standard example: $k = 2, n = 2m$, $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ with isolated singularity. For the more general case see e.g. [M] p. 103, and [CL].

In this paper we look at cohomology with integral coefficients.

Theorem 2.1: We have long exact sequences:

$$\ldots \to H^m(L \setminus K) \to H^m(F) \to H^{m+2-k}(F) \to H^{m+1}(L \setminus K) \to \ldots \text{ (Wang sequence)},$$

$$\ldots \to H^{m-1}(K) \to H^m(F, \partial F) \to H^m(F) \to H^m(K) \to \ldots,$$

$$\ldots \to H^m(L) \to H^m(F) \to H^{m-k+2}(F, \partial F) \to H^{m+1}(L) \to \ldots$$

Note that the second and third long exact sequences are the ones for the pair $(F, \partial F)$ and the pair (L, F): we can embed F in L.

For $k = 2$ the first and third sequences read:

$$\ldots \to H^m(L \setminus K) \to H^m(F) \stackrel{h^* - id}{\to} H^m(F) \to H^{m+1}(L \setminus K) \to \ldots \text{ and}$$

$$\ldots \to H^m(L) \to H^m(F) \stackrel{Var^*}{\to} H^m(F, \partial F) \to H^{m+1}(L) \to \ldots$$

Here $h : F \to F$ is “the” monodromy.

Proof. (i) For the Wang sequence, see Spanier [S] p. 456.

Note that $L \setminus K$ may be replaced by $X \cap D_\epsilon \cap f^{-1}(\partial B_\alpha)$, see Lemma 1.2, and

$$f : X \cap D_\epsilon \cap f^{-1}(\partial B_\alpha) \to \partial B_\alpha$$

is a locally trivial fibration.

(ii) In the second exact sequence we may replace K by ∂F; see Theorem 1.1b).

(iii) As for the third exact sequence, note that we may replace L by

$$X \cap \big((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha))\big)$$

Let D be an open "ball" in ∂B_α, $t \in D$. Then:

$$H^{m+1}(L, F) \simeq H^{m+1}\left(X \cap ((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha))), X \cap D_\epsilon \cap f^{-1}(\bar{D})\right)$$

$$\simeq H^{m+1}\left(X \cap ((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha))), X \cap ((D_\epsilon \cap f^{-1}(\bar{D})) \cup (S_\epsilon \cap f^{-1}(B_\alpha)))\right)$$

$$\simeq H^{m+1}\left(X \cap D_\epsilon \cap f^{-1}(\partial B_\alpha \setminus D), X \cap ((D_\epsilon \cap f^{-1}(\partial D)) \cup (S_\epsilon \cap f^{-1}(\partial B_\alpha \setminus D)))\right)$$

$$\simeq H^{m+1}((F, \partial F) \times (\partial B_\alpha \setminus D, \partial D)) \simeq H^{m+2-k}(F, \partial F)\,.$$

In fact, for the first isomorphism note that $L \approx X \cap ((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha)))$, similarly as in Lemma 1.2. Furthermore, F is a deformation retract of $X \cap D_\epsilon \cap f^{-1}(\bar{D})$.
For the second one, note that $f|S_\epsilon \cap f^{-1}(B_\alpha)$ is trivial, see Theorem 1.1b), so $S_\epsilon \cap f^{-1}(D)$ is a strong deformation retract of $S_\epsilon \cap f^{-1}(B_\alpha)$.
The third isomorphism is established by excision, the fourth one is due to the fact that

$$f : D_\epsilon \cap f^{-1}(\partial B_\alpha \setminus D) \to \partial B_\alpha \setminus D$$

is a trivial fibration. The last one follows from the Künneth formula. □

Since one cannot expect good connectivity properties in the real case, let us look at the Euler characteristic.

Corollary 2.2:

a) $\chi(L) = 0$ if n is even, $\chi(L) = 2\chi(F)$ if n is odd,

b) $\chi(K) = 0$ if $n - k$ is even, $\chi(K) = 2\chi(F)$ if $n - k$ is odd.

Proof. First let us observe the following: Suppose that M is a compact manifold with boundary of dimension m. Then $\chi(M, \partial M) = (-1)^m \chi(M)$. In particular, $\chi(M) = 0$ if M is closed and m is odd.
This is obvious by Poincaré duality, in the non-orientable case with coefficients in $\mathbb{Z}/2\mathbb{Z}$.
a) Suppose that n is even. Then L is a closed manifold of odd dimension, hence $\chi(L) = 0$. Therefore we assume now that n is odd. By the third exact sequence and Poincaré duality we have

$$\chi(L) = \chi(F) - (-1)^k \chi(F, \partial F) = \chi(F) - (-1)^n \chi(F) = 2\chi(F)$$

b) Similarly, $\chi(K) = 0$ if $n - k$ is even. So suppose that $n - k$ is odd. Then

$$\chi(K) = \chi(F) - \chi(F, \partial F) = \chi(F) - (-1)^{n-k} \chi(F) = 2\chi(F).$$

□

So $\chi(F)$ can be expressed by the Euler characteristic of a link except if k and n are both even.

3. The Milnor fibres of a real analytic function ($k = 1$)

Now let us switch to the case $k = 1$. Then we have two typical Milnor fibres: F_+ (resp. F_-), corresponding to F_t with $t > 0$ (resp. $t < 0$).

Theorem 3.1:
a) $H^m(L \setminus K) \simeq H^m(F_+) \oplus H^m(F_-)$.
b) We have long exact sequences:

$$\ldots \to H^{m-1}(K) \to H^m(F_+, \partial F_+) \to H^m(F_+) \to H^m(K) \to \ldots,$$
$$\ldots \to H^m(L) \to H^m(F_+) \oplus H^m(F_-) \to H^m(K) \to \ldots \text{ and}$$
$$\ldots \to H^m(L) \to H^m(F_+) \to H^{m+1}(F_-, \partial F_-) \to \ldots$$

The middle exact sequence is a Mayer-Vietoris sequence, of course. As a consequence,

$$\chi(L) + \chi(K) = \chi(F_+) + \chi(F_-).$$

Proof. $H^m(L, F_+) \simeq H^m(F_-, \partial F_-)$ by excision. The rest is clear. □

Corollary 3.2: If n is even, we have

$$\chi(F_+) = \chi(F_-), \chi(L) = 0, \chi(K) = 2\chi(F_+).$$

If n is odd,

$$\chi(L) = \chi(F_+) + \chi(F_-), \chi(K) = 0.$$

Proof. If n is even, $\chi(L) = 0$, hence $\chi(F_+) = -\chi(F_-, \partial F_-) = \chi(F_-)$. If n is odd, $\chi(K) = 0$. The rest is clear. □

It is difficult to calculate individual cohomology groups but:

Corollary 3.3: a) Suppose that $n = 2m + 1, m \geq 1$ and that F_+ and F_- have the homotopy type of a bouquet of m-spheres. Then $H^0(L) = \mathbb{Z}$, $H^l(L) = 0$ for $l \neq 0, m, 2m$, and $H^m(L)$ is free abelian. Furthermore $H^{2m}(L) \simeq \mathbb{Z}/2\mathbb{Z}$ if $m = 1$ and L is non-orientable, otherwise $H^{2m}(L) \simeq \mathbb{Z}$.
b) Suppose that $n = 2m + 2, m \geq 1$ and that F_+ or F_- has the homotopy type of a bouquet of m-spheres. Then $H^0(K) = \mathbb{Z}$, $H^l(K) = 0$ for $l \neq 0, m, 2m$, and $H^m(K)$ is free abelian. Furthermore $H^{2m}(K) \simeq \mathbb{Z}/2\mathbb{Z}$ if $m = 1$ and K is non-orientable, otherwise $H^{2m}(K) \simeq \mathbb{Z}$.

Proof. We know that $K \neq \emptyset$, otherwise F_+ and F_- are compact which gives the wrong homology.
a) The exact sequence

$$0 \to H^0(L) \to H^0(F_+) \oplus H^0(F_-) \to H^0(K)$$

shows that L is connected. This implies the statement for $m = 1$.
In the case $m \geq 2$ we know that F_+ and F_- are simply connected, hence orientable. So we have for $0 < l < 2m$ an exact sequence

$$H^{l-1}(F_+) \to H_{2m-l}(F_-) \to H^l(L) \to H^l(F_+) \to H_{2m-1-l}(F_-)$$

because $H_{2m-l}(F_-) \simeq H^l(F_-, \partial F_-)$.
For $l \neq m$ we deduce $H^l(L) = 0$. For $l = m$ we obtain

$$0 \to H_m(F_-) \to H^m(L) \to H^m(F_+) \to 0,$$

so $H^m(L)$ is free abelian. Of course, $H^{2m}(L) \simeq \mathbb{Z}$.
b) Assume that the hypothesis is true for F_+. Again, F_+ is orientable if $m \geq 2$. Note that $H_{2m-l}(F_+) \simeq H^{l+1}(F_+, \partial F_+)$.

Suppose first that F_+ is orientable. Then we have an exact sequence

$$H^0(F_+) \to H^0(K) \to H_{2m}(F_+).$$

Since $H_{2m}(F_+) = 0$ we obtain that K is connected.

If F_+ is non-orientable we have that $m = 1$, and the universal covering of F_+ is contractible. Therefore the orientation covering of F_+ has the homotopy type of a bouquet of 1-spheres, too. We conclude as before that its boundary is connected. So K is connected, too.

So we must only look at the case $m \geq 2$. For $0 < l < 2m$, we have an exact sequence

$$H_{2m+1-l}(F_+) \to H^l(F_+) \to H^l(K) \to H_{2m-l}(F_+) \to H^{l+1}(F_+)$$

For $l \neq m$ we have

$$H^l(F_+) = H_{2m-l}(F_+) = 0$$

hence $H^l(K) = 0$.

For $l = m$ we have an exact sequence

$$0 \to H^m(F_+) \to H^m(K) \to H_m(F_+) \to 0$$

which implies that $H^m(K)$ is free abelian. □

Example 3.4: a) $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ holomorphic with isolated singularity,

$$X := \mathbb{C}^{m+1} \cap \{Im\, g = 0\}, \quad f := Re\, g, \text{ and } \quad n = 2m + 1.$$

We obtain that $L := S_\epsilon \cap \{Im\, g = 0\}$ is a compact manifold of dimension $2m$,

$$H^0(L) = H^{2m}(L) = \mathbb{Z},$$

$H^m(L)$ free abelian of rank 2μ, $\mu =$ Milnor number of g.

b) $X = \mathbb{C}^{m+1}, f = Im\, g$, which leads with K instead of L to the same result as before, because the Milnor fibres of f and g have the same homotopy type. See Lemma 5.1 below.

4. Euler characteristic of the real Milnor fibre

Using resolution of singularities we can calculate the Euler characteristic of the Milnor fibre(s).

In the situation of section 2, we can put $Y := X \cap \{f_1 = \ldots = f_{k-1} = 0\}$. Then the Milnor fibres of $f_k : (Y, 0) \to (\mathbb{R}, 0)$ coincide with the one of $f : (X, 0) \to (\mathbb{R}^k, 0)$, so we can reduce to the case $k = 1$ with $F_+ \approx F_-$. So it is sufficient to look at the case $k = 1$ (cf. Example 3.4a).

Let us assume $k = 1$. Choose an embedded resolution $\pi : X' \to X$ of $f^{-1}(\{0\}) \subset X$. Then $(f \circ \pi)^{-1}(\{0\})$ is a divisor with normal crossing, it has a natural stratification. Let S_{li}, l being the codimension of the stratum, be the strata contained in $\pi^{-1}(\{0\})$. Locally at a point of this stratum, $f \circ \pi = \varepsilon x_1^{\nu_1} \cdot \ldots \cdot x_l^{\nu_l}$ with respect to suitable local coordinates, $\varepsilon = \pm 1$.

Put:

$\alpha_{li} := 2^{l-1}$ if there is a j such that ν_j is odd,

$\alpha_{li} := 2^l$ if $\nu_1, \ldots, \nu_l$ are even, $\varepsilon = 1$,

$\alpha_{li} := 0$ if $\nu_1, \ldots, \nu_l$ are even, $\varepsilon = -1$.

Theorem 4.1: $\chi(F_+) = \sum_{l,i} \alpha_{li} (-1)^{n-l} \chi(S_{li})$.

Proof. (cf. [CF] in the case $X = \mathbb{R}^n$) Let U_l be a suitable closed neighbourhood of the union of strata of codimension $\geq l$. More precisely, put

$$U_l := \{x \in X' \mid \psi_l(x) \leq \epsilon_l\},$$

where $\psi_l : X' \to [0, \infty[$ is a real analytic function whose zero set is the union of strata of codimension $\geq l$, and where $0 < \epsilon_1 \ll \epsilon_2 \ll \ldots \ll \epsilon_n \ll 1$, and suppose $0 < t \ll \epsilon_1$. Put $U^l := U_l \cup \ldots \cup U_n$. Then each connected component of $U_l \setminus U^{l+1}$ is the total space of a topological fibre bundle over $S_{li} \setminus U^{l+1}$, the fibre being the normal slice with respect to S_{li} for some i. Note that $S_{li} \setminus U^{l+1}$ has the same homotopy type as S_{li}. The normal slice N of S_{li} at p is homeomorphic to $\mathbb{R}^l$. Near p we can write $f \circ \pi$ as above. Then

$$N \cap \{f \circ \pi = t\} = \{x \in \mathbb{R}^l \mid \varepsilon x_1^{\nu_1} \cdot \ldots \cdot x_l^{\nu_l} = t\}, \quad 0 < t \ll 1.$$

If there is a j such that ν_j is odd, we may assume $j = l$, then we can write the right hand side as the graph of a function defined on $(\mathbb{R}^*)^{l-1}$. This set is the disjoint union of 2^{l-1} contractible components.

If all ν_j are even, $\varepsilon = -1$, the set is empty.

If all ν_j are even, $\varepsilon = 1$, we get the disjoint union of two graphs of functions defined on the same set as above, so we obtain 2^l contractible components.

Therefore the Euler characteristic of $N \cap \{f \circ \pi = t\}$, $t > 0$, is α_{li}.

Now

$$F_+ \sim D_\epsilon \cap X \cap \{f > 0\} \sim \pi^{-1}(D_\epsilon \cap X \cap \{f > 0\})$$

If we vary ϵ (resp. $\epsilon_1, \ldots, \epsilon_n$) we see that

$$\pi^{-1}(D_\epsilon \cap X \cap \{f > 0\}) \sim U^1 \cap \{f \circ \pi > 0\} \sim U^1 \cap \{f \circ \pi = t\}$$

Furthermore, $U^1 = \dot{\bigcup} (U_l - U^{l+1})$, hence

$$\chi(F_+) = \chi(\{f \circ \pi = t\} \cap U^1) = \sum_l \chi_c(\{f \circ \pi = t\} \cap (U_l \setminus U^{l+1}))$$

$$= \sum_{l,i} \alpha_{li} \chi_c(S_{li}) = \sum_{l,i} \alpha_{li} (-1)^{n-l} \chi(S_{li})$$

Here χ_c is the Euler characteristic with compact support. □

It is easier to calculate $\chi(F_+) + \chi(F_-)$:

Corollary 4.2: $\chi(F_+) + \chi(F_-) = \sum_{l,i} 2^l (-1)^{n-l} \chi(S_{li})$, and so, if $\chi(F_+) = \chi(F_-)$ (in particular if n is even), then

$$\chi(F_+) = \sum_{l,i} 2^{l-1} (-1)^{n-l} \chi(S_{li}).$$

The first statement of the corollary can also be proved directly, without using the local description of $f \circ \pi$: note that $(\mathbb{R}^*)^l$ has 2^l contractible components.

By the way, we can calculate $\chi(K)$ and $\chi(L)$ using the same resolution:

Let us denote by S'_{li} those strata S_{li} which are contained in the strict transform of $f^{-1}(0)$, i.e., in the closure of $\pi^{-1}(f^{-1}(\{0\}) \setminus \{0\})$, S''_{li} the remaining ones. Then:

$$\chi(K) = \sum 2^{l-1} (-1)^{n-l} \chi(S'_{li}),$$

$$\chi(L) = \sum 2^{l-1} (-1)^{n-l} \chi(S'_{li}) + \sum 2^l (-1)^{n-l} \chi(S''_{li})$$

which agrees with the formula $\chi(L) + \chi(K) = \chi(F_+) + \chi(F_-)$ proved before (Theorem 3.1).

In the case of L, note that in the normal slice we have to look at $N \setminus \pi^{-1}(0)$ which differs from $N \setminus (f \circ \pi)^{-1}(0)$ if we are at a point of the strict transform of $f = 0$: then we have 2^{l-1} instead of 2^l contractible components.

Using the formula for $\chi(K)$ we obtain an easier formula for $\chi(F_+)$ if n is even: Then

$$\chi(F_+) = \chi(F_-) = \sum 2^{l-2}(-1)^{n-l}\chi(S'_{li}),$$

because $\chi(K) = 2\chi(F_+)$.

5. Comparison of Milnor fibres of mappings (resp. functions)

There is another connection between the cases $k \geq 2$ and $k = 1$ in section 2 (resp. 3):

Let us take up the assumptions of section 2 (in particular, $k \geq 2$) and write $\chi(f)$ instead of $\chi(F)$. Similarly in 3: $\chi(f)_+$ instead of $\chi(F_+)$.

Lemma 5.1: For $0 < \alpha \ll \epsilon \ll 1$, the inclusion of $X \cap D_\epsilon \cap \{f_1 = \ldots = f_{k-1} = 0, f_k = \alpha\}$ in $X \cap D_\epsilon \cap f_k^{-1}(\alpha)$ is a homotopy equivalence.

Proof. Let ϕ, ψ be defined as in the proof of Lemma 1.2. Compare

$$X \cap D_\epsilon \cap \{\|f\| \leq \alpha, f_k > 0\}$$

with $X \cap B_\epsilon \cap \{f_k > 0\}$. Choose a vector field v such that, on $X \cap D_\epsilon \cap \{\|f\| \geq \alpha\}$:

$$d\phi_x(v(x)) = 1, d\psi_x(v(x)) > 0,$$

and near $f_k = 0 : (df_k)_x(v(x)) = 0$. This is possible: assume that we have a point p such that $d\psi_p = \lambda d\phi_p$ with $\lambda < 0$, we get a contradiction because of the Curve Selection Lemma. Similarly, suppose that near $f_k = 0$ there is a p, $\|f(p)\| \geq \alpha$, such that $d\psi_p = \lambda d\phi_p + \mu(df_k)_p$ with $\lambda \leq 0$ we would get also such a point with $f_k(p) = 0$, which contradicts the Curve Selection Lemma. So we obtain that

$$X \cap D_\epsilon \cap \{\|f\| \leq \alpha, f_k > 0\} \sim X \cap D_\epsilon \cap \{f_k > 0\}$$

Moreover, $f : X \cap D_\epsilon \cap \{\|f\| \leq \alpha, f_k > 0\} \to \{t \in B_\alpha \,|\, t_k > 0\}$ is a trivial fibration, so

$$X \cap D_\epsilon \cap \{\|f\| \leq \alpha, f_k > 0\} \sim X \cap D_\epsilon \cap \{f = (0, \ldots, 0, \alpha)\}$$

Now we can find a vector field w on $\{f_k > 0\}$ such that, on $X \cap D_\epsilon \cap \{f_k > 0\}$:

$$(df_k)_x(w(x)) = 1, d\psi_x(w(x)) > 0,$$

because of the Curve Selection Lemma. Therefore

$$X \cap D_\epsilon \cap \{f_k > 0\} \sim X \cap D_\epsilon \cap \{0 < f_k \leq \alpha\}.$$

Finally, X has an isolated singularity at 0, so $f_k : X \cap D_\epsilon \cap \{0 < f_k \leq \alpha\} \to]0, \alpha]$ is a trivial fibration, hence

$$X \cap D_\epsilon \cap \{0 < f_k \leq \alpha\} \sim X \cap D_\epsilon \cap \{f_k = \alpha\}.$$

□

In the case of $X = \mathbb{R}^n$ this is a consequence of a conjecture by J.Milnor [M], p. 100, see also [ADD].

Corollary 5.2: $\chi(f) = \chi(f_1)_+ = \chi(f_1)_- = \ldots = \chi(f_k)_+ = \chi(f_k)_-$.

Now let us turn to the special case $X = \mathbb{R}^n$. Then we have the following formula:

Theorem 5.3: (G.Khimshiashvili [K]) If $k = 1$, $\chi(f)_+ = 1 - (-1)^n deg_0 \nabla f$, where ∇f is the gradient of f and $deg_0 \nabla f$ is the topological degree of $\frac{\nabla f}{|\nabla f|} : S_\epsilon \to S_1$.

Replacing f by $-f$ we obtain that $\chi(f)_- = 1 - deg_0 \nabla f$
Note that L is a sphere in our case. This implies altogether:

Corollary 5.4: ([ADD])
a) $\chi(f) = 1 - deg_0 \nabla f_1 = \ldots = 1 - deg_0 \nabla f_k$.
b) If n is odd, $deg_0 \nabla f_1 = \ldots = deg_0 \nabla f_k = 0$, so $\chi(f) = 1$.

Proof. b) By the Corollary before, $\chi(f_i)_+ = \chi(f_i)_-$, so according to Khimshiashvili: $deg_0 \nabla f_i = 0$, so $\chi(f) = \chi(f_i)_+ = 1$. □

6. Global analogue

a) Now let us pass to the global case. Let X be a compactifiable real analytic (e.g. a real algebraic) subspace of $\mathbb{R}^N$ which is purely n-dimensional, $f : X \to \mathbb{R}^k$ a compactifiable real analytic mapping. Let C be the set of critical points of f; recall that singular points of X are automatically critical points of f. Assume that

(i) the set of critical points of f which are contained in $f^{-1}(\{0\})$ is compact,

(ii) for $0 < \alpha \ll 1$ the set $C \cap f^{-1}(B_\alpha \setminus \{0\})$ is closed in X, i.e. there is no convergent sequence $(p_n) \to p^*$ of critical points of f such that $f(p_n) \neq 0$ for all n, $p^* \in X$, $f(p^*) = 0$.

Then we get that for $0 < \alpha \ll \frac{1}{R} \ll 1$ the mapping

$$f : X \cap D_R \cap f^{-1}(B_\alpha \setminus \{0\}) \to B_\alpha \setminus \{0\}$$

is a locally trivial fibration:

Assume $R \gg 0$. Then X is smooth along $X \cap S_R$, S_R intersects X transversally, and $f|X \cap S_R$ has no critical point which is mapped to 0. Therefore $f|X \cap S_R$ has no critical points above B_α. Finally, f has no critical points in $X \cap D_R \cap f^{-1}(B_\alpha \setminus \{0\})$.

As at the beginning of section 4 we may reduce to the case $k = 1$. So assume $k = 1$; then we get fibres F_+ and F_-.

Let us fix a compactification $\bar{f} : \bar{X} \to \mathbb{R}$ and let $\pi : \bar{X}' \to \bar{X}$ be an embedded resolution of $\bar{f}^{-1}(0) \cup C \cup X_\infty \subset \bar{X}$ where $X_\infty := \bar{X} \setminus X$. We can achieve that

$$\pi : \pi^{-1}(f^{-1}(\{0\}) \setminus C) \to f^{-1}(\{0\}) \setminus C$$

is an isomorphism. Put $X' := \pi^{-1}(X)$. We have a natural stratification of $(\bar{f} \circ \pi)^{-1}(0)$ such that $\pi^{-1}(f^{-1}(\{0\}))$ is a union of strata. Locally at a point of such a stratum of codimension l in $\bar{X}'$, $\bar{f} \circ \pi = \varepsilon x_1^{\nu_1} \cdot \ldots \cdot x_\lambda^{\nu_\lambda}$, $\lambda \leq l$, with respect to suitable local coordinates, $\varepsilon = \pm 1$.
Put:

$\alpha_{li} := 2^{l-1}$ if there is a j such that ν_j is odd,

$\alpha_{li} := 2^l$ if $\nu_1, \ldots, \nu_\lambda$ are even, $\varepsilon = 1$,

$\alpha_{li} := 0$ if $\nu_1, \ldots, \nu_\lambda$ are even, $\varepsilon = -1$.

Then we have, similarly as in section 4:

Theorem 6.1: $\chi(F_+) = \sum_{l,i} \alpha_{li}(-1)^{l+1}\chi(S_{li})$, where the sum extends over all strata contained in $\pi^{-1}(f^{-1}(\{0\}))$.

Proof. Let U_l be a suitable closed neighbourhood of the union of $(\bar{X}' \setminus X') \cap (\bar{f} \circ \pi)^{-1}(\{0\})$ and all strata of $\pi^{-1}(f^{-1}(\{0\}))$ of codimension $\geq l$, $U^l := U_l \cup \dots U_{n+1}$. Then

$$\chi(F_+) = \chi((\bar{f} \circ \pi)^{-1}(\{t\}) \setminus U_{n+1}) = (-1)^{n-1}\chi_c((\bar{f} \circ \pi)^{-1}(\{t\}) \setminus U_{n+1}),$$

and

$$\chi_c((\bar{f} \circ \pi)^{-1}(\{t\}) \setminus U_{n+1}) = \sum_{l=1}^{n} \chi_c((\bar{f} \circ \pi)^{-1}(\{t\}) \cap U_l \setminus U^{l+1})$$

We continue similarly as in the proof of Theorem 4.1. □

We have a similar formula for $K := f^{-1}(0) \cap S_R, R \gg 0$:
$\chi(K) = \sum_{l,i} 2^{l-1}(-1)^{n-l}\chi(S_{li})$, where the sum extends to all strata contained in

$$(\bar{X}' \setminus X') \cap \overline{X' \cap (\bar{f} \circ \pi)^{-1}(0)}.$$

If n is even, this implies a simpler formula for $\chi(F_+) = \chi(F_-)$ because

$$\chi(K) = 2\chi(F_+) = 2\chi(F_-) :$$

$$\chi(K) = \chi(\partial F_+) = \chi(F_+) - \chi(F_+, \partial F_+) = 2\chi(F_+)$$

because of Poincaré duality. Similarly for F_-.

b) The fibration studied in **a**) is not so natural because it ignores vanishing cycles at infinity.

So let us suppose instead that X is a compactifiable real analytic space, $f : X \to \mathbb{R}^k$ compactifiable real analytic, and that f is a submersive mapping between smooth spaces above $B_\alpha \setminus \{0\}$ for $0 < \alpha \ll 1$. Let $\bar{f} : \bar{X} \to \mathbb{R}^k$ be a compactification of f. Put $X_\infty := \bar{X} \setminus X$. We can stratify $\bar{X}$ and $\mathbb{R}^k$ subanalytically so that X is a union of strata and $\bar{f}$ is a stratified mapping.

Let T be a stratum of $\mathbb{R}^k$ such that $T \neq \{0\}$, $0 \in \overline{T}$. Because of Thom's first isotopy lemma we know that $f : f^{-1}(T) \to T$ defines a locally trivial fibration.

We want to calculate the Euler characteristic of the typical fibre F of this fibration. Since T is subanalytic we can find by the Curve Selection Lemma a real analytic curve $p :]-c, c[\to \mathbb{R}^k$ such that $p(0) = 0$, $p(t) \in T$ for $t > 0$. We apply base change to f with respect to p. In this way we reduce to the case $k = 1$. We need only to look at F_+.

So let us look at the case $k = 1$. Then we obtain that f is a locally trivial fibration above $B_\alpha \setminus \{0\}$, we have two typical fibres F_+, F_-. Let π and α_{li} be defined as in subsection a).

Theorem 6.2: $\chi(F_+) = \sum_{l,i} \alpha_{li}(-1)^{l+1}\chi(S_{li})$, where the sum extends over all strata of $(\bar{f} \circ \pi)^{-1}(\{0\})$ which are not contained in the closure of $\pi^{-1}(X_\infty \setminus (\bar{f} \circ \pi)^{-1}(\{0\}))$.

Proof. Analogous to the one of Theorem 6.1. □

Again we can find a simpler formula if n is even. First fix t, $0 < t \leq \alpha$. For $R \gg \frac{1}{\alpha}$ we have that $f^{-1}(\{t\}) \cap D_R$ is a deformation retract of $f^{-1}(\{t\})$. Now we have a formula for the boundary:

$$\chi(f^{-1}(\{t\}) \cap S_R) = \sum_{l,i} \alpha_{li}(-1)^{n-l}\chi(S_{li}),$$

where the sum extends over all strata of $(\bar{f} \circ \pi)^{-1}(\{0\})$ which are contained in the closure of $\pi^{-1}(X_\infty \setminus (\bar{f} \circ \pi)^{-1}(\{0\}))$.

If n is even we have that $\chi(f^{-1}(\{t\}) \cap S_R) = 2\chi(f^{-1}(\{t\}) \cap D_R) = 2\chi(F_+)$.

c) Assume that hypothesis (i) of part a) as well as the hypothesis of b) are given. Then we have hypothesis (ii) of part a), too. The fibrations in a) and b) may be different due to the presence of vanishing cycles at infinity, as shown by the real version of the Broughton example. Here is a different example where the fibres F_+ and F_- in b) have a different Euler characteristic:

Put $X := \mathbb{R}^2$, $f : X \to \mathbb{R}$: $f(x, y) := -x(xy^2 - 1)$.

Then $f^{-1}(\{0\})$ is the disjoint union of $\{x = 0\}$, $\{y < 0, x = \frac{1}{y^2}\}$, $\{y > 0, x = \frac{1}{y^2}\}$;

for $t > 0$,

$$f^{-1}(\{t\}) = \left\{x \geq t, y = \pm\frac{\sqrt{x-t}}{x}\right\};$$

for $t < 0$, $f^{-1}(\{t\})$ is the disjoint union of $\left\{x > 0, y = \frac{\sqrt{x-t}}{x}\right\}$, $\left\{x > 0, y = -\frac{\sqrt{x-t}}{x}\right\}$ and $\left\{t \leq x < 0, y = \pm\frac{\sqrt{x-t}}{x}\right\}$.

So $\chi(f^{-1}(\{0\})) = 3$, $\chi(f^{-1}(\{t\})) = 1$ for $t > 0$ and $\chi(f^{-1}(\{t\})) = 3$ for $t < 0$. Note that f has no critical points, so the fibre in a) has the same Euler characteristic as $f^{-1}(\{0\})$. Altogether, 0 is not a critical value but an atypical one.

References:

[ADD] R. Araújo dos Santos, D. Dreibelbis, N. Dutertre: Topology of the real Milnor fiber for isolated singularities. In: Contemp. Math. **569**, 2012, pp. 67-75. DOI: 10.1090/conm/569/11242

[CF] G. Comte, G. Fichou: Grothendieck ring of semialgebraic formulas and motivic real Milnor fibres. arXiv: 1111.3181v4 (2012)

[CL] P.T. Church, K. Lamotke: Nontrivial polynomial isolated singularities. Indag. Math. **37**, 149-154 (1975) DOI: 10.1016/1385-7258(75)90027-X

[H] H.A. Hamm: On the Euler characteristic of the real Milnor fibre(s). Oberwolfach Reports (to appear)

[K] G. Khimshiashvili: On the local degree of a smooth map. Soobshch. Akad. Nauk Gruz. SSR **85**, 309-311 (1977)

[M] J. Milnor: Singular points of complex hypersurfaces. Princeton Univ. Press: Princeton 1968.

[S] E.H. Spanier: Algebraic topology. McGraw-Hill: New York 1966.

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