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CLASSIFICATIONS OF COMPLETELY INTEGRABLE IMPLICIT SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to Professor Shyuichi Izumiya on the occasion of his 60th birthday

Abstract. An implicit second order ordinary differential equation is said to be *completely integrable* if there exists at least locally an immersive two-parameter family of geometric solutions on the equation hypersurface like as in the case of explicit equations. An implicit equation may have an immersive one-parameter family of geometric solutions (or, singular solutions) and a geometric solution (or, an isolated singular solution). In this paper, we give a classification of types of completely integrable implicit second order ordinary differential equations and give existence conditions for such families of solutions.

1. Introduction

An implicit second order ordinary differential equation is given by the form

$$F(x,y,p,q)=0,$$

where F is a smooth function of the independent variable x, the function y, its first and second derivatives $p=dy/dx$ and $q=d^2y/dx^2$ respectively.

It is natural to consider $F=0$ as being defined on a subset in the space of 2-jets of smooth functions of one variable, $F:\mathcal{O}\to\mathbb{R}$ where $\mathcal{O}$ is an open subset in $J^2(\mathbb{R},\mathbb{R})$. Throughout this paper, we assume that 0 is a regular value of F. It follows that the set $F^{-1}(0)$ is a hypersurface in $J^2(\mathbb{R},\mathbb{R})$. We call $F^{-1}(0)$ the *equation hypersurface*. Let (x,y,p,q) be a local coordinate on $J^2(\mathbb{R},\mathbb{R})$ and $\xi\subset TJ^2(\mathbb{R},\mathbb{R})$ be the canonical contact system (the Engel structure) on $J^2(\mathbb{R},\mathbb{R})$. It is well-known that locally the contact system is given by the vanishing of the two 1-forms $\alpha_1=dy-pdx$ and $\alpha_2=dp-qdx$.

We now define the notion of solutions. A *smooth solution* (or a *classical solution*) of $F=0$ passing through a point z_0 is a smooth function germ $y=f(x)$ at a point t_0 such that

$$(t_0,f(t_0),f'(t_0),f''(t_0))=z_0 \quad \text{and} \quad F(x,f(x),f'(x),f''(x))=0.$$

In other words, there exists a smooth function germ $f:(\mathbb{R},t_0)\to\mathbb{R}$ such that the image of the 2-jet extension, $j^2f:(\mathbb{R},t_0)\to(J^2(\mathbb{R},\mathbb{R}),z_0)$, is contained in the equation hypersurface. It is easy to see that the map j^2f is an integral (Engel) immersion. More generally, a *geometric solution* of $F=0$ passing through a point z_0 is an integral immersion $\gamma:(\mathbb{R},t_0)\to(J^2(\mathbb{R},\mathbb{R}),z_0)$ such that the image of γ is contained in the equation hypersurface, namely, $\gamma'(t)\neq 0$, $\gamma^*\alpha_1=\gamma^*\alpha_2=0$ and $F(\gamma(t))=0$ for each $t\in(\mathbb{R},t_0)$.

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In this paper, the following notions are basic (cf. [3, 6, 10, 11, 12, 20]):

A *smooth complete solution on* $F^{-1}(0)$ *at* z_0 is defined by a two-parameter family of smooth function germs $y = f(t, r, s)$ such that

$$F\left(t, f(t,r,s), \frac{\partial f}{\partial t}(t,r,s), \frac{\partial^2 f}{\partial t^2}(t,r,s)\right) = 0$$

and the map germ $j^2_* f : (\mathbb{R} \times \mathbb{R}^2, (t_0, r_0, s_0)) \to (F^{-1}(0), z_0)$ defined by

$$j^2_* f(t,r,s) = \left(t, f(t,r,s), \frac{\partial f}{\partial t}(t,r,s), \frac{\partial^2 f}{\partial t^2}(t,r,s)\right)$$

is an immersion. It follows that the equation hypersurface is foliated locally by a two-parameter family of smooth solutions.

On the other hand, consider the corresponding definition for geometric solutions. We call $\Gamma : (\mathbb{R} \times \mathbb{R}^2, (t_0, r_0, s_0)) \to (F^{-1}(0), z_0)$ a *complete solution on* $F^{-1}(0)$ *at* z_0 if Γ is a two-parameter family of geometric solutions of $F = 0$ and

$$\operatorname{rank} \begin{pmatrix} \partial x/\partial t & \partial y/\partial t & \partial p/\partial t & \partial q/\partial t \\ \partial x/\partial r & \partial y/\partial r & \partial p/\partial r & \partial q/\partial r \\ \partial x/\partial s & \partial y/\partial s & \partial p/\partial s & \partial q/\partial s \end{pmatrix} (t_0, r_0, s_0) = 3,$$

where $\Gamma(t,r,s) = (x(t,r,s), y(t,r,s), p(t,r,s), q(t,r,s))$. This condition means that Γ is an immersion germ, that is, the equation hypersurface is foliated locally by a two-parameter family of geometric solutions. We say that an equation $F = 0$ is *smoothly completely integrable* (respectively, *completely integrable*) at z_0 if there exists a smooth complete solution (respectively, a complete solution) on $F^{-1}(0)$ at z_0.

In the study of implicit ODEs from the view point of singularity theory, there is a lot of research. For example, generic singularities and properties were given in the case of first order in [1, 2, 4, 5, 7, 8, 10, 17, 19], in the case of second order in [14, 15] and in the case of any order in [9] etc. This paper is focused on the theory of completely integrable implicit ODEs (cf. [18, 20, 21]). Especially, we shall classify types of completely integrable implicit second order ODEs. In §2, we give previous results for completely integrable implicit second order ODEs, for more detail see [3, 19, 20]. In §3, we divide types of completely integrable implicit second order ODEs into ten and give an existence condition for families of geometric solutions for each type. In §4, we give examples which are useful to understand the notions of complete solutions and results. Moreover, as an application of the results, we consider the confluent hypergeometric equations (the degenerate hypergeometric equations) from the view point of complete integrability (Example 4.5). In Appendix, we give a corresponding result for completely integrable implicit first order ODEs. These results had been essentially given by Shyuichi Izumiya ([11]).

All map germs and manifolds considered here are differential of class C^∞.

2. Basic notions and previous results

Let $F(x, y, p, q) = 0$ be an implicit second order ODE. We denote the total derivative of F by $F_X = F_x + pF_y + qF_p$, where F_x (respectively, F_y, F_p, F_q) is the partial derivative with respect to x (respectively, y, p, q).

We say that $F = 0$ is of *(second order) Clairaut type* (for short, *type C*) at z_0 if there exists a function germ $\alpha : (F^{-1}(0), z_0) \to \mathbb{R}$ such that

$$F_X|_{F^{-1}(0)} = \alpha \cdot F_q|_{F^{-1}(0)},$$

and of *reduced type* (for short, *type R*) at z_0 if there exists a function germ $\beta : (F^{-1}(0), z_0) \to \mathbb{R}$ such that

$$F_q|_{F^{-1}(0)} = \beta \cdot F_X|_{F^{-1}(0)}.$$

Note that we call $F = 0$ is of reduced type as of first order type in [20]. Then we have shown the following result.

Theorem 2.1. ([20])
(1) *$F = 0$ is smoothly completely integrable at z_0 if and only if $F = 0$ is of type C at z_0.*
(2) *$F = 0$ is completely integrable at z_0 if and only if $F = 0$ is either of type C or of type R at z_0.*

We say that a geometric solution $\gamma : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0)$ is a *singular solution* of $F = 0$ at z_0 if for any representative $\widetilde{\gamma} : I \to F^{-1}(0)$ of γ and any open subinterval $(a, b) \subset I$ at t_0, $\widetilde{\gamma}|_{(a,b)}$ is never contained in a leaf of a complete solution (cf. [3, 11, 13]).

Around $z \in F^{-1}(0)$ such that the contact plane ξ_z intersects $T_z F^{-1}(0)$ transversally, it is easy to see that a complete solution on $F^{-1}(0)$ exists by integrating the line field $\xi \cap TF^{-1}(0)$. We call points where transversality fails *contact singular points* and denote by $\Sigma_c = \Sigma_c(F)$ the set of contact singular points. It is easy to check that the contact singular set is given by

$$\Sigma_c = \{z \in J^2(\mathbb{R}, \mathbb{R}) |\ F(z) = 0, F_X(z) = 0, F_q(z) = 0\}.$$

From the definition of singular solutions, it is easy to see that a geometric solution

$$\gamma : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0)$$

t is a singular solution only if it is contained in Σ_c (cf. [21]). We also consider the subset $\Delta = \Delta(F) \subset \Sigma_c$ which is defined to be the set of points $z \in \Sigma_c$ such that $T_z F^{-1}(0)$ coincides with the kernel of $\alpha_1(z)$. Explicitly, it is given by $\Delta = \{z \in \Sigma_c |\ F_p(z) = 0\}$.

Now suppose that $F = 0$ is completely integrable at z_0 and Σ_c is a 2-dimensional manifold around z_0. We say that a map germ

$$\Phi : (\mathbb{R} \times \mathbb{R}, (t_0, a_0)) \to (\Sigma_c, z_0)$$

is a *complete solution on Σ_c at z_0* if Φ is an immersion germ and $\Phi(\cdot, a)$ is a geometric solution for each $a \in (\mathbb{R}, a_0)$, that is, an immersive one-parameter family of geometric solutions of $F = 0$. Moreover, we call Φ a *complete singular solution on Σ_c at z_0* if $\Phi(\cdot, a)$ is a singular solution for each $a \in (\mathbb{R}, a_0)$.

If ξ_z intersects $T_z\Sigma_c$ transversally in $T_z F^{-1}(0)$, then integrating the line field $\xi \cap T\Sigma_c$ yields a complete solution on Σ_c. We call a point where transversality does not hold a *second order contact singular point* and denote the set of such points by $\Sigma_{cc} = \Sigma_{cc}(F)$ (cf. [3, 20, 21]).

Conditions for existence of a complete solution on $F^{-1}(0)$ and a complete (singular) solution on Σ_c for implicit second order ODEs were given under a regularity condition.

Theorem 2.2. ([3]) *Suppose that 0 is a regular value of $F_q|_{F^{-1}(0)}$.*
(1) *$F = 0$ is completely integrable at z_0 if and only if $z_0 \notin \Sigma_c$ or Σ_c is a 2-dimensional manifold around z_0.*
(2) *Let $F = 0$ be completely integrable.*
(i) *The leaves of the complete solution on $F^{-1}(0)$ which meet Σ_c away from Δ intersect Σ_c transversally.*
(ii) *The leaves of the complete solution on $F^{-1}(0)$ which meet Δ are tangent to Σ_c.*
(3) *Let $F = 0$ be completely integrable and $\Sigma_c \neq \emptyset$.*
(i) *There exists a complete singular solution on Σ_c at z_0 if and only if $z_0 \notin \Sigma_{cc}$ or Σ_{cc} is a 1-dimensional manifold around z_0.*

(ii) *Suppose that* $F = 0$ *admits a complete singular solution on* Σ_c*. Then each leaf of the complete singular solution on* Σ_c *intersects* Σ_{cc} *transversally.*
(4) *Let* $F = 0$ *be completely integrable at* $z_0 \in \Sigma_c$*. If* $z_0 \in \Delta$*, then* Δ *is a* 1*-dimensional manifold around* z_0*.*

Theorem 2.3. ([20]) *Suppose that* 0 *is a regular value of* $F_X|_{F^{-1}(0)}$*.*
(1) $F = 0$ *is completely integrable at* z_0 *if and only if* $z_0 \notin \Sigma_c$ *or* Σ_c *is a* 2*-dimensional manifold around* z_0*.*
(2) *Let* $F = 0$ *be completely integrable.*
(i) *The leaves of the complete solution on* $F^{-1}(0)$ *which meet* Σ_c *away from* Δ *intersect* Σ_c *transversally.*
(ii) *The leaves of the complete solution on* $F^{-1}(0)$ *which meet* Δ *are tangent to* Σ_c*.*
(3) *Let* $F = 0$ *be completely integrable and* $\Sigma_c \neq \emptyset$*.*
(i) *There exists a complete solution on* Σ_c *at* z_0 *if and only if* $z_0 \notin \Sigma_{cc}$ *or* Σ_{cc} *is a* 1*-dimensional manifold around* z_0*.*
(ii) *Suppose that* $F = 0$ *admits a complete solution on* Σ_c*. Then each leaf of the complete solution on* Σ_c *intersects* Σ_{cc} *transversally.*

Remark 2.4. The important differences between Theorems 2.2 and 2.3 are (3) and (4). One is an existence condition for a complete singular solution on Σ_c and the other is only for a complete solution on Σ_c. Moreover, if $F = 0$ is completely integrable at $z_0 \in \Delta$ and 0 is a regular value of $F_q|_{F^{-1}(0)}$, then Δ is a 1-dimensional manifold around z_0. However, Δ is not necessarily a 1-dimensional manifold around z_0 when 0 is a regular value of $F_X|_{F^{-1}(0)}$, see Examples 4.1 and 4.4.

Proposition 2.5. ([18, 20]) *Let* $F = 0$ *be completely integrable at* $z_0 \in \Sigma_c$*.*
(1) *If* 0 *is a regular value of* $F_q|_{F^{-1}(0)}$*, then* $F = 0$ *is of type* C *at* z_0*.*
(2) *If* 0 *is a regular value of* $F_X|_{F^{-1}(0)}$*, then* $F = 0$ *is of type* R *at* z_0*.*

Proposition 2.6. ([20]) *Let* $F = 0$ *be completely integrable at* z_0 *and* Σ_c *be a* 2*-dimensional manifold around* z_0*. Then the second order singular set* Σ_{cc} *is contained in* Δ*.*

3. Completely integrable implicit second order ODEs

In this section, we analyse completely integrable implicit second order ODEs in detail. Let $F(x, y, p, q) = 0$ be an implicit second order ODE at z_0. If $z_0 \notin \Sigma_c$, then $F = 0$ satisfies either $F_q(z_0) \neq 0$ or $F_X(z_0) \neq 0$.

First we assume that $F_q(z_0) \neq 0$. By the implicit function theorem, $F = 0$ can be represented by an explicit equation at least locally. In this case, $F = 0$ is of type C at z_0 and we call this type C_q. Next we assume that $F_X(z_0) \neq 0$. Then $F = 0$ is of type R at z_0 and we call this type R_X. In both cases, there is a unique geometric solution passing through each point of $F^{-1}(0)$. It follows that there is a complete solution on $F^{-1}(0)$ and no singular solution.

By Theorem 2.1, a completely integrable ODE at z_0 is either of type C or of type R at z_0. If $z_0 \in \Sigma_c$, then $F = 0$ satisfies either $F_p(z_0) \neq 0$ or $F_y(z_0) \neq 0$ by the assumption that $F = 0$ is regular at z_0 (see §1). The main purpose of this paper is to classify types of the completely integrable implicit second order ODEs at a point in detail, and to give existence conditions for a complete (singular) solution on Σ_c for each type respectively. It is concluded that there are ten kinds of types, see Table 1.

Conditions				Type	Name
$z_0 \notin \Sigma_c$	$F_q(z_0) \neq 0$			C	C_q
	$F_X(z_0) \neq 0$			R	R_X
$z_0 \in \Sigma_c$	$F_p(z_0) \neq 0$	z_0 is a regular point of $F_q\|_{F^{-1}(0)}$		C	RC_p
		z_0 is a regular point of $F_X\|_{F^{-1}(0)}$		R	RR_p
	$F_y(z_0) \neq 0$,	z_0 is a regular point of $F_q\|_{F^{-1}(0)}$		C	RC_y
	$F_p(z_0) = 0$	z_0 is a regular point of $F_X\|_{F^{-1}(0)}$	$\Sigma_c = \Delta$	R	RR_y^1
			$\Sigma_c \supsetneq \Delta = \Sigma_{cc}$	R	RR_y^2
			$\Sigma_c \supsetneq \Delta \supsetneq \Sigma_{cc}$	R	RR_y^3
		z_0 is a singular point of $F_q\|_{F^{-1}(0)}$		C	SC_y
		and $F_X\|_{F^{-1}(0)}$		R	SR_y

Table 1. A classification of types of completely integrable implicit second order ODEs at z_0.

3.1. **On the types RC_p and RR_p.** If $z_0 \in \Sigma_c$ and $F_p(z_0) \neq 0$, by the implicit function theorem, there exists a smooth function $g : V \to \mathbb{R}$, where V is an open set in $\mathbb{R}^3$, such that in a neighbourhood of z_0, $(x, y, p, q) \in F^{-1}(0)$ if and only if $-p + g(x, y, q) = 0$. Thus we may assume without loss of generality that $F(x, y, p, q) = -p + g(x, y, q) = 0$. Under this notations, $F_q = g_q$ and $F_X = g_x + g \cdot g_y - q$. It follows that z_0 is a regular point of either $F_q|_{F^{-1}(0)}$ or $F_X|_{F^{-1}(0)}$.

If z_0 is a regular point of $F_q|_{F^{-1}(0)}$, then $F = 0$ is of type C at z_0 and Σ_c is a 2-dimensional manifold around z_0 by Proposition 2.5 and Theorem 2.2. We call this type RC_p. By $z_0 \notin \Delta$ and Proposition 2.6, we have $z_0 \notin \Sigma_{cc}$. Hence $F = 0$ has a complete singular solution on Σ_c at z_0.

On the other hand, suppose that z_0 is a regular point of $F_X|_{F^{-1}(0)}$. By Proposition 2.5 and Theorem 2.3, $F = 0$ is of type R at z_0 and Σ_c is a 2-dimensional manifold around z_0. We call this type RR_p. By $z_0 \notin \Delta$ and Proposition 2.6, we have $z_0 \notin \Sigma_{cc}$. Since the leaves of the complete solution which meet Σ_c away from Δ intersect Σ_c transversally, $F = 0$ has a complete singular solution on Σ_c at z_0.

3.2. **On the type RC_y.** If $z_0 \in \Sigma_c$ and $F_y(z_0) \neq 0$, again by the implicit function theorem, there exists a smooth function $f : U \to \mathbb{R}$, where U is an open set in $\mathbb{R}^3$, such that in a neighbourhood of z_0, $(x, y, p, q) \in F^{-1}(0)$ if and only if $-y + f(x, p, q) = 0$. Thus we may assume without loss of generality that $F(x, y, p, q) = -y + f(x, p, q) = 0$. Define the diffeomorphism $\phi : U \to F^{-1}(0), (x, p, q) \mapsto (x, f(x, p, q), p, q)$ and $u_0 = \phi^{-1}(z_0)$. Below, if $F_y(z_0) \neq 0$, we keep the notations of the above.

Suppose that z_0 is a regular point of $F_q|_{F^{-1}(0)}$. By Proposition 2.5 and Theorem 2.2, $F = 0$ is of type C at z_0 and Σ_c is a 2-dimensional manifold around z_0. We call this type RC_y. Moreover, $F = 0$ has a complete singular solution on Σ_c at z_0 if and only if $z_0 \notin \Sigma_{cc}$ or Σ_{cc} is a 1-dimensional manifold around z_0 by Theorem 2.2.

Remark 3.1. If Σ_{cc} is a 1-dimensional manifold around z_0, then $\Delta = \Sigma_{cc}$ and Σ_{cc} is an isolated singular solution passing through z_0 (see, [3, Proposition 1.4]). In this case, $F = 0$ have a two-parameter family of geometric solutions, a one-parameter family of singular solutions and an isolated singular solution passing through $z_0 \in \Sigma_{cc}$, see Example 4.2.

3.3. **On the type RR_y^1.** Let $z_0 \in \Sigma_c$ and $F_y(z_0) \neq 0$. Suppose that z_0 is a regular point of $F_X|_{F^{-1}(0)}$. By Proposition 2.5 and Theorem 2.3, $F = 0$ is of type R at z_0 and Σ_c is a 2-dimensional manifold around z_0. In this case, there are three types. First one is $\Sigma_c = \Delta$ around z_0 (type RR_y^1), second is $\Sigma_c \supsetneq \Delta = \Sigma_{cc}$ around z_0 (type RR_y^2), and the last is $\Sigma_c \supsetneq \Delta \supsetneq \Sigma_{cc}$ around z_0 (type RR_y^3). We may assume that $F_p(z_0) = 0$, namely, $z_0 \in \Delta$.

Let $F=0$ be of the type RR^1_y at z_0. By Theorem 2.3, $F=0$ has a complete solution of Σ_c at z_0 if and only if $z_0 \notin \Sigma_{cc}$ or Σ_{cc} is a 1-dimensional manifold around z_0. In this case, we have the following result, see Examples 4.1 and 4.4.

Theorem 3.2. *Let $F=0$ be of type RR^1_y at $z_0 \in \Delta$. If $z_0 \notin \Sigma_{cc}$, then there exists a unique geometric solution passing through z_0.*

Proof. We denote $F(x,y,p,q) = -y + f(x,p,q) = 0$. Since $F=0$ is of type R at z_0, there exists a smooth function germ $\alpha : (F^{-1}(0), z_0) \to \mathbb{R}$ such that

$$f_q = \alpha \cdot (f_x - p + qf_p). \tag{1}$$

A complete solution, $\Gamma : (\mathbb{R} \times \mathbb{R}^2, 0) \to (F^{-1}(0), z_0)$, is given by integrating the vector field $\phi_* X$, where $X : U \to TU$ is given by

$$X = (-\alpha, -\alpha \cdot q, 1)$$

(cf. [3, Lemma 3.1]). By (1), we have

$$(f_x - p + qf_p)_q = (\alpha_x + q\alpha_p) \cdot (f_x - p + qf_p) + \alpha \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p) + f_p.$$

It follows from the assumption $\Sigma_c = \Delta$ that

$$(f_x - p + qf_p)_q|_{\phi^{-1}(\Sigma_c)} = \alpha|_{\phi^{-1}(\Sigma_c)} \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)|_{\phi^{-1}(\Sigma_c)}.$$

In this case, a complete solution on Σ_c, $\Phi : (\mathbb{R} \times \mathbb{R}, 0) \to (\Sigma_c, z_0)$, is given by integrating the vector field $\phi_* Y$, where $Y : \phi^{-1}(\Sigma_c) \to T\phi^{-1}(\Sigma_c)$ is given by

$$Y = (-\alpha|_{\phi^{-1}(\Sigma_c)}, (-\alpha \cdot q)|_{\phi^{-1}(\Sigma_c)}, 1)$$

(cf. [20, Lemma 3.5]). It follows that $\Gamma|_{\Gamma^{-1}(\Sigma_c)} = \Phi$ and hence there is a geometric solution on Σ_c. Let $\gamma : (\mathbb{R}, t_0) \to (\Sigma_c, z_0); \gamma(t) = (x(t), y(t), p(t), q(t))$ be a geometric solution passing through z_0. Since $z_0 \notin \Sigma_{cc}$, we have $x'(t) + \alpha \cdot q'(t) = 0$ at t_0. It follows that we can reparametrise $\gamma(t)$ as $(x(t), y(t), p(t), t)$. By the analogous way in the proof of Lemma 3.2 in [21], we can show uniqueness of the geometric solution passing through z_0. □

Proposition 3.3. *Let $F=0$ be of type RR^1_y at $z_0 \in \Delta$. If Σ_{cc} is a 1-dimensional manifold around z_0, then Σ_{cc} is a singular solution passing through z_0.*

Proof. It is easy to see that Σ_{cc} is a geometric solution passing through z_0. By definition,

$$\phi^{-1}(\Sigma_c) = (f_x - p + qf_p)^{-1}(0)$$

and

$$\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)^{-1}(0).$$

To show that Σ_{cc} is not a leaf of the complete solution on $F^{-1}(0)$ (and on Σ_c) at z_0, it is sufficient to check that the scalar product of $\mathrm{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)$ and the vector field X is non-zero at u_0. Now

$$\begin{aligned} &\langle \mathrm{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p), (-\alpha, -\alpha \cdot q, 1)\rangle \\ &= -\alpha \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_x - \alpha \cdot q((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_p \\ &\quad + ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_q. \end{aligned} \tag{2}$$

It follows from (1) that (2) is equal to $2(f_{xp} + qf_{pp}) - 1$ at u_0. By the assumption $\Sigma_c = \Delta$, there exists a smooth function germ β such that $f_p = \beta \cdot (f_x - p + qf_p)$ at least locally. Differentiating this equality with respect to x and p, we get

$$f_{xp} = \beta_x \cdot (f_x - p + qf_p) + \beta \cdot (f_x - p + qf_p)_x$$

and

$$f_{pp} = \beta_p \cdot (f_x - p + qf_p) + \beta \cdot (f_x - p + qf_p)_p.$$

It follows that (2) is non-zero at u_0. □

3.4. **On the type RR_y^2.** Suppose that $F = 0$ is of type RR_y^2 at z_0. See Example 4.2. Then $\Sigma_c \supsetneq \Delta = \Sigma_{cc}$ around z_0. By Theorem 2.3, $F = 0$ has a complete solution on Σ_c at z_0 if and only if Σ_{cc} is a 1-dimensional manifold around z_0. In this case, we have the following result.

Theorem 3.4. *Let $F = 0$ be of type RR_y^2 at $z_0 \in \Delta$. $F = 0$ has a complete singular solution on Σ_c at z_0 if and only if Σ_{cc} is a 1-dimensional manifold around z_0.*

Proof. By Theorem 2.3, each leaf of the complete solution on $F^{-1}(0)$ which meet Σ_c away from Σ_{cc} intersect Σ_c transversally, and each leaf of the complete solution on Σ_c intersects Σ_{cc} transversally. Therefore the complete solution on Σ_c is the complete singular solution on Σ_c. □

By the definition of Σ_{cc},

$$(f_x - p + qf_p)_x + q(f_x - p + qf_p)_p = 0,\ (f_x - p + qf_p)_q = 0$$

at $z_0 \in \Sigma_{cc}$. Since z_0 is a regular point of $F_X|_{F^{-1}(0)}$, $(f_x - p + qf_p)_p \neq 0$ at z_0. The equation $F = 0$ satisfies either

$$\text{(i) } ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_q \neq 0$$

or

$$\text{(ii) } ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_q = 0$$

at z_0. It follows that z_0 is a regular point of $(f_x - p + qf_p)_x + q(f_x - p + qf_p)_p$, or of $(f_x - p + qf_p)_q$.

Proposition 3.5. *Let $F = 0$ be of type RR_y^2 at $z_0 \in \Delta$. Suppose that Σ_{cc} is a 1-dimensional manifold around z_0.*

(1) *If $F = 0$ satisfies the condition* (i)*, then each leaf of the complete solution on $F^{-1}(0)$ is intersects Σ_{cc} transversally and hence Σ_{cc} is a singular solution passing through z_0.*

(2) *If $F = 0$ satisfy the conditions* (ii) *and $F_{pq}|_{\Sigma_{cc}} \equiv 0$ around z_0, then each leaf of the complete solution on $F^{-1}(0)$ is tangent to Σ_{cc}. If $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Sigma_{cc}$ is a geometric solution, $\gamma(t)$ is represented by the form (a, b, c, t), where $a, b, c \in \mathbb{R}$. Moreover, $\gamma(t)$ is a leaf of the complete solution on $F^{-1}(0)$.*

Proof. (1) Since $\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)^{-1}(0)$, it is sufficient to check that the scalar product of $\mathrm{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)$ and the vector field X is non-zero at u_0. By the same calculations in Proposition 3.3,

$$\langle \mathrm{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p), (-\alpha, -\alpha \cdot q, 1)\rangle = 2(f_{xp} + qf_{pp}) - 1$$

at u_0. The condition (i) guarantees that $2(f_{xp} + qf_{pp}) - 1 \neq 0$ at u_0. Therefore each leaf of the complete solution on $F^{-1}(0)$ intersects Σ_{cc} transversally and hence Σ_{cc} is a singular solution passing through z_0.

(2) Since $\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_q)^{-1}(0)$, it is sufficient to check that the scalar product of $\mathrm{grad}(f_x - p + qf_p)_q$ and the vector field X is zero. By the direct calculations, the consequence follows from the condition $F_{pq}|_{\Sigma_{cc}} \equiv 0$ around z_0.

Let $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Sigma_{cc}$ be a geometric solution passing through z_0. By differentiating $f_p(x(t), p(t), q(t)) = 0$ with respect to t, we get

$$(f_{xp} + qf_{pp})(x(t), p(t), q(t)) \cdot x'(t) + f_{pq}(x(t), p(t), q(t)) \cdot q'(t) = 0.$$

By the condition (ii), we have $f_{xp}+qf_{pp}=1/2$ at u_0 and hence $x'(t)\equiv 0$. This means that $x(t)$ is constant on Σ_{cc} around z_0. Differentiating (1) with respect to p, we have

$$f_{pq}=\alpha_p\cdot(f_x-p+qf_p)+\alpha\cdot(f_x-p+qf_p)_p.$$

It follows that $\alpha|_{\Sigma_{cc}}\equiv 0$ around z_0. By the form of the vector field X (see, in the proof of Theorem 3.2), $\Gamma|_{\Gamma^{-1}(\Sigma_{cc})}=\gamma$. □

3.5. **On the type RR^3_y.** Suppose that $F=0$ is of type RR^3_y at z_0. See Example 4.3. Then $\Sigma_c\supsetneq\Delta\supsetneq\Sigma_{cc}$ around z_0. In this subsection, assume that Δ is a 1-dimensional manifold around z_0 and $z_0\notin\Sigma_{cc}$, since we consider complete solutions. By Theorem 2.3, $F=0$ has a complete solution on Σ_c at z_0. If Δ is not a geometric solution passing through z_0, the complete solution on Σ_c is the complete singular solution on Σ_c. On the other hand, if Δ is a geometric solution passing through z_0, we have the following result.

Proposition 3.6. *Let $F=0$ be of type RR^3_y at $z_0\in\Delta\setminus\Sigma_{cc}$. If $\gamma(t)=(x(t),y(t),p(t),q(t))\in\Delta$ is a geometric solution passing through z_0, then $\gamma(t)$ is represented by the form (a,b,c,t) where $a,b,c\in\mathbb{R}$. Moreover, $\gamma(t)$ is a leaf of both complete solutions on $F^{-1}(0)$ and Σ_c.*

Proof. Since $z_0\notin\Sigma_{cc}$, we have $(f_x-p+qf_p)_x+q(f_x-p+qf_p)_p\neq 0$ at u_0. Differentiating equalities $(f_x-p+qf_p)(x(t),p(t),q(t))=0$ and $f_p(x(t),p(t),q(t))=0$ with respect to t, we have

$$\begin{pmatrix}(f_x-p+qf_p)_x+q(f_x-p+qf_p)_p & (f_x-p+qf_p)_q\\ f_{xp}+qf_{pp} & f_{pq}\end{pmatrix}\begin{pmatrix}x'(t)\\ q'(t)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Since $\gamma(t)$ is a geometric solution, $(x'(t),q'(t))\neq(0,0)$ on Δ. Thus

$$\det\begin{pmatrix}(f_x-p+qf_p)_x+q(f_x-p+qf_p)_p & (f_x-p+qf_p)_q\\ f_{xp}+qf_{pp} & f_{pq}\end{pmatrix}=0$$

on Δ. It follows that $\alpha|_\Delta\equiv 0$ and hence $x'(t)\equiv 0$. This means that $x(t)$ is constant on Δ around z_0. By the forms of the vector field X for a complete solution on $F^{-1}(0)$ and of the vector field Y for a complete solution on Σ_c (which appeared in the proof of Theorem 3.2), it follows that $\Gamma|_{\Gamma^{-1}(\Delta)}=\Phi|_{\Phi^{-1}(\Delta)}=\gamma$. □

3.6. **On the type SC_y.** Suppose that $F=0$ is of type C at $z_0\in\Sigma_c$ and z_0 is a singular point of $F_q|_{F^{-1}(0)}$ and $F_X|_{F^{-1}(0)}$. We call this type SC_y. See Example 4.4.

Proposition 3.7. *Let $F=0$ be of type SC_y at z_0. If Σ_c is a 2-dimensional manifold around z_0, then $z_0\notin\Sigma_{cc}$.*

Proof. Let $F(x,y,p,q)=-y+f(x,p,q)=0$. Since $F=0$ is of type C at z_0, there is a function germ $\alpha:(F^{-1}(0),z_0)\to\mathbb{R}$ such that

$$f_x-p+qf_p=\alpha\cdot f_q. \tag{3}$$

By differentiating (3) with respect to p, we have $f_{xp}-1+qf_{pp}=\alpha_p\cdot f_q+\alpha\cdot f_{pq}$. Hence $f_{xp}+qf_{pp}=1$ at u_0. By a direct calculation,

$$(f_x-p+qf_p)_{xq}+q(f_x-p+qf_p)_{pq}=(f_{xq}+qf_{pq})_x+q(f_{xq}+qf_{pq})_p+f_{xp}+qf_{pp}. \tag{4}$$

On the other hand, by (3),

$$\begin{aligned}&(f_x-p+qf_p)_{xq}+q(f_x-p+qf_p)_{pq}\\ &=(\alpha_{xq}+q\alpha_{pq})\cdot f_q+\alpha_q\cdot(f_{qx}+qf_{pq})+(\alpha_x+q\alpha_p)\cdot f_{qq}+\alpha\cdot(f_{xqq}+qf_{pqq}).\end{aligned} \tag{5}$$

By definition, $\phi^{-1}(\Sigma_c)=f_q^{-1}(0)$. Since Σ_c is a 2-dimensional manifold around z_0, there is a regular function germ $g:(U,u_0)\to\mathbb{R}$ and a function germ $k:(U,u_0)\to(\mathbb{R},0)$ such that

$\phi^{-1}(\Sigma_c) = g^{-1}(0)$ and $f_q = k \cdot g$ at least locally. By a direct calculation, the right hand of (4) is given by

$$((k_x+qk_p)_x+q(k_x+k_p)_p)\cdot g+2(k_x+qk_p)\cdot(g_x+qg_p)+k\cdot((g_x+qg_p)_x+q(g_x+qg_p)_p)+f_{xp}+qf_{pp}.$$

Also the right hand of (5) is given by

$$\begin{aligned}&(\alpha_{xq}+q\alpha_{pq})\cdot k\cdot g+\alpha_q\cdot((k_x+qk_p)\cdot g+k\cdot(g_x+qg_p))+(\alpha_x+q\alpha_p)\cdot(k_q\cdot g+k\cdot g_q)\\&\quad+\alpha\cdot((k_{xq}+qk_{pq})\cdot g+k_q\cdot(g_x+qg_p)+(k_x+qk_p)\cdot g_q+k\cdot(g_{xq}+qg_{pq}))\,.\end{aligned}$$

If $z_0 \in \Sigma_{cc}$, then $g = g_x + qg_p = g_q = 0$ at u_0. This contradicts the fact that (4) = (5), namely 1=0 at u_0. □

Under the assumption of Proposition 3.7, it follows from $z_0 \notin \Sigma_{cc}$ that there is a complete solution on Σ_c at z_0. According to Theorem 3.11 in below, a geometric solution passing through z_0 on Σ_c is a singular solution for type C. Hence the complete solution on Σ_c is the complete singular solution on Σ_c at z_0.

3.7. **On the type SR_y.** Suppose that $F = 0$ is of type R at $z_0 \in \Sigma_c$ and z_0 is a singular point of $F_q|_{F^{-1}(0)}$ and $F_X|_{F^{-1}(0)}$. We call this type SR_y. We can also prove the following result by using the same arguments in the proof of Proposition 3.7, so we omit the proof.

Proposition 3.8. *Let $F = 0$ be of type SR_y at z_0. If Σ_c is a 2-dimensional manifold around z_0, then $z_0 \notin \Sigma_{cc}$.*

Moreover, we have the following result.

Proposition 3.9. *Let $F = 0$ be of type SR_y and not of type C at z_0. If Σ_c is a 2-dimensional manifold around z_0, then Δ is a 1-dimensional manifold around z_0. Moreover, Δ is not a geometric solution passing through z_0.*

Proof. By (1), $f_q = \alpha \cdot (f_x - p + qf_p)$ with $\alpha(z_0) = 0$. Since $\phi^{-1}(\Sigma_c) = (f_x - p + qf_p)^{-1}(0)$ is a 2-dimensional manifold around z_0, there exist a regular function germ $g : (U, u_0) \to (\mathbb{R}, 0)$ and a function germ $k : (U, u_0) \to (\mathbb{R}, 0)$ such that $f_x - p + qf_p = k \cdot g$ and $k^{-1}(0) \subset g^{-1}(0)$ at least locally. By a direct calculation, we have

$$(f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} = 1$$

at u_0. On the other hand,

$$(f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} = k_q \cdot (g_x + qg_p) + (k_x + qk_p) \cdot g_q$$

at u_0. Hence $k_q \cdot (g_x + qg_p) + (k_x + qk_p) \cdot g_q = 1$ at u_0. If $g_q(u_0) = 0$, then $k_q(u_0) \neq 0$. It follows that k is represented by $\lambda(x, p, q) \cdot (q - \mu(x, p))$ at least locally, where λ and μ are function germs with $\lambda(u_0) \neq 0$. Since $k^{-1}(0) \subset g^{-1}(0)$, $g(x, p, \mu(x, p)) = 0$. By differentiating this equality with respect to x and p, we have

$$g_x(x, p, \mu(x, p)) + \mu_x(x, p)g_q(x, p, \mu(x, p)) = 0$$

and

$$g_p(x, p, \mu(x, p)) + \mu_p(x, p)g_q(x, p, \mu(x, p)) = 0.$$

This contradicts the fact that g is regular at u_0. Therefore we have $g_q \neq 0$ at u_0.

By the definition of Δ, $\phi^{-1}(\Delta) = g^{-1}(0) \cap f_p^{-1}(0)$. To show that Δ is a 1-dimensional manifold around z_0, it is sufficient to show that the matrix

$$A = \begin{pmatrix} g_x & g_p & g_q \\ f_{xp} & f_{pp} & f_{pq} \end{pmatrix}$$

has rank 2 at u_0. Since $f_x - p + qf_p$ and f_q are singular at u_0, $f_{xp} + qf_{pp} = 1$ and $f_{pq} = 0$ at u_0. Therefore $\operatorname{rank} A = 2$ at u_0.

Next suppose that $\gamma : (\mathbb{R}, t_0) \to (\Delta, z_0); \gamma(t) = (x(t), y(t), p(t), q(t))$ is a geometric solution passing through z_0. By differentiating equalities $g(x(t), p(t), q(t)) = 0$ and $f_p(x(t), p(t), q(t)) = 0$ with respect to t, we have

$$\left(\begin{array}{cc} (g_x + qg_p)(x(t), p(t), q(t)) & g_q(x(t), p(t), q(t)) \\ (f_{xp} + qf_{pp})(x(t), p(t), q(t)) & f_{pq}(x(t), p(t), q(t)) \end{array}\right) \left(\begin{array}{c} x'(t) \\ q'(t) \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \end{array}\right).$$

Since the determinant of the matrix

$$\left(\begin{array}{cc} g_x + qg_p & g_q \\ f_{xp} + qf_{pp} & f_{pq} \end{array}\right)$$

does not vanish at t_0, $(x'(t), q'(t)) = (0, 0)$ at t_0. This contradicts the fact that $\gamma(t)$ is a geometric solution passing through z_0. □

As a conclusion, if $F = 0$ is of type SR_y, not of type C at z_0 and Σ_c is a 2-dimensional manifold around z_0, then there is a complete singular solution on Σ_c at z_0 by Propositions 3.8 and 3.9.

Finally, in this section, we give an important difference between type C and type R.

Lemma 3.10. *Let $F = 0$ be of type RC_y at z_0. If $z_0 \in \Delta \setminus \Sigma_{cc}$, then Δ is not a geometric solution passing through z_0.*

Proof. By Theorem 2.2, Δ is a 1-dimensional manifold around z_0. Suppose that

$$\gamma : (\mathbb{R}, t_0) \to (\Delta, z_0); \gamma(t) = (x(t), y(t), p(t), q(t))$$

is a geometric solution passing through z_0. Differentiating

$$f_p(x(t), p(t), q(t)) = 0 \quad \text{and} \quad f_q(x(t), p(t), q(t)) = 0$$

with respect to t, we have

$$\left(\begin{array}{cc} (f_{xp} + qf_{pp})(x(t), p(t), q(t)) & f_{pq}(x(t), p(t), q(t)) \\ (f_{xq} + qf_{pq})(x(t), p(t), q(t)) & f_{qq}(x(t), p(t), q(t)) \end{array}\right) \left(\begin{array}{c} x'(t) \\ q'(t) \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \end{array}\right).$$

Moreover, differentiating (3) with respect to p and q, $f_{xp} - 1 + qf_{pp} = \alpha_p \cdot f_q + \alpha \cdot f_{pq}$ and $f_{xq} + f_p + qf_{pq} = \alpha_q \cdot f_q + \alpha \cdot f_{qq}$ respectively. Then

$$\det \left(\begin{array}{cc} (f_{xp} + qf_{pp})(x(t), p(t), q(t)) & f_{pq}(x(t), p(t), q(t)) \\ (f_{xq} + qf_{pq})(x(t), p(t), q(t)) & f_{qq}(x(t), p(t), q(t)) \end{array}\right) = f_{qq}(x(t), p(t), q(t)).$$

The condition $z_0 \notin \Sigma_{cc}$ guarantees that $f_{qq} \neq 0$ at u_0. It follows that $(x'(t), q'(t)) = (0, 0)$ at t_0. This contradicts the fact that $\gamma(t)$ is a geometric solution passing through z_0. □

Theorem 3.11. *Let $F = 0$ be of type C at z_0. If $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Sigma_c$ is a geometric solution passing through z_0, then $\gamma(t)$ is the singular solution.*

Proof. First we assume that z_0 is a regular point of $F_q|_{F^{-1}(0)}$. If $z_0 \notin \Delta$, then $\gamma(t)$ is a singular solution passing through z_0 and hence we may regard that $\gamma(t) \subset \Delta$ by Theorem 2.2. Also if $z_0 \notin \Sigma_{cc}$, then $\gamma(t)$ is not a geometric solution passing through z_0 by Lemma 3.10. We may assume that $\gamma(t) \subset \Sigma_{cc}$. Then we can conclude that $\gamma(t)$ is a singular solution passing through z_0, see Remark 3.1.

Next we assume that z_0 is a singular point of $F_q|_{F^{-1}(0)}$. Also we may regard that $\gamma(t) \subset \Delta$. By differentiating $f_p(x(t), p(t), q(t)) = 0$ with respect to t,

$$(f_{xp} + qf_{pp})(x(t), p(t), q(t)) \cdot x'(t) + f_{pq}(x(t), p(t), q(t)) \cdot q'(t) = 0.$$

Since $f_{xp} - 1 + qf_{pp} = \alpha_p \cdot f_q + \alpha_p \cdot f_{pq}$, we have

$$(1 + \alpha \cdot f_{pq}(x(t), p(t), q(t))) \cdot x'(t) + f_{pq}(x(t), p(t), q(t)) \cdot q'(t) = 0.$$

By the assumption, $f_{pq}(u_0) = 0$. Hence $x'(t_0) = 0$ and $q'(t_0) \neq 0$. It follows from the form of smooth complete solution, $\gamma(t)$ is the singular solution passing through z_0. This completes the proof of Theorem 3.11. □

As a consequence, if $F = 0$ is of type C and there exists a geometric solution on the contact singular set, then uniqueness for geometric solutions does not hold.

4. Examples

We give examples of completely integrable second order ODEs. For more examples, refer to [3, Examples 5.1 and 5.2] etc.

Example 4.1. Let $F(x, y, p, q) = y + (1/2)p^2q^{2n+1} = 0$, where n is a natural number. In this case, $F_X = p(1+q^{2n+2})$ and $F_q = (1/2)(2n+1)p^2q^{2n}$. Hence $F = 0$ is of type R at $z_0 \in F^{-1}(0)$. Since 0 is a regular value of $F_X|_{F^{-1}(0)}$, and

$$\Sigma_c = \{(x, y, p, q) \mid y = p = 0\} = \Delta, \quad \Sigma_{cc} = \{(x, y, p, q) \mid y = p = q = 0\},$$

$F = 0$ is of type RR^1_y at $z_0 \in \Sigma_c$. By Theorems 2.3, 3.2 and Proposition 3.3, there exist a complete solutions on $F^{-1}(0)$ and Σ_c, and a singular solution. Indeed, the complete solutions $\Gamma : \mathbb{R} \times \mathbb{R}^2 \to F^{-1}(0), \Phi : \mathbb{R} \times \mathbb{R} \to \Sigma_c$ and the singular solution $\gamma : \mathbb{R} \to \Sigma_{cc}$ are given by

$$\Gamma(t, r, s) = \Big(-\frac{2n+1}{2} r \int (1 + t^{2n+2})^{-\frac{6n+5}{4(n+1)}} t^{2n} dt + s,$$
$$-\frac{1}{2} r^2 t^{2n+1} (1 + t^{2n+2})^{-\frac{2n+1}{2(n+1)}}, r(1 + t^{2n+2})^{-\frac{2n+1}{4(n+1)}}, t \Big),$$

$\Phi(t, a) = (a, 0, 0, t)$ and $\gamma(t) = (t, 0, 0, 0)$. We can observe that $\Gamma|_{\Gamma^{-1}(\Sigma_c)} = \Phi$.

Example 4.2. Let $F(x, y, p, q) = -y + pq^n - (n/(2n+1))q^{2n+1} = 0$, where n is a natural number. In this case, $F_X = -p + q^{n+1}$ and $F_q = -nq^{n-1}(-p + q^{n+1})$. Hence $F = 0$ is of type C and of type R for $n = 1$, and of type R for $n \geq 2$ at $z_0 \in F^{-1}(0)$. Since 0 is a regular value of $F_X|_{F^{-1}(0)}$ and

$$\Sigma_c = \left\{ (x, y, p, q) \mid y = \frac{n+1}{2n+1} q^{2n+1},\ p = q^{n+1} \right\},\ \Delta = \{(x, y, p, q) \mid y = p = q = 0\} = \Sigma_{cc},$$

$F = 0$ is of type RR^2_y at $z_0 \in \Delta$. Note that $F = 0$ is also of type RC_y at z_0 if $n = 1$. By Theorems 2.3 and 3.4, there exist a complete solution on $F^{-1}(0)$ and a complete singular solution on Σ_c. Moreover, $F = 0$ satisfies the condition (i) of Proposition 3.5 in §3.4, Σ_{cc} is an isolated singular solution. Indeed, the complete solution on $F^{-1}(0)$, the complete singular solution on Σ_c and the isolated singular solution are given by

$$\Gamma(t, r, s) = \left(t^n + r, \frac{n^2}{(n+1)(2n+1)} t^{2n+1} + st^n, \frac{n}{n+1} t^{n+1} + s, t \right),$$

$$\Phi(t, a) = \left(\frac{n+1}{n} t^n + a, \frac{n+1}{2n+1} t^{2n+1}, t^{n+1}, t \right) \text{ and } \gamma(t) = (t, 0, 0, 0).$$

If $n=1$, the complete solution on $F^{-1}(0)$ can be parametrised by

$$\Gamma(t,r,s)=\left(t,\frac{1}{6}t^3+\frac{1}{2}rt^2+st+rs-\frac{1}{3}r^3,\frac{1}{2}t^2+rt+s,t+r\right).$$

Example 4.3. Let

$$F(x,y,p,q)=-y+(1/2)x^2-(1/n)pq^n+(1/n)xq^n+(1/2n^2)q^{2n}-(1/n(2n+1))q^{2n+1}=0,$$

where n is a natural number. In this case, $F_X=x+(1/n)q^n-p-(1/n)q^{n+1}$ and $F_q=q^{n-1}F_X$. Since 0 is a regular value of $F_X|_{F^{-1}(0)}$ and

$$\Sigma_c=\left\{(x,y,p,q)\mid y=\frac{1}{2}x^2-\frac{1}{2n^2}q^{n+1}+\frac{n+1}{n^2(2n+1)}q^{2n+1}\right\},$$

$$\Delta=\left\{(x,y,p,q)\mid y=\frac{1}{2}x^2,p=x,q=0\right\},\ \Sigma_{cc}=\emptyset,$$

$F=0$ is of type RR_y^3 at $z_0\in\Delta$. Note that if $n=1$, then $F=0$ is also of type RC_y at z_0. By Theorem 2.3, there exist complete solutions on $F^{-1}(0)$ and Σ_c. Since Δ is not a geometric solution, the complete solution on Σ_c is the complete singular solution on Σ_c. The complete solution on $F^{-1}(0)$ and the complete singular solution on Σ_c at 0 are given by

$$\Gamma(t,r,s)=\left(-\frac{1}{n}t^n+r,\frac{1}{(n+1)(2n+1)}t^{2n+1}-\frac{1}{n}st^n+\frac{1}{2}r^2,-\frac{1}{n+1}t^{n+1}+s,t\right),$$

$$\Phi(t,a)=\left(x(t,a),\frac{1}{2}x(t,a)^2-\frac{1}{2n^2}t^{n+1}+\frac{n+1}{n^2(2n+1)}t^{2n+1},x(t,a)+\frac{1}{n}t^n-\frac{1}{n}t^{n+1},t\right),$$

where

$$x(t,a)=-\frac{1}{n}\left(\frac{n+1}{n}t^n+\frac{1}{n-1}t^{n-1}+\cdots+\frac{1}{2}t^2+t+\log|t-1|\right)+a.$$

Example 4.4. Let $F(x,y,p,q)=-y+xp-(1/2)x^2q+x^n=0$, where n is a natural number. In this case, $F_X=nx^{n-1}$ and $F_q=-(1/2)x^2$. Hence $F=0$ is of type R for $n=1$ and 2 at $z_0\in F^{-1}(0)$. Also $F=0$ is both types of C and R for $n=3$, and of type C for $n\geq 4$ at z_0.

First suppose that $n=1$. Since $F_X=1$, we have $\Sigma_c=\emptyset$. It follows that $F=0$ is of type R_X at z_0. The complete solution on $F^{-1}(0)$ at 0 is given by

$$\Gamma(t,r,s)=\left(\frac{2r}{1-rt},\frac{4r}{1-rt}\log|1-rt|+\frac{4r+2rs}{1-rt}+\frac{2r}{(1-rt)^2},2\log|1-rt|+\frac{2}{1-rt}+s,t\right).$$

Second suppose that $n=2$. Since 0 is a regular value of $F_X|_{F^{-1}(0)}$ and

$$\Sigma_c=\{(x,y,p,q)\mid x=y=0\}=\Delta,\ \Sigma_{cc}=\emptyset,$$

$F=0$ is of type RR_y^1 at $z_0\in\Delta$. The complete solutions on $F^{-1}(0)$ and Σ_c are given by

$$\Gamma(t,r,s)=\left(re^{\frac{t}{4}},\frac{r^2}{2}te^{\frac{t}{2}}-3r^2e^{\frac{t}{2}}+rse^{\frac{t}{4}},rte^{\frac{t}{4}}-4re^{\frac{t}{4}}+s,t\right),$$

$\Phi(t,a)=(0,0,a,t)$. We can observe that $\Gamma|_{\Gamma^{-1}(\Sigma_c)}=\Phi$.

Finally suppose that $n\geq 3$. Since 0 is a singular value of $F_q|_{F^{-1}(0)}$ and $F_X|_{F^{-1}(0)}$, $F=0$ is of type SC_y at $z_0\in\Delta$. We have

$$\Sigma_c=\{(x,y,p,q)\mid x=y=0\}=\Delta,\ \Sigma_{cc}=\emptyset.$$

The complete solution on $F^{-1}(0)$ and the complete singular solution on Σ_c are given by

$$\Gamma(t,r,s)=\left(t,\frac{2}{(n-2)(n-1)}t^n+\frac{1}{2}rt^2+st,\frac{2n}{(n-2)(n-1)}t^{n-1}+rt+s,\frac{2n}{n-2}t^{n-2}+r\right),$$

$\Phi(t,a)=(0,0,a,t)$. Note that if $n=3$, then $F=0$ is also of type SR_y at z_0.

Example 4.5. Let $F(x,y,p,q)=xq+(a-x)p-by=0$ be the confluent hypergeometric equations (the degenerate hypergeometric equations), where $a,b\in\mathbb{R}$, see in [16]. The equation have the confluent hypergeometric function as a solution. However, we can decide by using the results whether the equation have a complete solution or not. This is a new viewpoint for the equation as far as we know.

Since we consider the regular equation, we may assume that $b\neq 0$. By

$$F_X=q(1+a-x)-p(1+b) \quad \text{and} \quad F_q=x,$$

$$\Sigma_c=\{(x,y,p,q)\mid x=0, ap-by=0, q(1+a)-p(1+b)=0\}.$$

If $z_0\notin\Sigma_c$, then there exist a complete solution at z_0 and also a unique geometric solution passing through z_0. If $z_0\in\Sigma_c$ and $a=-1, b=-1$, then $F_X=q\cdot F_q$, Σ_c is a 2-dimensional manifold and $\Sigma_{cc}=\emptyset$. It follows that $F=0$ is of type RC_y at z_0. By Theorem 2.2, there exist a complete solution on $F^{-1}(0)$ and a complete singular solution on Σ_c. The complete solution on $F^{-1}(0)$ and the complete singular solution on Σ_c are given by

$$\Gamma(t,r,s)=\left(t, re^t+(1+t)s, re^t+s, re^t\right),\ \Phi(t,a)=(0,a,a,t).$$

If $z_0\in\Sigma_c$ and $a=-1, b\neq -1$ (respectively, $a\neq -1$), then Σ_c is a 1-dimensional manifold. Hence $F=0$ is not completely integrable at z_0.

Appendix A. Completely integrable implicit first order ODEs

In this appendix, we quickly review known results for the theory of completely integrable implicit first order ODEs

$$F(x,y,p)=0,\ p=dy/dx.$$

For more detail, see [10, 11, 12, 13, 19]. Assume that 0 is a regular value of F. We say that $F=0$ is completely integrable at a point if there exists an immersive one-parameter family of geometric solutions on $F^{-1}(0)$ at the point. The contact singular set $\Sigma_c=\Sigma_c(F)$ is given by

$$\Sigma_c=\{z\in J^1(\mathbb{R},\mathbb{R})\mid F(z)=0, F_X(z)=0, F_p(z)=0\}.$$

Here $F_X=F_x+pF_y$. We say that an equation $F=0$ is of *(first order) Clairaut type* (for short, *type C*) at z_0 if there exists a function germ $\alpha:(F^{-1}(0),z_0)\to\mathbb{R}$ such that

$$F_X|_{F^{-1}(0)}=\alpha\cdot F_p|_{F^{-1}(0)},$$

and of *reduced type* (for short, *type R*) at z_0 if there exists a function germ $\beta:(F^{-1}(0),z_0)\to\mathbb{R}$ such that

$$F_p|_{F^{-1}(0)}=\beta\cdot F_X|_{F^{-1}(0)},$$

In [11], it has been shown the following results.

Theorem A.1. ([11]) *Let $F(x,y,p)=0$ be an implicit first order ODE at z_0. The following are equivalent:*

(1) *$F=0$ is completely integrable at z_0.*
(2) *$F=0$ is either of type C or of type R at z_0.*
(3) *$z_0\notin\Sigma_c$ or Σ_c is a 1-dimensional manifold around z_0.*

Moreover, if Σ_c is a 1-dimensional manifold around z_0, then Σ_c is a singular solution of $F=0$ passing through z_0.

Now suppose that $z_0 \in \Sigma_c$. Since $F = 0$ is regular, $F_y(z_0) \neq 0$. By the implicit function theorem, there exists a smooth function $f : U \to \mathbb{R}$, where U is an open set in $\mathbb{R}^2$, such that in a neighbourhood of z_0, $(x, y, p) \in F^{-1}(0)$ if and only if $-y + f(x, p) = 0$. Thus we may assume without loss of generality that $F(x, y, p) = -y + f(x, p) = 0$. It follows that z_0 is a regular point of either $F_p|_{F^{-1}(0)}$ or $F_X|_{F^{-1}(0)}$. Therefore, completely integrable implicit first order ODEs have four kinds of types (cf. [19]), see Table 2.

Conditions			Type	Name
$z_0 \notin \Sigma_c$	$F_p(z_0) \neq 0$		C	C_p
	$F_X(z_0) \neq 0$		R	R_X
$z_0 \in \Sigma_c$	$F_y(z_0) \neq 0$	z_0 is a regular point of $F_p\|_{F^{-1}(0)}$	C	RC_y
		z_0 is a regular point of $F_X\|_{F^{-1}(0)}$	R	RR_y

Table 2. A classification of types of completely integrable implicit first order ODEs at z_0.

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