ON THE CONNECTION BETWEEN FUNDAMENTAL GROUPS AND PENCILS WITH MULTIPLE FIBERS

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INTRODUCTION

The study of the topology of complex projective (or quasiprojective) smooth varieties depends strongly on the knowledge of the topology of the complement of hypersurfaces in a projective space. Considering a projection, any smooth projective variety is a covering of a projective space of the same dimension ramified along a hypersurface. These coverings are measured by (finite index subgroups of) the fundamental group of the complement of the hypersurface. Using Lefschetz-Zariski theory, if we take a *generic* plane section the fundamental group of the complement does not change. As a consequence, for fundamental group purposes, one can restrict their attention to the study of complements of curves in the projective plane, as stated in the foundational paper by O. Zariski [26].

The richness of coverings for a space depends on its fundamental group. This is why we are mostly interested in curves $C \subset \mathbb{P}^2$ whose $\pi_1(\mathbb{P}^2 \setminus C)$ is non-abelian. The first known example is probably the curve formed by three lines $C := L_1 \cup L_2 \cup L_3$ intersecting at one point P. There is an easy way to compute this fundamental group; the pencil of lines through P is parametrized by $\mathbb{P}^1$; this pencil induces an epimorphism of $\mathbb{P}^2 \setminus C$ onto $\mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$ (the punctures corresponding to the three lines). Moreover, this map is a locally trivial fibration (with fiber isomorphic to $\mathbb{C}$) and hence $\pi_1(\mathbb{P}^2 \setminus C) \cong \pi_1(\mathbb{P}^1 \setminus \{p_1, p_2, p_3\})$, which is a free group of two generators.

The first known examples of irreducible curves whose fundamental groups are known to be non-abelian appeared in [26]. The first one corresponds to a hexacuspidal sextic, with its six cusps on a conic; the equation of such a curve is of the form $f_2^3 + f_3^2 = 0$, where f_j is a homogeneous polynomial of degree j. Its fundamental group is $\mathbb{Z}/2 * \mathbb{Z}/3$; in §2 we will see the relation between this group and the pencil generated by $f_2^3 = 0$ and $f_3^2 = 0$. This kind of examples have been generalized by various authors replacing $(2, 3)$ by (p, q). In the same paper, Zariski found the irreducible curve with smallest possible degree having a non-abelian fundamental group: the tricuspidal quartic. This example and many others appearing in the literature are also connected with pencils.

The precise connection with pencils can be stated as follows: a pencil defines a dominant morphism to a quasi-projective curve, inducing an epimorphism at the level of fundamental groups. The multiplicities of the fibers of the pencil induce an orbifold structure on the quasi-projective group, and the map defines an epimorphism onto the orbifold fundamental group. When such an orbifold fundamental group is non-abelian, then the original fundamental group has a surjection onto a non-abelian group. Such surjections coming from dominant maps will be referred to as *geometric* surjections.

The tricuspidal quartic is the only irreducible curve of degree 4 with a non-abelian fundamental group. The degree-five case was studied by A. Degtyarev [9]; he found exactly two irreducible

quintics with non-abelian fundamental groups. One of them, also studied by the first author [2], has an infinite fundamental group. In §2, we will study its relationship with a pencil. The question whether or not all non-abelian fundamental groups have a *geometric* surjection onto an orbifold group naturally arises. A positive result in this direction is given in [8] for certain roots of the Alexander polynomial. In addition, all the examples studied, up to now, supported an affirmative answer to this question.

In this paper we will show an explicit example of a non-abelian fundamental group whose complement admits no geometric surjections. This curve is one of the quintics referred to in the previous paragraph, which will be called the *projective Degtyarev curve* throughout this text. As a brief description, the projective Degtyarev curve has exactly three singular points of type $\mathbb{A}_4$; its fundamental group is finite and non-abelian. In Proposition 4.4, we prove that this group admits no geometric surjections. Once the group is computed, the proof is rather straightforward; it depends on the orders of the group and its abelianization and on the properties of orbifold groups.

If we add a tangent line to one of the singular points of the projective Degtyarev curve, the complement of the union in $\mathbb{P}^2$ is the complement of an affine curve, which will be called the *affine Degtyarev curve*. This affine curve has an infinite non-abelian fundamental group and non-trivial characteristic varieties (see §1 for the definition). Extending results of Arapura and others, it is known that irreducible components of positive dimension (for the fundamental group of a quasiprojective variety) are obtained as pull-back of irreducible components of characteristic varieties of orbifolds. A natural question arises: Is it also true for irreducible components of dimension 0 (isolated points)? Plenty of computations supported a positive answer: most quasiprojective groups satisfy the property for irreducible components of any dimension (see §2 for examples). The main Theorem 4.5 of this paper shows that the fundamental group of the complement of the affine Degtyarev curve does not satisfy this property. This is the only known example, up to now.

The paper is organized as follows. In §1, the concepts of orbifold and characteristic varieties are recalled, also some orbifold groups are studied. In §2, we relate non-abelian fundamental groups of the complements of curves (which are known in the literature) with orbifold morphisms (via pencils of curves). In §3, we describe Degtyarev curves and, in order to obtain a presentation for their fundamental groups, we compute a special braid monodromy. The fundamental groups are obtained in §4, where also the main results of the paper are stated and proved. Finally, further properties of the affine Degtyarev curve are sketched in §5.

1. ORBIFOLD GROUPS AND CHARACTERISTIC VARIETIES

The fundamental groups of oriented Riemann surfaces have been extensively studied. The fundamental group of a compact Riemann surface of genus g is

$$\pi_g := \left\langle a_i, b_i, 1 \le i \le g \,\middle|\, \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \right\rangle.$$

If C is a surface with genus g and $k > 0$ punctures then its fundamental group is free of rank $2g + k - 1$. We are going to extend this family by considering *orbifold groups*.

In this paper, we will refer to an *orbifold* X_φ as an orbifold Riemann surface, that is, a quasiprojective Riemann surface X with a function $\varphi : X \to \mathbb{N}$ with value 1 outside a finite number of points. The finite set $M_\varphi = \{x \in X \mid \varphi(x) > 1\}$ will be called the *set of orbifold points* and $\varphi(x)$ is the *orbifold index* of $x \in M_\varphi$.

We may think that a neighborhood of a point $P \in X_\varphi$ such that $\varphi(P) = n$ is the quotient of a disk (centered at P) by a rotation of angle $\frac{2\pi}{n}$. We will consider that a loop around P is trivial if its lifting bounds a disk. Following this idea, we define orbifold fundamental groups.

Definition 1.1. For an orbifold X_φ, let $p_1, \ldots, p_n$ the points such that $m_j := \varphi(p_j) > 1$. Then, the *orbifold fundamental group* of X_φ is

$$\pi_1^{\mathrm{orb}}(X_\varphi) := \pi_1(X \setminus \{p_1, \ldots, p_n\}) / \langle \mu_j^{m_j} = 1 \rangle$$

where μ_j is a meridian of p_j. We denote X_φ by $X_{m_1, \ldots, m_n}$.

Example 1.2. If X is a compact surface of genus g and type $X_{m_1, \ldots, m_n}$, then

$$\pi_1^{\mathrm{orb}}(X_\varphi) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g, \mu_1, \ldots, \mu_n \,\middle|\, \prod_{i=1}^{g} [a_i, b_i] = \prod_{j=1}^{n} \mu_j, \; \mu_j^{m_j} = 1 \; (j = 1, \ldots, n) \right\rangle,$$

where products are supposed to respect the order. If X is not compact and $\pi_1(X)$ is free of rank r, then

$$\pi_1^{\mathrm{orb}}(X_\varphi) = \left\langle a_1, \ldots, a_r, \mu_1, \ldots, \mu_n \,\middle|\, \mu_j^{m_j} = 1 \; (j = 1, \ldots, n) \right\rangle.$$

Definition 1.3. A dominant algebraic morphism $\rho : Y \to X$ between an algebraic manifold Y and a Riemann surface X defines an *orbifold morphism* $Y \to X_\varphi$ if for all $p \in X$, the divisor $\rho^*(p)$ has multiplicity $\varphi(p)$, that is, $\rho^*(p) = \varphi(p)D$, where D is a (possibly non-reduced) divisor in Y.

Proposition 1.4. *Let $\rho : Y \to X$ define an orbifold morphism $Y \to X_\varphi$. Then ρ induces a homomorphism $\rho_* : \pi_1(Y) \to \pi_1^{\mathrm{orb}}(X_\varphi)$. Moreover, if the generic fiber is connected, then ρ_* is surjective.*

Proof. Let $M_\varphi := \{x \in X \mid \varphi(x) > 1\}$; we consider the restriction mapping $\tilde{\rho} := \rho_| : Y \setminus \rho^{-1}(M_\varphi) \to X \setminus M_\varphi$. This map induces a morphism $\tilde{\rho}_* : \pi_1(Y \setminus \rho^{-1}(M_\varphi)) \to \pi_1(X \setminus M_\varphi)$ fitting in the following commutative diagram:

$$\begin{array}{ccc} \pi_1(Y \setminus \rho^{-1}(M_\varphi)) & \xrightarrow{\tilde{\rho}_*} & \pi_1(X \setminus M_\varphi) \\ i_* \downarrow & & \downarrow j_* \\ \pi_1(Y) & \xrightarrow{\rho_*} & \pi_1(X). \end{array}$$

The vertical mappings are induced by the inclusions. They are both surjective; the kernel of j_* is generated by the meridians of the points in M_φ while the kernel of i_* is generated by the meridians of the irreducible components of $\rho^{-1}(M_\varphi)$, i.e., the components of the pull-back divisor $\rho^*(M_\varphi)$.

Let us consider an irreducible component D of $\rho^*(M_\varphi)$ such that $\rho(D) =: x \in M_\varphi$. Let $n := \varphi(x)$; note that the multiplicity m_D of D in $\rho^*(M_\varphi)$ is a multiple of n. We can interpret m_D as follows. If μ_D denotes a meridian of D, then there is a meridian μ_x of x such that $\tilde{\rho}_*(\mu_D) = (\mu_x)^{m_D}$. Following Definition 1.1, it is easily seen that $\tilde{\rho}_*$ factorizes through a morphism (also called ρ_*) $\pi_1(Y) \to \pi_1^{\mathrm{orb}}(X_\varphi)$.

The above argument also works if one replaces M_φ by a finite set $M \supseteq M_\varphi$. In particular, one can choose M to be the bifurcation locus of ρ, i.e., the mapping is a differentiable locally trivial fibration outside M. If the fiber is generically connected, the long exact homotopy sequence of this fibration implies the surjectivity of $\tilde{\rho}_*$ (for M). The result follows. $\square$

Definition 1.5. A fundamental group $G := \pi_1(Y)$ of an algebraic manifold is said to *posses a geometric surjection* if Y possesses an orbifold morphism $Y \to X_\varphi$ whose generic fiber is connected, and such that $\pi_1^{\mathrm{orb}}(X_\varphi)$ is non-abelian.

We recall the notion of characteristic varieties and its relationship with orbifolds. We focus our attention on the characteristic varieties of quasiprojective manifolds, though they can be defined in general and depend only on the fundamental group. Let X be a connected topological space X, having the homotopy type of a finite CW-complex, and let $G := \pi_1(X, x_0)$, $x_0 \in X$ which will be omitted if it is not necessary. Recall that the space of characters of G is

$$(1.1) \qquad H^1(X; \mathbb{C}^*) = \mathrm{Hom}(H_1(X; \mathbb{Z}), \mathbb{C}^*) = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) =: \mathbb{T}_G.$$

Remark 1.6. Since G is finitely generated, then it is also the case for $H_1(X; \mathbb{Z})$. Let $n := \mathrm{rk}\, H_1(X; \mathbb{Z})$ and Tors_G be the torsion subgroup of $H_1(X; \mathbb{Z})$. Then $\mathbb{T}_G$ is an abelian complex Lie group with $|\mathrm{Tors}_G|$ connected components (each one is isomorphic to $(\mathbb{C}^*)^n$) satisfying the following exact sequence:

$$1 \to \mathbb{T}_G^{\mathbf{1}} \to \mathbb{T}_G \to \mathrm{Tors}_G \to 1,$$

where $\mathbb{T}_G^{\mathbf{1}}$ is the connected component containing the trivial character $\mathbf{1}$.

For a character $\xi \in \mathbb{T}_G$, we can construct a local system of coefficients $\mathbb{C}_\xi$ over X.

Definition 1.7. The k-th *characteristic variety* of X is the subvariety of $\mathbb{T}_G$, defined by:

$$\mathcal{V}_k(X) = \{\xi \in \mathbb{T}_G \mid \dim H^1(X, \mathbb{C}_\xi) \geq k\},$$

where $H^1(X, \mathbb{C}_\xi)$ is the cohomology with coefficients in the local system ξ. In some cases we will use the notation $\mathcal{V}_k(G)$ since it is independent of X as far as $\pi_1(X) \cong G$. The definition also applies to orbifolds replacing π_1 by π_1^{orb}.

The following result is straightforward.

Proposition 1.8. *Let $\varphi : G \to H$ be a group epimorphism. Then φ^* induces injections $\mathbb{T}_H \cong \varphi^* \mathbb{T}_H \hookrightarrow \mathbb{T}_G$ and $\mathcal{V}_j(H) \cong \varphi^* \mathcal{V}_j(H) \hookrightarrow \mathcal{V}_j(G)$.*

Remark 1.9. Let us explain how to compute these invariants. For the sake of simplicity, the twisted homology, instead of the cohomology, will be computed. Let us consider a finite CW-complex homotopy equivalent to X; for the sake of simplicity the CW-complex will be denoted X. Let $\pi : \tilde{X} \to X$ be the maximal abelian covering. Note that $\tilde{X}$ inherits a CW-complex structure. The group of automorphisms of π is $H_1(X; \mathbb{Z})$. The action of this Abelian group endows the chain complex $C_*(\tilde{X}; \mathbb{C})$ with a module structure over the ring $\Lambda := \mathbb{Z}[H_1(X; \mathbb{Z})]$. The differentials of the complex are Λ-homomorphisms. Moreover, $C_*(\tilde{X}; \mathbb{C})$ is a free Λ-module of finite rank. If we fix a character ξ, $\mathbb{C}$ has a natural Λ-module structure which is denoted by $\mathbb{C}_\xi$ (as the local system of coefficients). The twisted homology of X is the homology of the $C_*(X; \mathbb{C})^\xi := C_*(\tilde{X}; \mathbb{C}) \otimes_\Lambda \mathbb{C}_\xi$. Following this interpretation, it is not difficult to prove that the characteristic varieties are algebraic subvarieties of $\mathbb{T}_G$, defined with integer equations.

This i-th jumping loci of $C_*(\tilde{X}; \mathbb{C})$ with respect to $\_ \otimes_\Lambda \mathbb{C}_\xi$ can also be viewed as the zero locus of the i-th Fitting ideal of $H_1(\tilde{X}; \mathbb{C})$ or, analogously, the support of the module $\wedge^i H_1(\tilde{X}; \mathbb{C})$ over the ring Λ (see [17]).

Following the theory developed by various authors (Beauville [6], Arapura [1], Simpson [22], Budur [7], Delzant [11], Dimca [13]), the structure of characteristic varieties for quasiprojective manifolds can be stated as follows.

Theorem 1.10 ([4])**.** *Let Σ be an irreducible component of $\mathcal{V}_k(G)$, $k \geq 1$. Then one of the two following statements holds:*

- *There exists a surjective orbifold morphism $\rho : X \to C_\varphi$ and an irreducible component Σ_1 of $\mathcal{V}_k(\pi_1^{orb}(C_\varphi))$ such that $\Sigma = \rho^*(\Sigma_1)$.*
- *Σ is an isolated torsion point.*

Remark 1.11. In general, both G and its characteristic varieties are difficult to compute. For the complement of hypersurfaces in a projective space, Libgober [17] gave an alternative way of computing most components of the characteristic varieties from algebraic properties of the hypersurface without computing G.

Remark 1.12. Characteristic varieties can also be understood from Alexander-invariant point of view. Following Theorem 1.10, characteristic varieties are determined by finite-degree abelian coverings.

We compute the invariants for some orbifold groups.

Proposition 1.13. *Let G be the orbifold group of $\mathbb{P}^1_{2,5,10}$. Then G is a semidirect product of the fundamental group of a compact surface of genus 2 and $\mathbb{Z}/10\mathbb{Z}$. The torus $\mathbb{T}_G$ is μ_{10}, the group of 10-th roots of unity, $\mathcal{V}_1(G)$ consists of the primitive 10-th roots of unity and $\mathcal{V}_2(G) = \emptyset$.*

Proof. Let us consider the short exact sequence associated with the abelianization map ($\mathbb{Z}/10 := \langle t \mid t^{10} = 1 \rangle$ is G/G'). This sequence corresponds to an orbifold morphism, which is a ramified cyclic covering of degree 10 of $\mathbb{P}^1$. The ramification points correspond with the orbifold points whose ramification indices equal the orbifold index. Using Riemann-Hurwitz one checks that the covering space is a compact Riemann surface of genus 2. Since the meridian of an orbifold point of index 10 is of order 10 in G, then the exact sequence splits and we have a semidirect-group structure.

In order to compute $\mathcal{V}_1(G)$ we follow the construction outlined in Remark 1.9, applied to the CW-complex associated with the presentation of G given by $\langle x, y \mid x^2 = y^5 = (xy)^{10} = 1 \rangle$. Let us denote p the unique 0-cell, x, y the 1-cells and A, B, C the 2-cells (corresponding to the relations in the given order). Let us fix a character $\xi \in \mathbb{T}_G$. It is clear that $\mathbf{1} \notin \mathcal{V}_1(G)$. We can assume that $\zeta := \xi(t) \neq 1$. The complex $C_*(X; \mathbb{C})^\xi$ is given by

$$0 \longrightarrow \mathbb{C}^3 \xrightarrow{\partial_2} \mathbb{C}^2 \xrightarrow{\partial_1} \mathbb{C} \longrightarrow 0.$$

The matrix for ∂_1 is $\begin{pmatrix} \zeta^5 - 1 & \zeta^2 - 1 \end{pmatrix}$. In particular, $\dim \ker \partial_1 = 1$. The matrix for ∂_2 equals

$$\begin{pmatrix} \zeta^5 + 1 & 0 & \dfrac{\zeta^{10} - 1}{\zeta - 1} \\ 0 & \zeta^8 - \zeta^6 + \zeta^4 - \zeta^2 + 1 & \zeta^5 \dfrac{\zeta^{10} - 1}{\zeta - 1} \end{pmatrix}$$

In order to have non-trivial homology, this matrix must vanish and this happens only when ζ is a primitive 10-th root of unity. $\square$

Proposition 1.14. *Let G be the orbifold group of $\mathbb{P}^1_{2,2,5,5}$. Then G is an extension of $\mathbb{Z}/10\mathbb{Z}$ by the fundamental group of a compact surface of genus 4. The torus $\mathbb{T}_G$ is μ_{10}, the group of 10-th roots of unity, and both $\mathcal{V}_1(G)$ and $\mathcal{V}_2(G)$ consist of the primitive 10-th roots of unity.*

Proof. The short exact sequence associated with the abelianization map ($G/G' = \mathbb{Z}/10$) corresponds to a covering of the orbifold as in the proof of Proposition 1.13, and using Riemann-Hurwitz one obtains that the covering space is a compact Riemann surface of genus 4.

We compute the characteristic varieties as in the proof of Proposition 1.13 for the presentation of G given by $\langle x, y, z \mid x^5 = y^5 = z^2 = (xyz)^2 = 1 \rangle$. Let us denote p the unique 0-cell, x, y, z the 1-cells and A, B, C, D the 2-cells (corresponding to the relations in the given order). Let us fix a character $\xi \in \mathbb{T}_G$. It is clear that $\mathbf{1} \notin \mathcal{V}_1(G)$. We can assume that $\zeta := \xi(t) \neq 1$. The complex $C_*(X; \mathbb{C})^\xi$ is given by

$$0 \longrightarrow \mathbb{C}^4 \xrightarrow{\partial_2} \mathbb{C}^3 \xrightarrow{\partial_1} \mathbb{C} \longrightarrow 0.$$

The matrix for ∂_1 is $\begin{pmatrix} \zeta^2 - 1 & \zeta^2 - 1 & \zeta^5 - 1 \end{pmatrix}$. In particular, $\dim \ker \partial_1 = 2$. The matrix for ∂_2 equals

$$\begin{pmatrix} \zeta^8 - \zeta^6 + \zeta^4 - \zeta^2 + 1 & 0 & 0 & \zeta^5 + 1 \\ 0 & \bar{\zeta}^8 - \bar{\zeta}^6 + \bar{\zeta}^4 - \bar{\zeta}^2 + 1 & 0 & \zeta(\zeta^5 + 1) \\ 0 & 0 & \zeta^5 + 1 & \zeta^5 + 1 \end{pmatrix}$$

In order to have non-trivial homology, this matrix must have rank less than 2 and this happens only when ζ is a primitive 10-th root of unity. Moreover, in that case, the matrix vanishes. $\square$

2. Examples

In this section, we will present a collection of examples of curves with non-abelian fundamental groups and geometric surjections and its relationship with characteristic varieties.

Remark 2.1. If $Y := \mathbb{P}^2 \setminus \mathcal{C}$ admits an orbifold morphism $Y \to X_\varphi$, then the non-singular compactification $\bar{X}$ of X is $\mathbb{P}^1$.

Remark 2.2. The easiest examples of curves with non-abelian fundamental groups and geometric surjections come from hyperplane (or line) arrangements. If a line arrangement $\mathcal{A}$ has a point P of multiplicity $k \geq 3$, then the pencil of lines through P defines a morphism $\rho : \mathbb{P}^2 \setminus \mathcal{A} \to X$, where X is a k-punctured projective line. We have an epimorphism $\rho_* : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}) \to \pi_1(X)$ and the latter is a free group of rank $k - 1$ (hence non abelian).

The following result is well known for specialists.

Proposition 2.3. *The following three assertions are equivalent:*

(1) *The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ is non abelian,*
(2) *The arrangement $\mathcal{A}$ has a point of multiplicity at least 3,*
(3) *The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ has a geometric surjection.*

Proof. By the remark above, it is obvious that (2) implies (1) and (3). Also, by definition, (3) implies (1). Hence it is enough to prove that (1) implies (2). Note that, if (2) does not hold, then $\mathcal{A}$ is an arrangement in general position. Either we choose a particular example (e.g. a real arrangement) and a braid monodromy argument implies immediately the abelianity or we use Hattori's topological description of arrangements of hyperplanes in general position [16]. It is also the starting point of *Zariski's proof* of *Zariski's conjecture* in [26] (we thank the referee for pointing this out to us). $\square$

The argument used in Remark 2.2 can be easily generalized when, instead of considering three (or more) incident lines, one considers three (or more) fibers of any pencil of curves in $\mathbb{P}^2$. Of course, any such example corresponds to curves with at least three irreducible components. The notion of orbifold allows for wider generalizations of this concept to curves with any number of irreducible components (for example to irreducible curves).

As stated in the Introduction, the first example of this kind is rather old, see [26]. Let us consider a conic C_2 of equation $f_2 = 0$ and a cubic C_3 of equation $f_3 = 0$. Let us assume that they do not have common components and they are not *multiple* lines. Let C be a curve of equation $f_2^3 - f_3^2$. Note that the mapping $\rho : \mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{[1 : 1]\}$ given by $[x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2]$ is well defined (all the base points of the pencil are in C) and surjective. This mapping induces an orbifold map onto a 1-punctured Riemann sphere with two *orbifold* points of multiplicities 2 and 3 (at $[0 : 1]$ and $[1 : 0]$ respectively). Thus according to Proposition 1.4, one obtains an epimorphism $\pi_1(\mathbb{P}^2 \setminus C)$ onto $\mathbb{Z}/2 * \mathbb{Z}/3$.

Proposition 2.4. *Let G be the orbifold fundamental group of $\mathbb{C}_{2,3}$. Then, $\mathbb{T}_G = \mu_6$, $\mathcal{V}_1(G)$ consists of the 6-th primitive roots of unity and $\mathcal{V}_2(G) = \emptyset$. In particular, any curve with equation $f_2^3 - f_3^2 = 0$ has non-trivial characteristic varieties.*

The proof of this Proposition follows easily from the above arguments.

Remark 2.5. For generic choices of f_2 and f_3 this epimorphism is in fact an isomorphism (this is actually the case originally considered by Zariski in [26]). However, this is not the case, for instance, when C is reducible (since $b_1(\mathbb{P}^2 \setminus C) > 1$). Even if C is irreducible one may not necessarily have an isomorphism for several reasons: either there are few non-generic fibers in the pencil (e.g., a sextic with six cusps and four ordinary nodes) or there are several pencils (a sextic with nine cusps).

These examples can be generalized if we replace $(2,3)$ by any coprimes (p,q), see Oka [21], Némethi [20] and Dimca [12]. In such cases, the fundamental group of a generic curve with equation $f_p^q + f_q^p = 0$ is $\mathbb{Z}/p * \mathbb{Z}/q$. Also Zariski [26] considered another interesting example where the target orbifold is compact.

Let us consider the tricuspidal quartic C_4 with equation $f_4 = 0$. It is not hard to prove that we can choose

$$(2.1) \qquad f_4 := x^2 y^2 + y^2 z^2 + x^2 z^2 - 2xyz(x + y + z).$$

and $\operatorname{Sing}(C_4) = \{[1:0:0], [0:1:0], [0:0:1]\}$. The curve C_4 is parametrized by

$$(2.2) \qquad [t:s] \mapsto [t^2 s^2 : (t-s)^2 s^2 : t^2(t-s)^2]$$

and its singular points correspond to $[t:s] = [0:1], [1:1]$, and $[1:0]$. Let $P \in C_4$ be a smooth point with parameter $\alpha \equiv [\alpha : 1]$ and let L_t be the tangent line to C_4 at P, with equation $f_1 = 0$, where

$$(2.3) \qquad f_1 := (\alpha - 1)^3 x - \alpha^3 y + z.$$

Let C_2 be the conic passing through the singular points of C_4 and tangent to C_4 at P. Since five (non-degenerate) conditions are imposed, such a conic is unique. As before, let $f_2 = 0$ be the equation of C_2, where

$$(2.4) \qquad f_2 := \alpha(\alpha - 1)xy - (\alpha - 1)xz + \alpha yz.$$

We consider now a cubic C_3 having a nodal point at P (one of the branches tangent to C_4 at P) and tangent to C_4 at the three cuspidal points. Counting the conditions it is easy to prove that only one such cubic exists, with equation $f_3 = 0$, where

$$(2.5)$$
$$f_3 := -(\alpha - 2)(2\alpha - 1)(\alpha + 1)xyz - \alpha^3 xy^2 - xz^2 - (\alpha - 1)^3 x^2 y + yz^2 + (\alpha - 1)^3 x^2 z + \alpha^3 y^2 z.$$

Lemma 2.6. $f_4 f_1^2 = f_3^2 - 4f_2^3.$

A straightforward computation provides a proof of this Lemma, which easily results in the following:

Proposition 2.7. *The fundamental group of $\mathbb{P}^2 \setminus C_4$ possesses a geometric surjection onto $\mathbb{P}^1_{2,2,3}$.*

Remark 2.8. Zariski proved in [26] that $\pi_1(\mathbb{P}^2 \setminus C_4)$ is a non-abelian group of order 12. The above mapping induces a central extension of $\mathbb{D}_6$ (dihedral group of order 6) whose kernel is cyclic of order 2. Note that there is an epimorphism from $\pi_1(\mathbb{P}^2 \setminus (C_4 \cup L_t))$ onto the orbifold group of a 1-punctured Riemann sphere with two multiple points $(2,3)$. For a generic P it is possible to prove that $\pi_1(\mathbb{P}^2 \setminus (C_4 \cup L))$ equals $\mathbb{B}_3$. There is a particular case corresponding to the bitangent line L_b. In this case there are two such mappings and $\pi_1(\mathbb{P}^2 \setminus (C_4 \cup L_b))$ is the Tits-Artin group of a triangle.
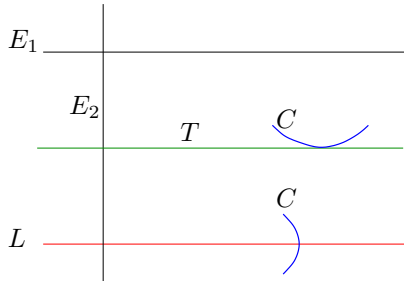
FIGURE 1. Cremona transformation

In [9], Degtyarev proved that only two irreducible curves of degree 5 have non-abelian funda-
mental groups. One of them is extensively studied in §3. The other one was also studied by the
first author in [2]. It is a rigid curve with one point of type $\mathbb{A}_6$ and three cuspidal points (it is
the dual curve of the quartic with one $\mathbb{A}_6$). Let C_5 be this curve (with equation $f_5 = 0$). In [2]
this group was shown to be non-abelian by finding an epimorphism from an actual presentation
of $\pi_1(\mathbb{P}^2 \setminus C_5)$ onto the triangle group of type 2, 3, 7, which is the orbifold group of $\mathbb{P}^1_\varphi$ with three
multiple points of these orders. In fact, one has the following:

Proposition 2.9. *The fundamental group of $\mathbb{P}^2 \setminus C_5$ possesses a geometric surjection onto $\mathbb{P}^1_{2,3,7}$.*

Proof. The three summands of Lemma 2.6, which are polynomials of degree six, obviously belong
to a pencil of sextics and, hence, they define a map outside the base points. For a particular
parameter (a primitive 6-th root of unity, $L_t = L_b$ is the bitangent of C_4. We are going to
consider the Cremona transformation $\rho : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ associated with the net of conics having
three infinitely near points in common with C_5 at P, the singular point of type $\mathbb{A}_6$. Let us
describe this transformation. After blowing up these three infinitely near points one obtains
a rational surface X with a morphism $\sigma_1 : X \to \mathbb{P}^2$. Let us denote the three exceptional
components (in order of appearance) by E_1, E_2, and T, and finally the tangent line of C at P
by L (see Figure 1).

Convention 2.10. For birational morphisms, we keep the notation of a curve for its strict
transform unless otherwise stated.

In X one has $E_1 \cdot E_1 = -2$ and $E_1 \cdot E_1 = T \cdot T = L \cdot L = -1$. Since L and T are *combinatorially
equivalent*, one can consider the birational morphism $\sigma_2 : X \to \mathbb{P}^2$ obtained as the composition
of the contractions of L, E_2 and E_1. The resulting surface is rational with Euler characteristic 3
and hence it is a projective plane. It is not hard to prove that $\rho = \sigma_2^{-1} \circ \sigma_1$. Let us denote
$\tilde{C} := \rho(C)$. Note that $\tilde{C}$ is a tricuspidal quartic and T is its unique bitangent line L_b, one point
$\tilde{P}$ comes from the infinitely near point of C at P and the other one Q comes from the other
intersection point of C and L.

We consider the pencil defined by the orbifold map of Proposition 2.7, where the base point
is $\tilde{P}$. Let C_3 be the cubic of equation (2.4) such that $2C_3$ is in the pencil. Following C_3 by σ_2
and σ_1, $C_6 := \rho^*(C_3)$ is a sextic with only one singular point at P (with two branches, one of
type $\mathbb{A}_6$ and a smooth branch with maximal contact with the singular branch). With the same
ideas, if C_2 is the conic of equation (2.5) such that $3C_2$ is in the pencil, then $C_4 := \rho^*(C_2)$ is a
quartic with an $\mathbb{A}_6$ singular point at P.

Finally $\rho^*(\tilde{C} + 2T) = C + 7L$. We have a pencil of degree 12 containing the fibers $2C_6$, $3C_4$
and $C + 7L$. This pencil produces the desired morphism. □

One can find more examples in the literature: Degtyarev [9], Flenner-Zaĭdenberg [14], and Tono [24]. In what follows, the last two families will be described. We start with some definitions.

Definition 2.11. A *Hirzebruch* surface is a rational surface X with a morphism $\pi : X \to \mathbb{P}^1$ which is a holomorphic (or algebraic) fibration with fiber $\mathbb{P}^1$. Such a surface is either $\Sigma_0 := \mathbb{P}^1 \times \mathbb{P}^1$ or it has a unique section S_n with negative self-intersection $-n$, $n > 0$; in that case π is unique and X is denoted by Σ_n (Σ_1 is the blowing-up of one point in $\mathbb{P}^2$).

For any Hirzebruch surface X there is a family of birational maps which are called *elementary Nagata transformations*. They are obtained as follows. Let us consider $\pi : X \to \mathbb{P}^1$, $P \in X$ and $F := \pi^{-1}(\pi(P))$; we consider the blowing-up $\sigma : \hat{X} \to X$ of P, with exceptional component $\tilde{F}$. Since $(F \cdot F)_X = 0$, we have that $(F \cdot F)_{\hat{X}} = -1$. By Castelnuovo criterion, we can blow down F and we obtain a new Hirzebruch surface $\tilde{X}$ where $\tilde{F}$ is a fiber.

Definition 2.12. An elementary Nagata transformation is said to be *positive* (resp. *negative*) if P belongs (resp. does not belong) to a section with non-positive self-intersection. For a positive one, one goes from Σ_n to Σ_{n+1}; for a negative one, from Σ_n to Σ_{n-1}.

In [3], the first author computed the fundamental group of Flenner-Zaĭdenberg curves and showed when it is non-abelian using orbifold groups. We show here that this can also be geometrically proved. In order to construct these curves, we start with a smooth conic C with two tangent lines L_1 and L_2, intersecting at some point P. After blowing up P one obtains $\pi : \Sigma_1 \to \mathbb{P}^1$ with exceptional component E. Let L_3 be another line in the pencil through P which intersects C at two points Q_1 and Q_2. Let us fix two positive integers a, b. After performing a positive elementary Nagata transformations at the point corresponding to the fiber of L_1 and b at the point corresponding to the fiber of L_2 one obtains a Hirzebruch surface Σ_{a+b+1}. One can then perform $a + b$ negative elementary Nagata transformations on the fiber corresponding to L_3 and based at a point in C (say Q_2 for the first one). After this process, E can be blown down which turns our surface into $\mathbb{P}^2$. The curve $C_{a,b}$ obtained has degree $d := a + b + 2$ and three singular points of type $\mathbb{A}_{2a}$, $\mathbb{A}_{2b}$, and a third one with local equation $u^{d-2} = v^{d-1}$.

Proposition 2.13. *The fundamental group of $\mathbb{P}^2 \setminus C_{a,b}$ possesses a geometric surjection onto $\mathbb{P}^1_{2, a+b, c}$, where $c := \gcd(2a + 1, 2b + 1)$.*

Proof. It is enough to follow the pencil of conics generated by $L_1 + L_2$ and C through the above transformations. We obtain a pencil of curves of degree $2(d - 1)$, where one fiber is $(2a + 1)\tilde{L}_1 + (2b + 1)\tilde{L}_2$ (they are the lines corresponding to the fibers of L_1 and L_2). The fiber containing $C_{a,b}$ is of the form $C_{a,b} + (d - 2)\tilde{L}_3$. Finally the double line in the pencil becomes a double curve of degree $d - 1$. $\qquad\square$

In [24], K. Tono describes all rational unicuspidal curves such that its complement in $\mathbb{P}^2$ has logarithmic Kodaira dimension 1. The construction given in [24, Theorem 1] shows that the complement of these curves have non-abelian fundamental group. Any other known rational unicuspidal curve has abelian fundamental group (for the complement).

Proposition 2.14. *For any Tono's curve C their fundamental group possesses a geometric surjection onto $\mathbb{P}^1_{\mu_A, \mu_G, n(C)}$, where $\mu_A, \mu_G \geq 2$ and the number $n(C)$ is the opposite of the self-intersection of the strict transform of C after the minimal embedded resolution of its unique singular point. This number is at least 2.*

Proof. It is enough to consider the construction of [24, Theorem 1] where a pencil is obtained with two multiple fibers $\mu_A A$ and $\mu_G G$ and a reducible fiber of the form $C + n(C)B$, where B is either a line (type I) or a smooth conic (type II). $\qquad\square$

Example 2.15. The curves of type I are parametrized by two integers $n, s \geq 2$. The curve C has degree $(n+1)^2(s-1)+1$, where $n(C) = n$, $\mu_A = n+1$ and $\mu_G = (n+1)(s-1)+1$. For $n = s = 2$, we obtain the multiplicities $2, 3, 4$; in fact, one can compute that this group is finite.

3. Degtyarev curves

Let us consider a projective Degtyarev curve, i.e., a plane projective curve of degree 5 such that $\mathrm{Sing}(C)$ consists of three points, and for each point $P \in \mathrm{Sing}(C)$ the germ (C, P) is topologically equivalent to an $\mathbb{A}_4$-singularity, i.e. with local equation $v^2 - u^5 = 0$; note that in this case, the germs are also analytically equivalent.

Most of the following properties appear in [9] and [19], but we include for the sake of completeness.

Properties 3.1. *Let* $C \subset \mathbb{P}^2$ *be a projective Degtyarev curve. Then:*

(D1) *The curve C is irreducible.*

(D2) *The tangent line L of C at a singular point P satisfies $(L \cdot C)_P = 4$.*

(D3) *Two Degtyarev projective curves are projectively equivalent.*

(D4) *The subgroup of projective transformations preserving C is cyclic of order 3.*

(D5) *The curve C is autodual.*

Proof. Since the three singular points are locally irreducible, (D1) is true. For (D2), note that $4 \leq (L \cdot C)_P \leq 5$. Let us assume that $(L \cdot C)_P = 5$; considering L as the line at infinity, $C \setminus L$ is an affine curve homeomorphic to $\mathbb{C}$. This case is discarded using Zaĭdenberg-Lin Theorem [25] and (D2) results.

In order to prove (D3), there are two approaches. The direct approach consists of computing the equations of the curve C fixing the position of the singular points and some of their tangent lines. The second method is quite simple and worth describing here: Let C_1, C_2 be two projective Degtyarev curves. By Bézout's Theorem, the singular points are not aligned; and hence, after a projective transformation, one may assume that $\mathrm{Sing}(C_1) = \mathrm{Sing}(C_2) =: S$. Assuming that $S := \{[1:0:0], [0:1:0], [0:0:1]\}$, one can perform a standard Cremona transformation $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ based on the three singular points and defined by $\psi([x:y:z]) = [yz:xz:xy]$.

Geometrically, this rational map is obtained by blowing-up the three vertices of S (obtaining a rational surface X_ℓ) and then blowing down the strict transforms of the lines joining the points of S (which have self-intersection -1 in X_ℓ). One can easily compute that $\tilde{C}_i := \psi(C_i)$ is a tricuspidal quartic. It is well known that there is only one tricuspidal quartic, up to projective transformation, therefore, after a suitable change of coordinates, one may assume $\tilde{C}_1 = \tilde{C}_2 =: \tilde{C}$, where $\tilde{C}$ is the curve with equation given in (2.1). The tricuspidal quartic satisfies the following properties. Let $\mathrm{Sing}(\tilde{C}) = \{P_1, P_2, P_3\}$; there are three points $Q_1^\ell, Q_2^\ell, Q_3^\ell \in \tilde{C}$, $\ell = 1, 2$ such that P_i, Q_j^ℓ, Q_k^ℓ are aligned for all the possibilities with $\#\{i, j, k\} = 3$. Let $\mathcal{A}_\ell$ be the arrangements of curves given by $\tilde{C}$ and the lines joining Q_i^ℓ and Q_j^ℓ.

The curve $\tilde{C}$ is parametrized as in (2.2) and the singular points $P_1 = [0:1:0]$, $P_2 = [1:0:0]$, and $P_3 = [0:0:1]$ correspond to $[t:s] = [0:1], [1:1]$, and $[1:0]$. It is not hard to check that $A_\ell := (\alpha_\ell, 2 + \alpha_\ell, -\alpha_\ell)$ are affine parameters of $(Q_1^\ell, Q_2^\ell, Q_3^\ell)$. The last condition implies that $\alpha_\ell^2 + \alpha_\ell - 1 = 0$. If $\alpha_1 = \alpha_2$ then $\mathcal{A}_1 = \mathcal{A}_2$.

The group of projective transformations fixing $\tilde{C}$ is the group of the permutation of the coordinates. The mapping $[x:y:z] \overset{\sigma}{\mapsto} [x:z:y]$ induces $[t:s] \mapsto [s:t]$ in the parametrization, and $[x:y:z] \overset{\tau}{\mapsto} [y:z:x]$ induces $[t:s] \mapsto [s:s-t]$.

Let us assume that $\alpha_1 \neq \alpha_2$ Applying the projective transformation σ, results in two operations on A_ℓ: the permutation $(1, 3)$, and the change of parameters. Thus, $\sigma(A_1) = (-\alpha_1^{-1}, (\alpha_1 + 2)^{-1}, \alpha_1^{-1}) = A_2$, which implies $\sigma(\mathcal{A}_1) = \mathcal{A}_2$.

Note that any projective transformation sending $\mathcal{A}_1$ to $\mathcal{A}_2$ lifts to an isomorphism $X_1 \to X_2$ and this isomorphism induces a projective transformation of the source $\mathbb{P}^2$, hence (D3) results.

In order to prove (D4) one can use a similar argument on the projective transformations fixing C (this last property was communicated to the authors by C.T.C. Wall).

The property (D5) follows from (D2) and Plücker generalized formulæ, see [19]. More precisely, given a curve D and a point $P \in D$, the order of the curve is the degree of its dual curve of $\check{D}$:

$$\deg(\check{D}) = \deg(D)(\deg(D) - 1) - \sum_{P \in D} (\mu(C, P) - 1 + m(C, P)).$$

This formula implies that $\deg(\check{D}) = 5$. The dual of a singular point of type $\mathbb{A}_4$ is either of the same type or of type $\mathbb{E}_8$ (in case the tangent line has multiplicity of intersection 5 with the curve at the singular point). Thus (D5) holds. $\qquad\square$

Remark 3.2. Note that any two projective Degtyarev curves are isotopic. Using the direct approach, we can give a symmetric equation:

$$\left(7 + 3\sqrt{5}\right)(x^3z^2 + x^2y^3 + y^2z^3) + \left(2\sqrt{5} + 6\right)(x^3yz + xy^3z + xyz^3) +$$
$$+ 2(x^3y^2 + x^2z^3 + y^3z^2) + \left(33 + 11\sqrt{5}\right)(x^2yz^2 + x^2y^2z + xy^2z^2) = 0.$$

Note that the permutation of two variables comes from the Galois transformation in $\mathbb{Q}(\sqrt{5})$. The curve also admits an equation with rational coefficients; in that case one of the singular points has rational coordinates but the other two are conjugate in $\mathbb{Q}(\sqrt{5})$:

(3.1)
$$z^2y^3 - z(33xz + 2x^2 + 8z^2)y^2 + (21z^2 + 21xz - x^2)(z^2 + 11xz - x^2)y + (x - 18z)(z^2 + 11xz - x^2)^2 = 0$$

Properties 3.1 imply that the affine Degtyarev curve is also *rigid*, i.e. any two affine Degtyarev curves are projectively equivalent, and in particular, they are isotopic. In order to study its complement, it is convenient to assume that the line corresponds to the line at infinity and hence it is enough to consider the complement of the affine curve whose equation is obtained from (3.1) by taking $z = 1$.

The fundamental group of the projective Degtyarev curve was computed in [9]. Here we will compute the fundamental group of the affine curve and also show how to recover the group of the projective curve. In order to compute the group we will use the braid monodromy associated with the projection $(x, y) \mapsto x$. Note that the discriminant of the equation (3.1) (with $z = 1$) is (up to a constant) $x(x^2 - 11x - 1)^5$. Since the three roots are real and the projection is $3 : 1$ with *enough* real roots, the real picture in Figure 2 contains all the required information to obtain the braid monodromy (the dotted lines represent the real part of the complex conjugate roots).

The braid monodromy is defined as a representation $\nabla_0 : \pi_1(\mathbb{C} \setminus \{0, a_+, a_-\}; x_0) \to \mathbb{B}_3$. The source is a free group of rank three generated by:

$$\mu_+ := \alpha_+ \cdot \beta_+ \cdot \gamma_+ \cdot \alpha_+^{-1}, \quad \mu_0 := \alpha_+ \cdot \beta_+ \cdot \alpha_0 \cdot \beta_0 \cdot \gamma_0 \cdot \alpha_0^{-1} \cdot \beta_+^{-1} \cdot \alpha_+^{-1} \text{ and}$$

$$\mu_- := \alpha_+ \cdot \beta_+ \cdot \alpha_0 \cdot \beta_0 \cdot \alpha_- \cdot \beta_- \cdot \gamma_- \cdot \alpha_-^{-1} \cdot \beta_0^{-1} \cdot \alpha_0^{-1} \cdot \beta_+^{-1} \cdot \alpha_+^{-1}.$$

Figure 3 shows a geometric basis of $\pi_1(\mathbb{C} \setminus \{0, a_+, a_-\}; x_0)$. The braids are obtained by considering the way the roots with respect to y move when the parameters move along x. We follow these conventions:

($\mathbb{B}1$) In order to draw the braids we consider the projection onto the real axis.
($\mathbb{B}2$) When two points have the same real part, we perturb the projection such that positive imaginary parts go to the right and negative imaginary parts go to the left.
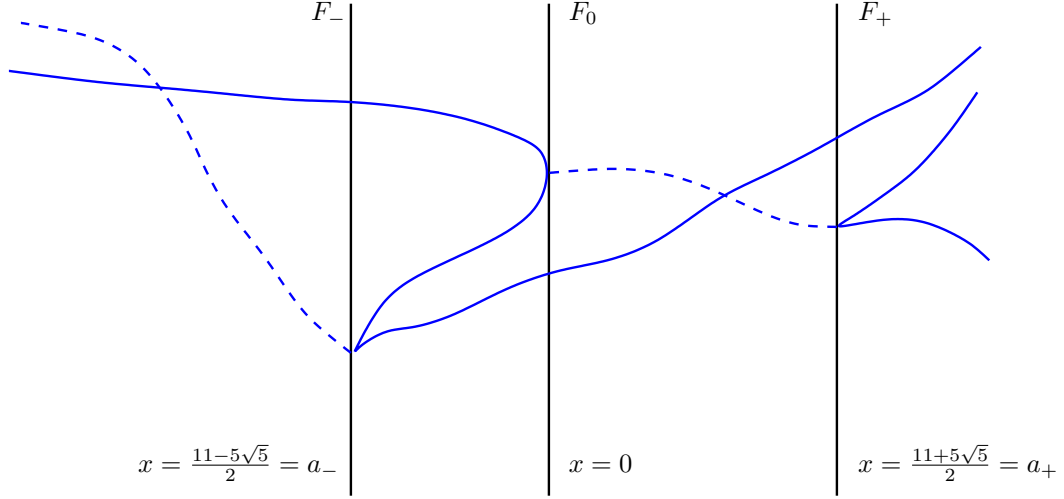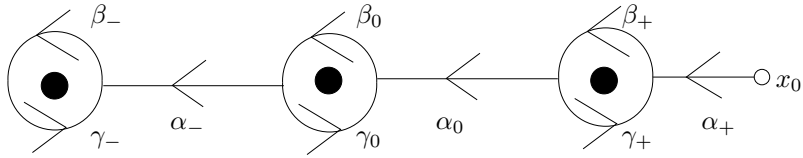($\mathbb{B}3$) Roots will be numbered from right to left.

FIGURE 2. Real picture of the affine Degtyarev curve

FIGURE 3. Paths in $\mathbb{C} \setminus \{0, a_+, a_-\}$

Paths	Braids
α_+	1
β_+	σ_2^2
γ_+	σ_2^3
α_0	$\sigma_1^{-1}\sigma_2$
β_0	1
γ_0	σ_1
α_-	1
β_-	σ_2^2
γ_-	σ_2^3

TABLE 1. Braids

($\mathbb{B}4$) The above conventions give a canonical way to identify open braids with closed braids.

Using the standard Artin generators of the braid groups, the braids obtained from following the paths in $\mathbb{C} \setminus \{0, a_+, a_-\}$ shown in Figure 3 are presented in Table 1.

Proposition 3.3. *The braid monodromy for the chosen projection of the affine Degtyarev curve is given by:*

$$\nabla_0(\mu_+) = \sigma_2^5, \quad \nabla_0(\mu_0) = (\sigma_2^2\sigma_1^{-1}\sigma_2) * \sigma_1, \quad \nabla_0(\mu_-) = (\sigma_2^2\sigma_1^{-1}\sigma_2\sigma_1) * \sigma_2^5 = \sigma_2^2 * \sigma_1^5,$$

where $a * b := aba^{-1}$.

4. Groups of Degtyarev curves

In order to compute the fundamental groups we apply the Zariski-van Kampen method. Let us consider the vertical line F of equation $x = x_0$. The set $F \setminus C$ is of the form $\{x_0\} \times \mathbb{C} \setminus \{y_1, y_2, y_3\}$, where $y_1, y_2, y_3 \in \mathbb{R}$. We choose a *big* real number y_0 in order to fix $(x_0, y_0) =: p_0$ as the base point. The free group $\pi_1(F \setminus C; p_0)$ has a free basis g_1, g_2, g_3 constructed as in Figure 3. The natural action of $\mathbb{B}_3$ on the free group $\mathbb{F}_3$ is expressed in this case as

$$(4.1) \qquad g_i^{\sigma_j} := \begin{cases} g_{i+1} & \text{if } i = j, \\ g_{i+1} * g_i & \text{if } i = j+1, \\ g_i & \text{if } i \neq j, j+1. \end{cases}$$

Proposition 4.1. *The fundamental group of the affine Degtyarev curve has a presentation*

$$(4.2) \qquad \left\langle g_1, g_2, g_3 \;\middle|\; g_i^{\nabla_0(\mu_j)} = g_i, i = 1, 2, 3, j = -, 0, + \right\rangle.$$

In this presentation, a meridian of the line at infinity is (up to conjugation) $\left(g_3(g_2 g_1)^2 \right)^{-1}$. In particular, a presentation for the projective Degtyarev curve is

$$(4.3) \qquad \left\langle g_1, g_2, g_3 \;\middle|\; (4.2), \quad g_3 = (g_2 g_1)^{-2} \right\rangle.$$

Proof. The first presentation is a consequence of the Zariski-van Kampen method by means of the braid monodromy. In order to prove the second one may consider a small deformation of the vertical line F. It will intersect the curve at five points. Three of them are close to (x_0, y_i), $i = 1, 2, 3$, and the other two ones lie in the *real* branches which go *faster* to infinity. The boundary of a big disk in this line is the inverse of a meridian of the line at infinity. $\qquad \square$

Remark 4.2. Proposition 4.1 provides right presentations of the group, but they may be quite cumbersome to work with by hand. Even if one wants to work with them with computer programs, like `GAP`[15], the presentations could be intractable. There are several ways around this problem

(P1) The presentation (4.2) works if we replace the braid monodromy ∇_0 for a conjugate. For example, conjugating the braids in Proposition 3.3 by σ_2^2 produces simpler braids and hence a simpler presentation of the group.

(P2) Instead of finding a good braid to perform the conjugation in (P1) by inspection, one can try to interpret this conjugation in a geometric way. Changing the base point in $\mathbb{C} \setminus \{0, a_+, a_-\}$ might produce simpler braids. For example choosing a real number $\tilde{y}_0 \in (a_-, 0)$ as a base point, one obtains the following as braid monodromy (for the new generators of the group):

$$(4.4) \qquad \tilde{\mu}_+ \mapsto (\sigma_2^{-1}\sigma_1) * \sigma_2^5, \quad \tilde{\mu}_0 \mapsto \sigma_1, \quad \tilde{\mu}_- \mapsto \sigma_2^5.$$

These braids have been obtained by conjugation of the ones in Proposition 3.3 by $\sigma_2^2 \sigma_1 \sigma_2^{-1}$.

(P3) If g is a meridian of the line at infinity obtained using a braid monodromy ∇_0, then, for a braid monodromy $(\nabla_0)^\tau := \tau^{-1} \nabla_0 \tau = (\tau^{-1}) * \nabla_0$, a meridian of the line at infinity is g^τ.

(P4) There is another geometric way to reduce the presentation. Note that among the relations $(g_j)^{\sigma_2^5} = g_j$, $j = 1, 2, 3$, one only needs to keep the relation given by $j = 2$. First of all, the relation for $j = 1$ is trivial; secondly $(g_3 g_2)^\tau = g_3 g_2$ and hence one of them is redundant. In the general case, this can be summarized as follows:

- Let us consider the action (4.1) (replacing 3 by n) of $\mathbb{B}_n$ on the free group with basis $g_1, \ldots, g_n$; let us consider a braid $\tau \in \mathbb{B}_n$ which can be decomposed as $\tau = \tau_1 \cdots \tau_r$, where τ_j involves only a set of n_j consecutive strings and $n = \sum_{j=1}^{r} n_j$. Then, among the relations $g_j^\tau = g_j$, we only need to consider $s := \sum_{j=1}^{r}(n_j - 1) = n - r$, disregarding one for each block of strings. Let J_τ be the chosen subset of indices.
- If $\beta = (\tau)^\sigma$, and τ can be decomposed as above, then the set of relations $g_j^\beta = g_j$, $j = 1, \ldots, n$, is equivalent to $(g_j^\sigma)^\tau = g_j^\sigma$, $j \in J$.

For example, in our case the presentation (4.2) can be reduced to have 3 relators.

Proposition 4.3. *The group G of the affine Degtyarev curve has a presentation:*

$$(4.5) \qquad \left\langle x, y \,\middle|\, xyxyx = yxyxy, [x, yxy^{-1}xyxy^{-1}xy] = 1 \right\rangle$$

A presentation of the group $G_\mathbb{P}$ of the projective Degtyarev curve is obtained from (4.5) by adding $x^5 = 1$. It turns out that $G_\mathbb{P}$ is a group of order 320 with the following properties:

- ($G_\mathbb{P}1$) $G_\mathbb{P}/G_\mathbb{P}'$ *is cyclic of order 5.*
- ($G_\mathbb{P}2$) *The center $Z(G_\mathbb{P})$ is the Klein group of order 4.*
- ($G_\mathbb{P}3$) *The group $G/Z(G_\mathbb{P})$ is a semidirect product of $(\mathbb{Z}/2)^4$ by $\mathbb{Z}^5$, where the action of a generator of $\mathbb{Z}^5$ cyclically permutes a generator system $h_1, \ldots, h_5$ of order 2 elements of $(\mathbb{Z}/2)^4$ satisfying $\sum h_i \equiv 0$.*

Proof. The presentation of G is obtained using the braid monodromy 4.4 and Remark 4.2(P4), where $x = g_1, g_2$ and $y = g_3$; note that x and y are conjugate. In order to obtain the presentation of $G_\mathbb{P}$ the relation of the line at infinity needs to be added. This is a *complicated* product of five conjugates of x. If one types this presentation in GAP, the output is that $G_\mathbb{P}$ has order 320 and that x is an element of order 5. Also according to GAP, the order of the quotient of G obtained by adding the relation $x^5 = 1$ is 320. These facts give the presentation of the statement. The properties of $G_\mathbb{P}$ are either trivial or easily computed using GAP. $\qquad \square$

Proposition 4.4. *The group $G_\mathbb{P}$ possesses no geometric surjections.*

Proof. The only properties needed for this are the size of both the group $G_\mathbb{P}$ and its abelianization. Let us assume that $G_\mathbb{P}$ possesses a geometric surjection. Since it is finite, the orbifold group must be finite. The only orbifolds having a finite non-abelian fundamental group are those of type $\mathbb{P}^1_{a,b,c}$, with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$ (the so-called *spherical* orbifolds): either $\mathbb{P}^1_{2,2,n}$, $n \geq 3$, or $\mathbb{P}^1_{2,3,m}$, $m = 3, 4, 5$. Since the order of the orbifold group must divide 320, the only possibilities are $(2, 2, n)$, where $n|160$. The group is dihedral and its abelianization is either $\mathbb{Z}/2$ or $(\mathbb{Z}/2)^2$. Since the abelianization of $G_\mathbb{P}$ is of order 5, the result follows. $\qquad \square$

We finish this section with the main result of this paper. We are going to compute the characteristic varieties of the complement of the affine Degtyarev curve and we will prove that these components cannot come from the characteristic varieties of an orbifold.

Theorem 4.5. *Let $\mathbb{T}_G = \mathbb{C}^*$ be the character torus of G. Then $\mathcal{V}_1(G)$ is the set containing 1 and the 10-th primitive roots of unity, whereas $\mathcal{V}_2(G) = \emptyset$. Therefore there is no geometric surjection of G onto an infinite orbifold group.*

Since finite group orbifolds do not have characteristic varieties, the following Corollary holds.

Corollary 4.6. *No irreducible component of $\mathcal{V}_1(G)$ is obtained as the pull-back of an irreducible component of the $\mathcal{V}_1(\Gamma)$ where Γ is an orbifold group.*

Proof of Theorem 4.5. We are going to change the presentation (4.5), by taking a new generator t satisfying $y = xt$:

$$(4.6) \qquad \left\langle x, t \mid xtx^2tx = tx^2tx^2t, [x, txt^{-1}xtxt^{-1}xt] = 1 \right\rangle$$

It is clear that $1 \in \mathcal{V}_1(G) \setminus \mathcal{V}_2(G)$ since the non-twisted homology has rank 1. Let us consider a non-trivial character $\xi \in \mathbb{T}_G$, which is identified by the image $1 \neq \zeta$ of a positive generator of $\mathbb{Z}$. One can associate a CW-complex with the presentation (4.6) with one 0-cell p, two 1-cells x, t and two 2-cells A, B (corresponding to the relations). Then, the complex $C_*(X; \mathbb{C})^\xi$ with which to compute the twisted homology is

$$0 \longrightarrow \mathbb{C}^2 \xrightarrow{\partial_2} \mathbb{C}^2 \xrightarrow{\partial_1} \mathbb{C} \longrightarrow 0.$$

The matrix for ∂_1 is $\begin{pmatrix} \zeta - 1 & 0 \end{pmatrix}$. In particular, $\dim \ker \partial_1 = 1$ and hence $\mathcal{V}_2(G) = 0$. The matrix for ∂_2 equals

$$\begin{pmatrix} 0 & 0 \\ 1 - \zeta + \zeta^2 - \zeta^3 + \zeta^4 & (1 - \zeta + \zeta^2 - \zeta^3 + \zeta^4)(\zeta - 1) \end{pmatrix}.$$

The homology is non trivial if and only if the matrix vanishes and hence $\mathcal{V}_1(G)$ is as in the statement.

Since we are working with the complement of an affine (hence projective) curve, if G admits a geometric surjection onto an infinite orbifold group, the orbifold must be over a rational curve. Since the abelianization has rank 1, the rational curve must be either $\mathbb{C}$ or $\mathbb{P}^1$. Any dominant morphism with target $\mathbb{C}$ can be considered as dominant on $\mathbb{P}^1$ and we treat only this case.

One needs to consider only orbifolds over $\mathbb{P}^1$ whose fundamental groups are infinite, have cyclic abelianizations and admit the 10-th primitive roots of unity in their characteristic varieties. In particular, the abelianization must be of the type $\mathbb{Z}/n\mathbb{Z}$, where 10 divides n.

Let us prove that any such orbifolds admit dominant morphisms in $\mathbb{P}^1_{2,5,10}$ and $\mathbb{P}^1_{2,2,5,5}$. It is not hard to prove (see, e.g., [4] for details) that for a prime p, the abelianization of G has non-trivial p-factors if at least two orbifold points have indices divisible by p. Using the identity mapping, we obtain dominant morphisms in either the above orbifolds or $\mathbb{P}^1_{10,10}$. We need to exclude the case where only a dominant morphism in $\mathbb{P}^1_{10,10}$ exists. In this case, $\mathbb{P}^1_{10n_1,10n_2,n_3,\ldots,n_r}$, $\gcd(n_j, 10) = 1$, $j = 1, \ldots, r$. We proved in [4] that no element of order 10 is in the characteristic varieties of this orbifold, and hence, these orbifolds do not satisfy the claim of the statement.

The properties of $\mathcal{V}_2$ allow us to discard $\mathbb{P}^1_{2,2,5,5}$, see Proposition 1.14. Let us assume that there is a geometric surjection onto the orbifold $\mathbb{P}^1_{2,5,10}$. Proposition 1.13 does not provide a direct obstruction in terms of $\mathcal{V}_1$. Moreover, the kernel of the abelianization map is the fundamental group K_2 of a compact Riemann surface of genus 2, see Proposition 1.13.

Note that $(xy)^5 = (x^2t)^5$ is a central element and the group K generated by this element defines an injection in G/G'. Following [10], if $G_0 := G/K$, the groups G'_0 and G' are isomorphic and hence G' is finitely presented. Using the Reidemeister-Schreier method, we find the following presentation:

$$(4.7) \qquad G' = \langle t_0, t_1, t_2, t_3, t_4 \mid t_{n+1}t_{n+3} = t_n t_{n+2} t_{n+4}, B_n = B_{n+1} \rangle,$$

where $B_n := t_n t_{n+1}^{-1} t_{n+2} t_{n+3}^{-1} t_{n+4}$ and $x * t_n = t_{n+1}$. Note that $x^{10} * t_n = t_{n+10} = A * t_n$, where $A := t_n t_{n+2} t_{n+4} t_{n+6} t_{n+8}$ for any n. This guarantees that the above presentation is finite. Summarizing, one can deduce that the kernel K_1 of the epimorphism onto $\mathbb{Z}/10$ equals $\mathbb{Z} \times G'$. Note that the rank of K_1 equals 5 and the rank of K_2 equals 4, so no contradiction arises.

According to `GAP` the next quotients of the lower central series have ranks 5 and 16 for K_2, and 2 and 0 (order 5) for K_1 and hence such an epimorphism cannot exist. $\qquad \square$

5. FURTHER PROPERTIES OF THE AFFINE DEGTYAREV CURVE

The affine Degtyarev curve is related with elliptic fibrations as follows. In order to work in a projective setting, one can first consider the projective Degtyarev curve, and fix a singular point P. We will denote by L the tangent line of C at P, and the remaining singular points by $P_\pm$. Let $\sigma : \Sigma_1 \to \mathbb{P}^2$ be the blow-up of P where E denotes the exceptional component. Strict transforms will follow Convention 2.10.

Each generic fiber of Σ intersects C at three points. There are four exceptions; three of them can be seen in Figure 2 and they are denoted by F_+, F_0, and F_-. The fourth one is L, which intersects C at two points: one is smooth and transversal and the other one is the infinitely near point of P in E, which is of type $\mathbb{A}_2$. In order to separate C and E we perform a positive elementary Nagata transformation $\rho : \Sigma_1 \dashrightarrow \Sigma_2$ on the fiber corresponding to L. The fiber which replaces L is denoted by F_∞. Note that F_∞ intersects C at two points: one of them corresponds to the blow-down of L and the other one is a point with a generic tangency. In particular, the combinatorics of the intersections at F_0 and F_∞ coincides.

Remark 5.1. Properties 3.1 imply the rigidity of this arrangement of curves in Σ_2. In particular, once the four fibers are ordered the cross-ratio of their images in $\mathbb{P}^1$ provides an invariant of the arrangement. The existence of an automorphism of Σ_2 preserving C and exchanging the two fibers containing the singular points can be easily checked. As a consequence of the cross-ratio argument, the two tangent fibers must also be exchanged. This automorphism defines a birational map of $\mathbb{P}^2$ which is related to the two solutions in $\mathbb{Q}(\sqrt{5})$ exhibited in the proof of Property 3.1(D3).

Let us consider the minimal resolution Z of the double covering of Σ_2 ramified at $C + E$. The ruling of Σ_2 induces a morphism $\rho : Z \to \mathbb{P}^1$ such that the generic fiber is elliptic. The only singular fibers are the preimages of F_+, F_- (of type I_5 in Kodaira notation), F_0, and F_∞ (of type I_1). These elliptic fibrations have been extensively studied in [18]. Once a section is fixed (e.g. the preimage of E), the set of sections has an abelian group structure (inherited by the structure on the fibers) which is called the Mordell-Weil group. Note that the involution associated with the double covering is defined by taking the opposite. It is known that the Mordell-Weil group of Z is cyclic of order 5.

Let us consider a conic C_1 tangent to C both at P and at another singular point and transversal to the third singular point. The preimage of C_1 by the double covering has two irreducible components which are denoted by E_1 and $-E_1$: they are opposite sections in the Mordell-Weil group. Interchanging the two singular points, one obtains the remaining two sections E_2 and $-E_2$ of Z.

Let us recall that G denotes the fundamental group of the complement of the affine Degtyarev curve, i.e. $\mathbb{P}^2 \setminus (C \cup L) = \Sigma_2 \setminus (C \cup E \cup L_\infty)$.

Remark 5.2. Despite Proposition 4.4, note that its affine version, $G = \pi_1(\mathbb{P}^2 \setminus (C \cup L))$ *does* posses a geometric surjection onto the orbifold over $\mathbb{P}^1_{2,2,5}$, since G admits an epimorphism onto the dihedral group of order 10, see for instance [5].

In order to construct this morphism, we may use the ideas in [23]. The mapping is obtained by a pencil of rational curves of degree 10, with the following non-reduced fibers:

- A smooth conic C_2 of multiplicity 5 such that $(C \cdot C_2)_{P_+} = 2$, $(C \cdot C_2)_{P_-} = 4$ and $(C \cdot C_2)_P = 4$.
- A quintic C_5 of multiplicity 2 such that $(C \cdot C_5)_{P_+} = 5$ (P_+ is a smooth point of C_5), $(C \cdot C_2)_{P_-} = 10$ (P_- is a singular point of C_5 of type $\mathbb{A}_4$), and $(C \cdot C_2)_P = 10$ (P is a singular point of C_5 of type $\mathbb{D}_6$).

- The curve $C + L + 2D_2$ where D_2 is a smooth conic such that $(C \cdot D_2)_{P_+} = 0$, $(C \cdot D_2)_{P_-} = 5$, and $(C \cdot D_2)_P = 4$.

We finish this section by describing some properties of the group G. For a point $Q \in C$, the local fundamental group $\pi_1^{\mathrm{loc}}(C, Q)$ of C at Q is $\pi_1(\mathbb{B}_Q \setminus C)$, where $\mathbb{B}_Q$ is a Milnor ball. The inclusion $\mathbb{B}_Q \setminus C \hookrightarrow \mathbb{C}^2 \setminus C$ induces a conjugacy class of subgroups (since the base point is not fixed) which will be called the *image of the local fundamental group*.

Proposition 5.3. *Let $P_\pm$ be the two singular points of the affine Degtyarev curve.*

(a) *The images of the local fundamental groups at P_+ and P_- are the whole group G.*
(b) *The center of G contains an abelian free subgroup of rank* 2.

Proof. The property about the image of the local fundamental group at P_- is obvious from the presentation (4.5). For P_+ it can be deduced using `GAP`. As a consequence we obtain two central elements (the images of the central elements of the local fundamental groups). The last property can be deduced by studying some quotients of subgroups of G. $\qquad\square$

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