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$\mathcal{A}_0$-SUFFICIENCY OF JETS FROM $\mathbb{R}^2$ TO $\mathbb{R}^2$

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Dedicated to professor Andrew du Plessis on his 60th birthday

Abstract. An r-jet $z \in J^r(2,2)$ is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(2,2)$ if every C^r realization of z is topologically right-left equivalent to z. We give sufficient conditions for $\mathcal{A}_0$-sufficiency in $\mathcal{E}_{[r]}(2,2)$. For a certain class of jets, we prove that our sufficient conditions are also necessary. Finally, we use the techniques developed in the course of the proofs of these results to give sufficient conditions for a 1-parameter family of C^r plane-to-plane map-germs to be topologically trivial.

1. Introduction

Let $\mathcal{E}_{[r]}(n,p)$ denote the set of C^r-map-germs $(\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$. Let $\omega : (\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$ be an r-jet. We say that ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(n,p)$ if, for any C^r-germ $f : (\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$ with $j^r f(0) = \omega$, there exist germs of homeomorphisms $h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ and $k : (\mathbb{R}^p,0) \to (\mathbb{R}^p,0)$ such that $f = k \circ \omega \circ h$.

The study of sufficiency of jets started with the classical papers of Kuiper [7], Kuo [8], [9] and Bochnak and Łojasiewicz [3]. In these papers the sufficiency of r-jets in $\mathcal{E}_{[r]}(n,1) = \mathcal{E}_{[r]}$ and $\mathcal{E}_{[r+1]}$ with respect to $\mathcal{R}_0$-equivalence and the sufficiency of r-jets in $\mathcal{E}_{[r+1]}(n,p)$ with respect to $\mathcal{V}$-equivalence were studied, and necessary and sufficient conditions for sufficiency were given. (Two map-germs f, g are $\mathcal{R}_0$-equivalent if there exists a germ of homeomorphism h such that $f = g \circ h$, and they are $\mathcal{V}$-equivalent if $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic.) In these cases the necessary and sufficient condition was formulated in terms of a Łojasiewicz inequality. This Łojasiewicz inequality implies that every representative of the jet is, in some sense, non-singular outside 0.

In this article we will study $\mathcal{A}_0$-sufficiency of jets, and we will only consider jets from $\mathbb{R}^2$ to $\mathbb{R}^2$. The nice geometric conditions we expect for representatives of such jets are that they only have fold singularities outside the origin and that they do not have singular double points. We must therefore put up Łojasiewicz inequalities avoiding such singularities outside 0, and hopefully such Łojasiewicz inequalities will be necessary and sufficient conditions for $\mathcal{A}_0$-sufficiency of plane-to-plane jets. We have however not been able to prove this in general. Let $\omega : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be a singular r-jet (identified with a polynomial map of degree $\leq r$) with singular set $\Sigma(\omega)$. Assume that ω is not the zero jet and that 0 is not isolated in $\Sigma(\omega)$. Then $\Sigma(\omega)$ is a 1-dimensional algebraic set. It follows that for small balls $B(0,\rho)$ around 0, 0 is in the closure of all components of $(\Sigma(\omega) - \{0\}) \cap B(0,\rho)$ and the number of such components are independent of the radius ρ. Let $C_1, \ldots, C_N$ be these components. By the curve selection lemma, we can find analytic curves $\gamma_i : [0,\epsilon) \to \mathbb{R}^2$ for $i = 1, \ldots, N$ with $\gamma_i(0) = 0$ and $\gamma_i(0,\epsilon) \subset C_i$. Let $\mathbf{n}_i = \lim_{t \to 0^+} \gamma_i'(t)/\left\|\gamma_i'(t)\right\|$. If all the $\mathbf{n}_i$ are distinct, we say that $C_1, \ldots, C_N$ have different tangent directions at 0. Assume that $C_1, \ldots, C_N$ have different tangent directions at 0. For such jets, we prove that there exist two Łojasiewicz inequalities which together are necessary and sufficient

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conditions for sufficiency. This result is Theorem 2.2 in Section 2. If we drop the hypothesis about the tangent directions, we can prove that our inequalities are sufficient conditions, but we have not suceeded in proving the necessity of both of these inequalities in the general case. For jets ω such that two components C_i and C_j of $\Sigma(\omega) - \{0\}$ have the same tangent direction at 0, the distance between points in C_i and C_j may be small compared to the distance to 0. This makes perturbation arguments complicated.

If we consider jets ω where 0 is isolated in $\Sigma(\omega)$, we can discard the second Łojasiewicz inequality, and the first Łojasiewicz inequality will be a necessary and sufficient condition for $\mathcal{A}_0$-sufficiency. In fact, it turns out that this inequality is a necessary and sufficient condition for $\mathcal{R}_0$-sufficiency in $\mathcal{E}_{[r]}(2,2)$ for such jets.

The statement of Theorem 2.2 in Section 2 below is a generalized and improved version of a theorem announced without proof in the article [5]. Also Theorem 2.3 is announced without proof in [5].

The article is organized in the following way: In Section 2 we introduce some notation and formulate the main results of the article. In Section 3 we write down the equations in the jet space for certain sets of singular 1- and 2-jets, and we find expressions for distance functions from jets to these singular sets. These distance functions will be used throughout the article. We also discuss the smoothness of one of these distance functions and we prove Propostion 2.1 and Theorem 2.3 of Section 2. In Section 4 we prove that the two Łojasiewicz inequalities formulated in Theorem 2.2 in Section 2 are stable in the sense that all C^r-representatives of a jet satisfying the inequalities also satisfy similar inequalities. We also derive a number of geometrical consequences of our Łojasiewicz inequalities.

In Section 5, we prove that the two Łojasiewicz inequalities of Theorem 2.2 imply sufficiency of the jet. In Section 6, we point out that every jet has a nice realization which has at most only fold singularities outside 0, and avoids singular double points. We then prove that if some of the inequalities of Theorem 2.2 are not satisfied, then we can find another bad realization of the jet having singularities which are topologically different from the singularities of the nice representative. (When we here consider the failure of the second Łojasiewicz inequality, we consider only jets ω such that the tangent directions at 0 of the components $C_1, \dots, C_N$ are distinct.) This will prove that the Łojasiewicz inequalities are necessary for sufficiency of the jet and therefore complete the proof of Theorem 2.2.

In Section 7 we give examples of sufficient and non-sufficient jets.

Finally, in Section 8, we look at germs of one-parameter families of C^r-maps and state sufficient conditions for such families to be topologically trivial. The conditions are analogous to those satisfied by one-parameter families of C^r-realizations of sufficient jets.

2. The Main Theorem

Let $J^1(2,2)$ be the set of 1-jets $(\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$. An element $z \in J^1(2,2)$ can be identified with a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ and thus with a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or (when we find it convenient) a vector $(a,b,c,d) \in \mathbb{R}^4$. Let $J^2(2,2)$ be the set of 2-jets $(\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$. An element $z \in J^2(2,2)$ can be identified with a polynomial map

$$z(x,y) = (ax + by + ex^2 + 2fxy + gy^2, cx + dy + hx^2 + 2ixy + jy^2).$$

Now $J^2(2,2)$ can be identified with $\mathbb{R}^{10}$ by identifying z with the tuple $(a,b,\dots,j)$ and we can therefore consider the splitting

$$(L,H) = (L_z,H_z) = ((a,b,c,d),(e,f,g,h,i,j)) \in \mathbb{R}^4 \times \mathbb{R}^6.$$

Consider the set $\mathcal{E}_{[r]}(2,2)$ of C^r-germs $f : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$. Let $r \geq 2$ and let $f : U \to \mathbb{R}^2$ be a representative of a germ in $\mathcal{E}_{[r]}(2,2)$. For $p \in U$ we can define $j^1f(p) \in J^1(2,2)$ and $j^2f(p) \in J^2(2,2)$ as the 1- and 2-jet, respectively, of $f\big((x,y)+p\big) - f(p)$ at $(x,y) = 0$. For any $f \in \mathcal{E}_{[r]}(2,2)$ we can consequently define germs $j^1f : (\mathbb{R}^2,0) \to J^1(2,2)$ and $j^2f : (\mathbb{R}^2,0) \to J^2(2,2)$ and thus define the germ (L_f, H_f) by $(L_f,H_f)(p) = (L_{j^2f(p)}, H_{j^2f(p)})$. Let $\Gamma \subset J^2(2,2)$ be defined by

$$\Gamma = \{\, (a,\dots,j) \,|\, ad-bc = 0, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} aj-bi-cg+df \\ -ai+bh+cf-de \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \},$$

via our identifications. We will see in Section 3 that Γ is the set of singular 2-jets which are not folds.

Let $\omega \in J^r(2,2)$ be a singular jet which we identify with a polynomial map $\omega : \mathbb{R}^2 \to \mathbb{R}^2$ of degree $\leq r$. Assume that 0 is not isolated in $\Sigma(\omega)$ and that ω is not the zero jet. Since $\Sigma(\omega)$ is algebraic, it follows that there exists $\rho_0 > 0$ such that when $0 < \rho < \rho_0$ then $(\Sigma(\omega)-\{0\}) \cap B(0,\rho)$ (where $B(0,\rho) \subset \mathbb{R}^2$ is the open ball with center 0 and radius ρ) is non-singular, has finitely many topological components, 0 is in the closure of each component, and the number of such components is independent of ρ (this follows for example from the results of chapter 2 of [11]). We denote these components by $C_1,\dots,C_N$ with no reference to the ball $B(0,\rho)$. As explained in the introduction, these curves have a well defined tangent direction at the origin.

For each $\epsilon > 0$, define

$$H_\epsilon = \left\{ p \,|\, d(j^1\omega(p),\Sigma) \leq \epsilon \left\|p\right\|^{r-1} \right\}.$$

Here $\Sigma \subset J^1(2,2)$ is the set of singular 1-jets, $d(j^1\omega(p),\Sigma)$ denotes the distance $\inf\{\left\|j^1\omega(p)-z\right\| \mid z \in \Sigma\}$, where $\|\cdot\|$ is the usual Euclidean norm when 1-jets are identified with vectors in $\mathbb{R}^4$ (when points in some finite dimensional linear spaces are identified with vectors in Euclidean spaces $\|\cdot\|$ will always (unless otherwise stated) denote the Euclidean norm via the identification). For every $\epsilon > 0$, H_ϵ is a closed semialgebraic set with $\Sigma(\omega) \subset H_\epsilon$ (this is a consequence of Proposition 2.2.8 of [1] and the Tarski-Seidenberg Theorem).

We now have the following proposition:

Proposition 2.1. *Let $r \geq 2$ and $\omega \in J^r(2,2)$ be a singular, non-zero jet such that 0 is not isolated in $\Sigma(\omega)$. Let Γ, ρ_0, $C_1,\dots,C_N$ and H_ϵ be as explained above. Consider the following condition:*

(I) *There is a neighbourhood U of 0 and constants $C > 0$ such that if $p \in U$ and $(L,H) \in \Gamma$, then*

$$\|L_\omega(p) - L\| + \|H_\omega(p) - H\| \, \|p\| \geq C \left\|p\right\|^{r-1}.$$

Assume that condition (I) *is satisfied. Then there exists $\epsilon_0 > 0$ such that if ρ_0 above is sufficiently small, then the following is satisfied: For each open ball $B(0,\rho) \subset \mathbb{R}^2$ with center 0 and radius $\rho < \rho_0$, and for each ϵ, $0 < \epsilon < \epsilon_0$, $(H_\epsilon - \{0\}) \cap B(0,\rho)$ has exactly N connected components, and we can label the components of $(H_\epsilon - \{0\}) \cap B(0,\rho)$ by $H_1,\dots,H_N$, such that $C_i \subset H_i$.*

Now we have:

Theorem 2.2 (Main Theorem)**.** *Let $r > 2$ and let $\omega \in J^r(2,2)$ be a jet as described in Proposition 2.1 . Let Γ, $C_1,\dots,C_N$ and H_ϵ be as defined above and assume that condition* (I) *of Proposition 2.1 is satisfied. Let ρ_0 and ϵ_0 be as in the conclusion of 2.1. Consider the following* *condition :*

(II) *There exist $\rho > 0$ with $\rho < \rho_0$ and $\epsilon > 0$ with $\epsilon < \epsilon_0$ and a constant $C > 0$ such that if H_i and H_j, $i \neq j$ are components of $(H_\epsilon - \{0\}) \cap B(0,\rho)$ and $p \in H_i \cup \{0\}$ and $q \in H_j \cup \{0\}$ then*

$$\|\omega(p) - \omega(q)\| \geq C(\|p\|^{r-1} + \|q\|^{r-1})\, \|p - q\|\,.$$

Assume also that the condition (II) *above is satisfied, then ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(2,2)$.*

Moreover, the condition (I) *of Proposition 2.1 is a necessary condition for $\mathcal{A}_0$-sufficiency in $\mathcal{E}_{[r]}(2,2)$ for all jets in $J^r(2,2)$ with $r > 2$, and if we consider singular, non-zero jets ω where 0 is not isolated in $\Sigma(\omega)$, and where all the components $C_1, \ldots, C_N$ of $\Sigma(\omega) - \{0\}$ have different tangent directions at 0, then condition* (II) *above is also a necessary condition for $\mathcal{A}_0$-sufficiency in $\mathcal{E}_{[r]}(2,2)$.*

Remark 1. One may conjecture that (I) together with (II) is equivalent to $\mathcal{A}_0$-sufficiency for all jets with non isolated critical point at 0. In fact one may sharpen this, and restrict (II) to $\Sigma(\omega)$ and conjecture that (I) together with this restricted version of (II) is equivalent to $\mathcal{A}_0$-sufficiency for all such jets. In two preprints [12] and [13], the second author has verified this conjecture for jets where all the components $C_1, \ldots, C_N$ of $\Sigma(\omega) - \{0\}$ have different tangent directions at 0, for jets of rank 1 and for weighted homogeneous jets. In fact for homogeneous jets, $\mathcal{A}_0$-sufficiency is equivalent to the geometrical condition that the jets only have fold singularities outside 0 and have no singular double points. The proofs of these results given in [12] and [13] depend however heavily on the results and techniques given in this article.

For jets ω where 0 is isolated in $\Sigma(\omega)$ we have the following sufficiency theorem:

Theorem 2.3. *Let $\omega \in J^r(2,2)$ with $r \geq 2$ be a singular jet and assume that there exists a neighborhood U of 0 such that $\Sigma(\omega) \cap U = \{0\}$. Then ω is $\mathcal{R}_0$-sufficient in $\mathcal{E}_{[r]}(2,2)$ if and only if ω satisfies the condition* (I) *in Proposition 2.1.*

Remark 2. Let $\omega = (f, g)$. Note that $\mathcal{R}_0$-sufficiency is by [2] (or [14]) equivalent to an inequality $d(\nabla f(p), \nabla g(p)) \geq C||p||^{r-1}$, in fact in [2] it is proven that this inequality also is equivalent to $\mathcal{A}_0$-sufficiency for jets with an isolated critical point at 0. We will see in Subsection 4.1 below that this inequality is trivially equivalent to the inequality $d(j^1\omega(p), \Sigma) \geq C||p||^{r-1}$. The left hand side of the inequality (I) in Proposition 2.1 is a sort of measure of the distance from the jet $j^2\omega(p)$ to the set of singular 2-jets which are not folds. So a priori, this is a much weaker inequality than the inequality $d(j^1\omega(p), \Sigma) \geq C||p||^{r-1}$, but we will show in Subsection 4.1 that these two inequalities actually are equivalent for jets ω with $\Sigma(\omega) = \{0\}$, proving Theorem 2.3. Together with the conclusion of Theorem 2.2 we thus get that $\mathcal{R}_0$-sufficiency, $\mathcal{A}_0$-sufficiency and (I) are equivalent conditions for jets in $J^r(2,2)$ with an isolated critical point at 0.

3. Folds

As remarked above, the left hand side of the inequality (I) of Proposition 2.1 somehow measures the distance from the 2-jet $j^2\omega(p)$ to the set of singular jets which are not folds. To see this we first have to study fold points and make some estimates in both $J^1(2,2)$ and $J^2(2,2)$.

By definition, a mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ has a fold singularity at a point p if $j^1F(p) \in \Sigma^1$, where Σ^1 is the set of jets of rank 1, $j^1F \pitchfork \Sigma^1$ at p and $\ker DF(p) + T_p\Sigma(F) = \mathbb{R}^2$. We say that a jet $z = (a, \ldots, j) \in J^2(2,2)$ is a fold if the associated polynomial mapping $z(x,y) = (f(x,y), g(x,y)) = (ax + \cdots + gy^2, cx + \cdots + jy^2)$ has a fold singularity at 0.

We want to describe the set of folds in $J^2(2,2)$ explicitly. Since the Jacobian matrix of z at 0 is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $ad-bc=0$ is the equation of the singular jets Σ in $J^1(2,2)$. Consider the mapping $(a,b,c,d) \mapsto ad-bc$. When $(a,b,c,d) \in \Sigma^1$ the gradient of this mapping, $(d,-c,-b,a)$, will be a normal vector of Σ^1 at (a,b,c,d). Then $j^1 z \pitchfork \Sigma^1$ if and only if at least one of $(\frac{\partial}{\partial x} j^1 z)(0)$, $(\frac{\partial}{\partial y} j^1 z)(0)$ is not perpendicular to $(d,-c,-b,a)$, that is $\left(\begin{smallmatrix} ai-bh-cf+de \\ aj-bi-cg+df \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. On the other hand, we have $Jz(x,y) = (\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x})(x,y)$, and a direct computation gives us that

$$\nabla Jz(0) = 2\begin{pmatrix} ai-bh-cf+de \\ aj-bi-cg+df \end{pmatrix}.$$

For $j^1 z \pitchfork \Sigma^1$, the vector $\left(\begin{smallmatrix} ai-bh-cf+de \\ aj-bi-cg+df \end{smallmatrix}\right)$ is therefore a normal vector to $\Sigma(z)$ at 0. The vector $\left(\begin{smallmatrix} aj-bi-cg+df \\ -ai+bh+cf-de \end{smallmatrix}\right)$ will consequently span $T_0\Sigma(z)$, and the condition $\ker Dz(0) + T_0\Sigma(z) = \mathbb{R}^2$ is obviously equivalent to

$$Dz(0)\begin{pmatrix} aj-bi-cg+df \\ -ai+bh+cf-de \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} aj-bi-cg+df \\ -ai+bh+cf-de \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus we see that the set

$$\Gamma = \{\, (a,\dots,j) \mid ad-bc=0, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} aj-bi-cg+df \\ -ai+bh+cf-de \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$$

is the set of singular 2-jets which are not folds.

3.1. **Distance from a jet to Σ in $J^1(2,2)$.** Let F, G be nonnegative functions. We will use the notation $F \sim G$ if there are constants $s,t>0$ such that $sF \le G \le tF$. Consider a jet $z \in J^1(2,2)$ identified with a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. By $\|M\|$, we mean the standard Euclidean norm $\|M\| = (a^2+b^2+c^2+d^2)^{\frac{1}{2}}$. Our first task will be to estimate the distance $d(z,\Sigma)$ from a z to $\Sigma \subset J^1(2,2)$.

Suppose $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a singular jet realizing the distance R from $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to Σ. It is clear that X is an element of Σ^1. A normal vector to Σ^1 at $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} D & -C \\ -B & A \end{smallmatrix}\right)$, so there is a t with

$$M - X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix} = t\begin{pmatrix} D & -C \\ -B & A \end{pmatrix}$$

giving

$$\det M = t\left\|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\|^2, \quad R = |t|\cdot\left\|\begin{pmatrix} D & -C \\ -B & A \end{pmatrix}\right\| = \frac{|ad-bc|}{\left\|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\|}.$$

Now, suppose $\|\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)\| \ge \|\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)\|$. Since $\left(\begin{smallmatrix} a & 0 \\ c & 0 \end{smallmatrix}\right) \in \Sigma$,

$$R \le \left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} b \\ d \end{pmatrix}\right\| \le \frac{1}{\sqrt{2}}\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\|.$$

The same argument can be applied if $\|\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)\| \le \|\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)\|$, so in any case,

$$R \le \frac{1}{\sqrt{2}}\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\|.$$

By the triangle inequality,

$$(1-\frac{1}{\sqrt{2}})\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\| \le \left\|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\| \le (1+\frac{1}{\sqrt{2}})\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\|. \tag{3.1}$$

So from this and from the expression for R above, we get that

$$(2-\sqrt{2})\frac{|Jz(x,y)|}{\|Dz(x,y)\|} \le R = d\big(j^1 z(x,y),\Sigma\big) \le (2+\sqrt{2})\frac{|Jz(x,y)|}{\|Dz(x,y)\|}, \tag{3.2}$$

and hence,

$$\frac{|Jz(x,y)|}{\|Dz(x,y)\|} \sim d\big(j^1 z(x,y),\Sigma\big) \tag{3.3}$$

for every non-zero jet $z \in J^r(2,2)$.

3.2. **Distance from a singular jet to Γ in $J^2(2,2)$.** Let $z = (a,b,\dots,j) \in J^2(2,2)$ with $ad - bc = 0$. Let

$$E = E_z = \{\omega \in \Gamma \mid L_\omega = (a,b,c,d)\}.$$

We want to estimate distance $d(z,E)$, i.e. the distance from a singular 2-jet z to the set of singular 2-jets with the same linear part as z satisfying the equation

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} aj - bi - cg + df \\ -ai + bh + cf - de \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.4}$$

If $a = b = c = d = 0$, then the distance is 0 of course. Suppose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is singular and non-zero. E is the linear subspace $\mathbb{R}^6$ with coordinates $(e,\dots,j)$ satisfying

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} aj - bi - cg + df \\ -ai + bh + cf - de \end{pmatrix} \\ &= \begin{pmatrix} (e,f,g,h,i,j)\cdot(-bd, ad+bc, -ac, b^2, -2ab, a^2) \\ (e,f,g,h,i,j)\cdot(-d^2, 2cd, -c^2, bd, -ad-bc, ac) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So $E = \mathrm{sp}\{v_1, v_2\}^\perp$, where

$$\begin{aligned} v_1 &= (-bd, ad+bc, -ac, b^2, -2ab, a^2) \\ v_2 &= (-d^2, 2cd, -c^2, bd, -ad-bc, ac). \end{aligned}$$

If $H_z = (e,f,g,h,i,j)$, then the distance we are seeking is the length of the projection $\mathbf{p}_E$ of (e,f,g,h,i,j) onto $E^\perp$. We notice that since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is singular, v_1 and v_2 are linearly dependent, and assuming that none of them are zero (otherwise, the expressions simplify),

$$\mathbf{p}_E = \frac{1}{2}\left(\frac{L_1}{\|v_1\|^2}v_1 + \frac{L_2}{\|v_2\|^2}v_2\right),$$

and so the distance R is

$$R = \|\mathbf{p}_E\| = \frac{1}{2}\left(\frac{|L_1|}{\|v_1\|} + \frac{|L_2|}{\|v_2\|}\right).$$

Suppose $\sup\{a^2,b^2,c^2,d^2\} \in \{a^2,b^2\}$ and put $N = \left\|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right\|^2$. It is easily seen that

$$\frac{1}{16}N^2 \le (\sup\{a^2,b^2,c^2,d^2\})^2 \le \|v_1\|^2 \le 12(\sup\{a^2,b^2,c^2,d^2\})^2 \le 12N^2,$$

and we get

$$\frac{1}{4}N \le \|v_1\| \le 2\sqrt{3}N. \tag{3.5}$$

In this case, $R = \frac{|L_1|}{\|v_1\|}$ and

$$\frac{|L_1|}{2\sqrt{3}N} \le R \le \frac{4|L_1|}{N}. \tag{3.6}$$

Similarly, if $\sup\{a^2, b^2, c^2, d^2\} \in \{c^2, d^2\}$,

$$\frac{|L_2|}{2\sqrt{3}N} \leq R \leq \frac{4|L_2|}{N}. \tag{3.7}$$

Notice that the left inequalities in (3.6) and (3.7) hold without the assumptions regarding which elements are realizing $\sup\{a^2, b^2, c^2, d^2\}$. By adding the left sides of the inequalities (3.6) and (3.7) we get $(|L_1| + |L_2|)/(2\sqrt{3}N) \leq 2R$. Also, one of the inequalities on the right side of either (3.6) or (3.7) must hold, so certainly $2R \leq 8(|L_1| + |L_2|)/N$. We get

$$\frac{\left\|\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}\right\|}{2\sqrt{3}N} \leq \frac{|L_1| + |L_2|}{2\sqrt{3}N} \leq 2R \leq 8\frac{|L_1| + |L_2|}{N} \leq 16\frac{\left\|\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}\right\|}{N}. \tag{3.8}$$

From this we see that

$$\frac{\left\|\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}\right\|}{N} = \frac{\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} aj - bi - cg + df \\ -ai + bh + cf - de \end{pmatrix}\right\|}{\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\|^2} \sim R = d(z, E_z). \tag{3.9}$$

In the language of partial derivatives and differentials of a C^r mapping f with $p \in \Sigma(f)$, inequality (3.9) reads

$$d(H_f(p), E_{j^2 f(p)}) \sim \frac{\left\|Df(p)\begin{pmatrix} \frac{\partial}{\partial y} Jf(p) \\ -\frac{\partial}{\partial x} Jf(p) \end{pmatrix}\right\|}{\|Df(p)\|^2}. \tag{3.10}$$

3.3. **Smoothness of the distance function and proofs of Proposition 2.1 and Theorem 2.3.** Let $\omega \in J^r(2,2)$. Before we can prove Proposition 2.1, we have to investigate the smoothness properties of the distance map we are about to define. Let $d : \mathbb{R}^2 \to \mathbb{R}$ be the map $p \mapsto d(p) = d(j^1\omega(p), \Sigma)$. We want information about where d is smooth. To this end, let $d' : J^1(2,2) \to \mathbb{R}$ be the map $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d(A, \Sigma) = \inf\{\|A - X\| \mid X \in \Sigma\}$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in J^1(2,2) \setminus \Sigma$. Consider $B = \{Y \mid A - Y \in \Sigma\}$. Then $Y \in B$ if and only if there exists $\mathbf{w} \in \mathbb{R}^2$ with $\|\mathbf{w}\| = 1$ such that $A\mathbf{w} = Y\mathbf{w}$. Since $\|Y\mathbf{w}\| \leq \|Y\|$, we get that

$$\inf\{\|A\mathbf{w}\| \mid \|\mathbf{w}\| = 1\} \leq d'(A).$$

On the other hand, let $\lambda = \inf\{\|A\mathbf{w}\| \mid \|\mathbf{w}\| = 1\}$ and let $\mathbf{w} = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ be a unit vector such that $\lambda = \|A\mathbf{w}\|$. Let Y be the matrix given by $Y\mathbf{w} = A\mathbf{w}$ and $Y\left(\begin{smallmatrix} -v \\ u \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Then $\|Y\| = \|A\mathbf{w}\|$ and it follows that

$$d'(A) = \inf\{\|A\mathbf{w}\| \mid \|\mathbf{w}\| = 1\}. \tag{3.11}$$

From this we see that

$$d'(A) = (\inf\{|\mathbf{w}^T A^T A\mathbf{w}|\,;\, \|\mathbf{w}\| = 1\})^{\frac{1}{2}} = (\inf\{|\beta|\,;\, \beta \,\text{eigenvalue of}\, A^T A\})^{\frac{1}{2}}.$$

Calculating the eigenvalues of the symmetric matrix $A^T A$, we find that

$$d'(A) = \frac{1}{\sqrt{2}}\sqrt{\|A\|^2 - \sqrt{\|A\|^4 - 4(\det A)^2}}.$$

If we want to find an explicit expression for $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \Sigma$ such that $d'(A) = \|A - X\|$, we can use the method of Lagrange multipliers. The coordinates of X have to satisfy the following equations:

$$x - a = \lambda w \tag{3.12}$$
$$y - b = -\lambda z \tag{3.13}$$
$$z - c = -\lambda y \tag{3.14}$$
$$w - d = \lambda x \tag{3.15}$$
$$xw - yz = 0. \tag{3.16}$$

Analyzing this system, we find that if $|\det(A)| < \frac{1}{2}\|A\|^2$ (note that the inequality $|\det(A)| \leq \frac{1}{2}\|A\|^2$ holds for any A), then $\lambda \neq \pm 1$ and then the solution of the above system is given by

$$x = \frac{a + \lambda d}{1 - \lambda^2}, \quad y = \frac{b - \lambda c}{1 - \lambda^2}, \quad z = \frac{c - \lambda b}{1 - \lambda^2}, \quad w = \frac{d + \lambda a}{1 - \lambda^2}. \tag{3.17}$$

where λ is given by

$$\lambda_1 = \frac{-\|A\|^2 + \sqrt{\|A\|^4 - 4(\det A)^2}}{2 \det A} \quad \text{or} \quad \lambda_2 = \frac{-\|A\|^2 - \sqrt{\|A\|^4 - 4(\det A)^2}}{2 \det A},$$

and X is given by (3.17) with $\lambda = \lambda_1$. From the expression of d' above we see that d' is smooth when $\det A \neq 0$ and $|\det A| \neq \frac{1}{2}\|A\|^2$. d is consequently smooth on the complement of the set $\Sigma(\omega) \cup \{p \mid |J\omega(p)| = \frac{1}{2}\|D\omega(p)\|^2\}$. Denote this complement by V. Let

$$S = \{p = (x, y) \in V \mid \nabla d(p) \cdot (y, -x) = 0\}.$$

Then

$$S = \{p \in V \mid d|_{\{q \in V \mid \|q\| = \|p\|\}} \text{ has a stationary point at } p\}.$$

From the definition of V and the expression of d' given above it follows that S is a semialgebraic set. Now we have the following lemma:

Lemma 3.1. *Assume ω satisfies condition (I) of 2.1, then there is a neighborhood U of 0 and a $C > 0$ such that*

$$d(p) = d(j^1\omega(p), \Sigma) \geq C\|p\|^{r-1}$$

when $p \in S \cap U$.

Proof. Consider the set

$$\begin{aligned} D = &\{(p, A) \subset S \times \Sigma \mid \left\|j^1\omega(p) - A\right\| \leq \left\|j^1\omega(q) - B\right\| \\ &\text{for all } q \in S \text{ with } \|p\| = \|q\| \neq 0 \text{ and } B \in \Sigma\}. \end{aligned}$$

An application of the Tarski-Seidenberg Theorem shows that D is semialgebraic. Assume that the inequality of the lemma is not satisfied. Then $(0, j^1\omega(0)) \in \overline{D}$ and the curve selection lemma implies that we can find an analytic curve $\tilde{\gamma} : [0, \delta) \to \mathbb{R}^2 \times \Sigma$ with $\tilde{\gamma}((0, \delta)) \subset D$ and $\tilde{\gamma}(0) = (0, j^1\omega(0))$. Let $\tilde{\gamma}(t) = (\gamma(t), A(t))$. We must have that $\left\|j^1\omega(\gamma(t)) - A(t)\right\| = o(\|\gamma(t)\|^{r-1})$.

Let

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}.$$

Then

$$j^1\omega(\gamma(t)) - A(t) = s(t)\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|}$$

where $|s(t)| = \left\|j^1\omega(t) - A(t)\right\|$. For each t let $\beta_t(u)$ be a curve such that $\beta_t(0) = \gamma(t)$, $\|\beta_t'(u)\| = 1$ and $\|\beta_t(u)\| = \|\gamma(t)\|$ for each u. Let $A_t(u) \in \Sigma$ be such that $d(j^1\omega(\beta_t(u)), \Sigma) = A_t(u)$. It is clear that $A_t(u) \in \Sigma^1$, and since $A_t(u)$ is given by equation (3.17) with $\lambda = \lambda_1$, it is clear that $A_t(u)$ is unique and smooth in u for small u. Moreover, $A_t(0) = A(t)$. By construction, $\left\|j^1\omega(\beta_t(u)) - A_t(u)\right\|^2$ must have a stationary point for $u = 0$. So

$$\begin{aligned}
&\frac{d}{du}\left\|j^1\omega(\beta_t(u)) - A_t(u)\right\|^2|_{u=0} \\
&= 2\left(\frac{d}{du}j^1\omega(\beta_t(u))|_{u=0} - \frac{d}{du}A_t(u)|_{u=0}\right)\cdot(j^1\omega(\gamma(t)) - A(t)) = 0.
\end{aligned}$$

(Here "$\cdot$"denotes the standard Euclidean inner product in $J^1(2,2)$ identified with $\mathbb{R}^4$ via the coordinates (a, b, c, d).)

Now, $\frac{d}{du}A_t(u)|_{u=0} \in T_{A(t)}\Sigma^1$, and since $j^1\omega(\gamma(t)) - A(t)$ is a normal vector to $T_{A(t)}\Sigma^1$, we get that

$$\left(\frac{d}{du}j^1\omega(\beta_t(u))|_{u=0}\right)\cdot(j^1\omega(\gamma(t)) - A(t)) = 0.$$

So

$$\left(\frac{d}{du}j^1\omega(\beta_t(u))|_{u=0}\right)\cdot\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} = \left(Dj^1\omega(\gamma(t))w(t)\right)\cdot\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} = 0,$$

where $w(t)$ is the unit vector $\frac{d}{du}\beta_t(u)|_{u=0}$.

Let $\|\gamma(t)\| \sim t^l$ and $|s(t)| = \left\|j^1\omega(\gamma(t)) - A(t)\right\| \sim t^q$. Then $q > l(r-1)$. Since we have that

$$\frac{d}{dt}\left\|j^1\omega(\gamma(t)) - A(t)\right\|^2 \sim t^{2q-1},$$

we get that

$$\left(\frac{d}{dt}(j^1\omega(\gamma(t)) - A(t))\right)\cdot(j^1\omega(\gamma(t)) - A(t)) \sim t^{2q-1}$$

and consequently that

$$\frac{d}{dt}(j^1\omega(\gamma(t)) - A(t))\cdot\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} \sim t^{q-1}.$$

Since $\frac{d}{dt}A(t) \in T_{A(t)}\Sigma^1$, and $\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}$ is a a normal vector to $T_{A(t)}\Sigma^1$, we must have

$$\frac{d}{dt}(j^1\omega(\gamma(t)))\cdot\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} \sim t^{q-1}.$$

Now $\frac{d}{dt}(j^1\omega(\gamma(t))) = Dj^1\omega(\gamma(t))\gamma'(t)$. Let $v(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$. Since $\|\gamma'(t)\| \sim t^{l-1}$, we get that

$$t^{q-l} \sim (Dj^1\omega(\gamma(t))v(t))\cdot\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} = o(t^{l(r-1)-l}) = o(\|\gamma(t)\|^{r-2}).$$

Let us consider $v(t)$ and $w(t)$ above as two unit vectors in $T_{\gamma(t)}\mathbb{R}^2$. Since $\gamma(t)$ is analytic, $v(t) = \frac{\gamma'(t)}{\|\gamma(t)\|}$ and $w(t)\cdot\gamma(t) = 0$, we must have $v(t)\cdot w(t) \to 0$ as $t \to 0$. Let $e_1(t) = \frac{\partial}{\partial x}\circ\gamma(t)$ and $e_2(t) = \frac{\partial}{\partial y}\circ\gamma(t)$, we must then have $e_1(t) = s_1(t)v(t)+p_1(t)w(t)$ and $e_2(t) = s_2(t)v(t)+p_2(t)w(t)$, where $|s_i(t)| < 2$ and $|p_i(t)| < 2$ for small t. From this and from above we get that

$$(Dj^1\omega(\gamma(t))e_1(t)) \cdot \frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t)\end{pmatrix}}{\|A(t)\|} = o(\|\gamma(t)\|^{r-2})$$

and

$$(Dj^1\omega(\gamma(t))e_2(t)) \cdot \frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t)\end{pmatrix}}{\|A(t)\|} = o(\|\gamma(t)\|^{r-2}).$$

For fixed t, write $j^2\omega(\gamma(t))$ as in Section 2 in the form

$$j^2\omega(\gamma(t)) = \left(\tilde{a}(t)x + \tilde{b}(t)y + \cdots + \tilde{g}(t)y^2, \tilde{c}(t)x + \tilde{d}(t)y + \cdots + \tilde{j}(t)y^2\right).$$

Then

$$Dj^1\omega(\gamma(t))e_1(t) = 2\begin{pmatrix} \tilde{e}(t) & \tilde{f}(t) \\ \tilde{h}(t) & \tilde{i}(t)\end{pmatrix}.$$

We thus get that

$$\begin{aligned} &2\begin{pmatrix} \tilde{e}(t) & \tilde{f}(t) \\ \tilde{h}(t) & \tilde{i}(t)\end{pmatrix} \cdot \frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t)\end{pmatrix}}{\|A(t)\|} \\ =&2\frac{a(t)\tilde{i}(t) - b(t)\tilde{h}(t) - c(t)\tilde{f}(t) + d(t)\tilde{e}(t)}{\|A(t)\|} \\ =&o(\|\gamma(t)\|^{r-2}). \end{aligned}$$

In a similar way we get that

$$2\frac{a(t)\tilde{j}(t) - b(t)\tilde{i}(t) - c(t)\tilde{g}(t) + d(t)\tilde{f}(t)}{\|A(t)\|} = o(\|\gamma(t)\|^{r-2}).$$

Let $\tilde{z}(t)$ be the singular 2- jet with $L_{\tilde{z}(t)} = (a(t), b(t), c(t), d(t))$ and $H_{\tilde{z}(t)} = H_\omega(\gamma(t)) = \left(\tilde{e}(t), \tilde{f}(t), \tilde{g}(t), \tilde{h}(t), \tilde{i}(t), \tilde{j}(t)\right)$. From above it is clear that

$$\frac{\left\|\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t)\end{pmatrix}\begin{pmatrix} a(t)\tilde{j}(t) - b(t)\tilde{i}(t) - c(t)\tilde{g}(t) + d(t)\tilde{f}(t) \\ -a(t)\tilde{i}(t) + b(t)\tilde{h}(t) + c(t)\tilde{f}(t) - d(t)\tilde{e}(t)\end{pmatrix}\right\|}{\left\|\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t)\end{pmatrix}\right\|^2} = o(\|\gamma(t)\|^{r-2}).$$

From (3.9) it is then clear that there exists a jet $z(t) = (L_{z(t)}, H_{z(t)}) \in \Gamma$ with $L_{z(t)} = (a(t), b(t), c(t), d(t))$ such that $\left\|H_\omega(\gamma(t)) - H_{z(t)}\right\| = o(\|\gamma(t)\|^{r-2})$. It follows that

$$\left\|L_\omega(\gamma(t)) - L_{z(t)}\right\| + \left\|H_\omega(\gamma(t)) - H_{z(t)}\right\|\|\gamma(t)\| = o(\|\gamma(t)\|^{r-1})$$

contradicting (I). □

Lemma 3.2. *Assume ω satisfies condition (I) of 2.1 with neighbourhood U and constant $C > 0$, then*

$$\|D\omega(p)\| \geq C\,\|p\|^{r-1}.$$

when $p \in U$.

Proof. It is clear that $(0, H_\omega(p)) \in \Gamma$ (where 0 is the zero-jet in $J^1(2,2)$) for each p, so

$$\|D\omega(p)\| = \|L_\omega(p) - 0\| + \|H_\omega(p) - H_\omega(p)\| \, \|p\| \geq C \, \|p\|^{r-1} .$$

and the lemma follows. □

Proof of Proposition 2.1. Let ω be as in Proposition 2.1 satisfying condition (I). As pointed out above, the function d is smooth at points p which are not singular and satisfy $|J\omega(p)| \neq \frac{1}{2} \|D\omega(p)\|^2$. Let the radius ρ_0 in the statement of Proposition 2.1 also be chosen so small that the conclusions of Lemma 3.1 and Lemma 3.2 hold when $U = B(0,\rho)$ and $0 < \rho < \rho_0$. From Lemma 3.2 it then follows that if d is not smooth at p and p is a regular point, then $|J\omega(p)| = \frac{1}{2} \|D\omega(p)\|^2 \geq \frac{C}{2} \|D\omega(p)\| \, \|p\|^{r-1}$, where C is given in Lemma 3.2. So, if $\epsilon < \frac{2-\sqrt{2}}{2} C$, it follows from inequality (3.2) that d is smooth in $(H_\epsilon - \Sigma(\omega)) \cap B(0,\rho)$ when $\rho < \rho_0$. Also assume that $\epsilon < C$ where this time C is the constant of Lemma 3.1. It follows that $(H_\epsilon - \{0\}) \cap B(0,\rho)$ contains no points in S when $\rho < \rho_0$.

The set $(H_\epsilon - \{0\}) \cap B(0,\rho)$ is semialgebraic and has consequently finitely many connected components, and each component C_i is contained in one such component. If ρ_0 is chosen small enough, we may apply Theorem 9.3.6 of [1], and conclude that $H_\epsilon \cap \overline{B(0,\rho)}$ is homeomorphic to the cone with vertex 0 and basis $H_\epsilon \cap \{p \mid \|p\| = \rho\}$. Since this basis is semialgebraic, and hence a finite union of closed segments and isolated points, it follows that each component of $(H_\epsilon - \{0\}) \cap \overline{B(0,\rho)}$ is a cone with the vertex 0 removed and with basis either a closed segment or a point of the circle $\{p \mid \|p\| = \rho\}$. Consider such a component H_k of $(H_\epsilon - \{0\}) \cap B(0,\rho)$ and a point $p \in H_k$. Assume H_k contains none of the components C_i. If p is an isolated point in $H_k \cap \{q \mid \|q\| = \|p\|\}$, then p is a local minimum of the function $d|_{\{q \mid \|q\| = \|p\|\}}$. If p is not isolated, then p is a point in $H_k \cap \{q \mid \|q\| = \|p\|\}$ and this set is a 1-dimensional compact curve which also must contain a local minimum of the function $d|_{\{q \mid \|q\| = \|p\|\}}$ in its interior. Since H_k does not contain any of the curves C_i, $d|_{\{q \mid \|q\| = \|p\|\}}$ is smooth at this local minimum so this minimum must be a point in S. From above we have that this is impossible.

If H_k contains two components C_i, C_j of $\Sigma(\omega) - \{0\}$, then $H_k \cap \{q \mid \|q\| = \|p\|\}$ contains a 1-dimensional compact curve such that the end-points of this curve are singular points and the interior points are non-singular. Then the function $d|_{\{q \mid \|q\| = \|p\|\}}$ must have a local maximum at an interior point of this curve. Again, this point must be a point in S which is impossible. We therefore conclude that it is impossible that a component of $(H_\epsilon - \{0\}) \cap B(0,\rho)$ contains several or no components of $\Sigma(\omega) - \{0\}$. This completes the proof of Proposition 2.1. □

Proof of Theorem 2.3. We will need the following lemma.

Lemma 3.3. *Let ω be a jet with $\Sigma(\omega) = \{0\}$ (as a set germ at 0). Consider the following inequality:*
There exist a constant C and a neighbourhood U of 0 such that

$$d(p) = d(j^1\omega(p), \Sigma) \geq C \, \|p\|^{r-1} \tag{I$'$}$$

for $p \in U$. Then (I$'$) is equivalent with the inequality (I) of Proposition 2.1.

Proof of Lemma 3.3. Assume that the inequality (I$'$) is not satisfied. Then we can find a sequence $p_n \to 0$ such that $d(j^1\omega(p_n), \Sigma) = o(\|p_n\|^{r-1})$. If p is a point such that $|J\omega(p)| =$

$\frac{1}{2}\|D\omega(p)\|^2$, then it follows from the estimates in (3.2) and Lemma 3.2 that

$$d(j^1\omega(p),\Sigma) \geq (2-\sqrt{2})\frac{|J\omega(p)|}{\|D\omega(p)\|} =$$
$$\frac{2-\sqrt{2}}{2}\|D\omega(p)\| \geq \frac{(2-\sqrt{2})}{2}C\|p\|^{r-1},$$

where C is given in Lemma 3.2. It follows from this and the existence of the sequence p_n that the function $d|_{\{p\mid \|p\|=\rho\}}$ must have an absolute minimum at points p where d is smooth, hence in the set S, when ρ is sufficiently small. Let p be such a point. From Lemma 3.1 it follows however that if $\|p\|$ is small then $d(p) \geq C\|p\|^{r-1}$ for some C independent of p, and since $d|_{\{p\mid \|p\|=\rho\}}$ attains an absolute minimum at p this contradicts the existence of the sequence p_n. So (I′) must be satisfied.
Let $\pi_1^2 : J^2(2,2) \to J^1(2,2)$ be the canonical projection. Then $\Gamma \subset (\pi_1^2)^{-1}(\Sigma)$ and from this, the implication (I′)⇒(I) is obvious. □

Let f and g be the components of ω. As pointed out in Remark 2, it follows from Lemma 3.3 that we only need to prove the equivalence of the inequality $d(j^1\omega(p),\Sigma) \geq C\|p\|^{r-1}$ and the inequality $d(\nabla f(p),\nabla g(p)) \geq C\|p\|^{r-1}$ of [2] (or [14]). From Subsection 3.2, we have $d(j^1\omega(p),\Sigma) \sim \frac{|J\omega(p)|}{\|D\omega(p)\|}$. From the definition in [2], we get that

$$d(\nabla f(p),\nabla g(p)) = \min\{||\nabla f(p) - \frac{\nabla f(p)\cdot\nabla g(p)}{||\nabla g(p)||^2}\nabla g(p)||, ||\nabla g(p) - \frac{\nabla g(p)\cdot\nabla f(p)}{||\nabla f(p)||^2}\nabla f(p)||\}.$$

If say, $\|\nabla f(p)\| \geq \|\nabla g(p)\|$, then a straightforward calculation shows that

$$d(\nabla f(p),\nabla g(p)) = \frac{|J\omega(p)|}{\|\nabla f(p)\|} \geq \frac{|J\omega(p)|}{\|D\omega(p)\|} \geq \frac{1}{\sqrt{2}}d(\nabla f(p),\nabla g(p)),$$

hence

$$d(\nabla f(p),\nabla g(p)) \sim \frac{|J\omega(p)|}{\|D\omega(p)\|}$$

and consequently

$$d(\nabla f(p),\nabla g(p)) \sim d(j^1\omega(p),\Sigma).$$

The conclusion of Theorem 2.3 follows from this. □

4. Stability of the Lojasiewicz inequalities

In this section we prove that the Lojasiewicz inequalities (I) of 2.1 and (II) of 2.2 are in some sense stable under perturbations of the jet by C^r- mappings with r-jet vanishing to r-th order at 0, and we derive some important geometrical consequences of the two Lojasiewicz inequalities.

4.1. **Lojasiewicz inequality (I)..** From now on, let $\omega = (f,g) \in J^r(2,2)$ for some $r \geq 2$ and with 0 not isolated in $\Sigma(\omega)$. Let $\tilde{\omega} = (\tilde{f},\tilde{g})$ be a C^r map with $j^r\tilde{\omega}(0) = 0$. For $t \in \mathbb{R}$, put $\omega_t(p) = \omega(p) + t\tilde{\omega}(p) = (f_t, g_t)$. Also, let $\epsilon > 0$ and let U be a neighbourhood of $0 \in \mathbb{R}^2$.

Lemma 4.1. *Assume that ω satisfies the condition (I) of Proposition 2.1 for some neighbourhood U of 0 and some constant $C > 0$. Then there are constants $0 < C' < C$ and $\epsilon > 0$ and a neighbourhood U' of 0 such that if $t \in (-\epsilon, 1+\epsilon)$, then condition (I) with constant C' holds for ω_t in U'.*

Proof. Let $(L,H) \in \Gamma$. By the triangle inequality,

$$\|L_{\omega_t}(p) - L\| \geq \|L_\omega(p) - L\| - |t| \, \|L_{\tilde{\omega}}(p)\| \geq \|L_\omega(p) - L\| - (1+\epsilon) \|L_{\tilde{\omega}}(p)\| \, ,$$

and similarly,

$$\|H_{\omega_t}(p) - H\| \geq \|H_\omega(p) - H\| - (1+\epsilon) \|H_{\tilde{\omega}}(p)\| \, .$$

Hence,

$$\begin{aligned} &\|L_{\omega_t}(p) - L\| + \|H_{\omega_t}(p) - H\| \, \|p\| \\ \geq &\|L_\omega(p) - L\| - (1+\epsilon) \|L_{\tilde{\omega}}(p)\| + \|H_\omega(p) - H\| - (1+\epsilon) \|H_{\tilde{\omega}}(p)\|) \, \|p\| \\ \geq &\frac{C}{2} \|p\|^{r-1} \end{aligned}$$

when U' is so small that $\frac{\|L_{\tilde{\omega}}(p)\|}{\|p\|^{r-1}} \leq \frac{C}{4(1+\epsilon)}$ and $\frac{\|H_{\tilde{\omega}}(p)\|}{\|p\|^{r-2}} \leq \frac{C}{4(1+\epsilon)}$. Such a neighbourhood U' exists for any $\epsilon > 0$ since $j^r\tilde{\omega}(0) = 0$ implies that $\|L_{\tilde{\omega}}(p)\| = o(\|p\|^{r-1})$ and that $\|H_{\tilde{\omega}}(p)\| = o(\|p\|^{r-2})$. Putting $C' = \frac{C}{2}$ completes the proof. $\square$

4.2. **Stability of Lojasiewicz inequality (II).** Let ω and ω_t be as in Subsection 4.1, but assume that $r > 2$. We assume that ω satisfies condition (I) of Proposition 2.1 and that U, C and ϵ are so small that by Lemma 4.1, (I) also is satisfied for ω_t, $t \in (-\epsilon, 1+\epsilon)$. Let $F : U \times (-\epsilon, 1+\epsilon) \to \mathbb{R}^3$ be the 1-parameter unfolding of ω given by $(p,t) \mapsto (\omega_t(p), t)$.

Lemma 4.2. *There are constants $C', \epsilon > 0$ such that if $t \in (-\epsilon, 1+\epsilon)$ and $p \in U \cap (\Sigma(\omega_t) \setminus \{0\})$, then*

$$\frac{\|\nabla J\omega_t(p)\|}{\|D\omega_t(p)\|} \geq C' \|p\|^{r-2} \, . \tag{4.1}$$

Proof. For $p \in U \cap (\Sigma(\omega_t) \setminus \{0\})$ we can choose H such that $(L_{\omega_t}(p), H) \in \Gamma$. Inequality (I) implies that for $t \in I = (-\epsilon, 1+\epsilon)$,

$$\|H_{\omega_t}(p) - H\| \, \|p\| \geq C \|p\|^{r-1} \, . \tag{4.2}$$

Choose H of this type, minimizing the distance $\|H_{\omega_t}(p) - H\|$. It follows from Schwartz inequality and (3.8) that

$$\frac{\|\nabla J\omega_t(p)\|}{\|D\omega_t(p)\|} \geq \frac{\left\| D\omega_t(p) \begin{pmatrix} \frac{\partial}{\partial y} J\omega_t(p) \\ -\frac{\partial}{\partial x} J\omega_t(p) \end{pmatrix} \right\|}{\|D\omega_t(p)\|^2} \geq \frac{1}{8} \|H_{\omega_t}(p) - H\| \geq \frac{C}{8} \|p\|^{r-2} \, . \tag{4.3}$$

The lemma follows by choosing $C' \leq \frac{C}{8}$. $\square$

Let $F_0 = F|_{(U \setminus \{0\}) \times (-\epsilon, 1+\epsilon)}$. It is easily seen that $JF(p,t) = J\omega_t(p)$. Thus, Lemma 4.2 implies that 0 is a regular value of JF_0 and we can conclude that $\Sigma(F_0)$ is a 2-dimensional C^{r-1} submanifold of $\mathbb{R}^3$. Define a vector field $\mathbf{v}$ on $\Sigma(F)$ by

$$\mathbf{v}(p,t) = \begin{cases} (0,0,1), & \text{if } p = 0 \\ \frac{\mathbf{p}_T(p,t)}{[\mathbf{p}_T(p,t)]_t}, & \text{otherwise,} \end{cases}$$

where $\mathbf{p}_T$ means the projection of $\mathbf{k} = (0,0,1)$ into the tangent plane of the manifold $\Sigma(F_0)$ and v_t denotes the t-component of any vector v. Notice that $\mathbf{v}_t \equiv 1$ on $\Sigma(F)$.

Lemma 4.3. $\|\mathbf{v}(p,t) - (0,0,1)\| = o\big(\|p\|\big)$.

Proof. In block-form the matrix of DF reads

$$DF = \begin{pmatrix} D\omega_t & \tilde{\omega} \\ 0 & 1 \end{pmatrix}.$$

As mentioned above, we see that $JF = 0 \Leftrightarrow J\omega_t = 0$. Put $h(p,t) = J\omega_t(p)$. Then $\Sigma(F) = h^{-1}(0)$, and hence, $\nabla h(p,t) \perp T_{(p,t)}\Sigma(F_0)$.

Let $\mathbf{p}_N(p,t)$ be the projection of $\mathbf{k} = (0,0,1)$ onto $\mathrm{sp}\{\nabla h(p,t)\}$. The projection $\mathbf{p}_T(p,t)$ of $\mathbf{k}$ into $T_{(p,t)}\Sigma(F_0)$ is

$$\mathbf{p}_T = \mathbf{k} - \mathbf{p}_N = \mathbf{k} - \frac{\frac{\partial h}{\partial t}}{\|\nabla h\|^2}\nabla h.$$

The t-component of $\mathbf{p}_T$ equals $\frac{\|\nabla J\omega_t\|^2}{\|\nabla h\|^2}$. Thus,

$$\mathbf{v} = \frac{\|\nabla h\|^2}{\|\nabla J\omega_t\|^2}\mathbf{k} - \frac{\frac{\partial h}{\partial t}}{\|\nabla J\omega_t\|^2}\nabla h.$$

Using that $\mathbf{v}_t = 1$, we get

$$\|\mathbf{v} - \mathbf{k}\| = \frac{\left|\frac{\partial h}{\partial t}\right|}{\|\nabla J\omega_t\|}.$$

Now, $\frac{\partial h}{\partial t} = \frac{\partial}{\partial t} J\omega_t$, where

$$J\omega_t = J\omega + t\left(\frac{\partial f}{\partial x}\frac{\partial \tilde{g}}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \tilde{g}}{\partial x} + \frac{\partial \tilde{f}}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial \tilde{f}}{\partial y}\frac{\partial g}{\partial x}\right) + t^2 J\tilde{\omega}.$$

From Lemma 3.2 we have that $\|D\omega(p)\| \geq C\|p\|^{r-1}$. Since

$$J\tilde{\omega}(p) = o(\|p\|^{r-1}\cdot\|p\|^{r-1})$$

and

$$\left(\frac{\partial f}{\partial x}\frac{\partial \tilde{g}}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \tilde{g}}{\partial x} + \frac{\partial \tilde{f}}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial \tilde{f}}{\partial y}\frac{\partial g}{\partial x}\right)(p) = o(\|p\|^{r-1})\,\|D\omega(p)\|,$$

we can conclude that $\frac{\partial h}{\partial t}(p,t) = o(\|p\|^{r-1})\,\|D\omega(p)\|$. By rearranging the terms of (4.1) we obtain

$$\frac{1}{\|\nabla J\omega_t(p)\|} \leq \frac{\|p\|^{2-r}}{C'\,\|D\omega_t(p)\|}.$$

Combining all this and the fact that $\|D\omega_t(p)\| = \|D\omega(p)\| + o(\|p\|^{r-1})$, we get

$$\begin{aligned}\|\mathbf{v}(p,t) - (0,0,1)\| &= \frac{\left|\frac{\partial h}{\partial t}(p,t)\right|}{\|\nabla J\omega_t(p)\|} \\ &= o(\|p\|^{r-1})\cdot\|D\omega(p)\|\cdot\frac{\|p\|^{2-r}}{C'\,\|D\omega_t(p)\|} \\ &= o(\|p\|).\end{aligned}$$

□

We are now going to extend the vector field $\mathbf{v}$ to a vector field ξ defined and continuous on all of $U \times (-\epsilon, 1+\epsilon)$. For simplicity, let $I = (-\epsilon, 1+\epsilon)$ and $U_0 = U - \{0\}$. Recall that $F_0 = F|_{U_0\times I}$.

Let $q \in U_0 \times I$, If $q \in \Sigma(F_0)$, we can find an open neighbourhood V of q in $\mathbb{R}^3$ and a C^{r-1}-diffeomorphism $\Phi : V \to W$ of V onto an open neighbourhood W of the origin in $\mathbb{R}^3$ such that $\Phi(V \cap \Sigma(F_0)) = W \cap (\mathbb{R}^2 \times \{0\})$. In W, define a vector field $\mathbf{v}_\Phi$ by

$$\mathbf{v}_\Phi(x,y,z) = D\Phi(\Phi^{-1}(x,y,0))\mathbf{v}(\Phi^{-1}(x,y,0)).$$

Now, put $V_q = V$ and define

$$\mathbf{w}_q(p,t) = D\Phi^{-1}(\Phi(p,t))\mathbf{v}_\Phi(\Phi(p,t))$$

for $(p,t) \in V_q$. When $q \in U_0 \times I - \Sigma(F)$, put $V_q = U_0 \times I - \Sigma(F)$ and define $w_q = (0,0,1)$ on V_q. Gluing these locally defined vector fields together by a partition of unity argument and scaling the resulting vector field such that the t-component becomes identically 1, we get a vector field ξ defined on $U_0 \times I$ extending $\mathbf{v}$. If the V_q's corresponding to points $q \in \Sigma(F_0)$ are chosen small enough, we obtain

$$\|\xi(p,t) - (0,0,1)\| = o(\|p\|). \tag{4.4}$$

We can extend ξ to all of $U \times I$ by defining $\xi(0,0,t) = (0,0,1)$.

This new vector field ξ is continuous, and by construction, ξ is C^{r-2} on $U_0 \times I$. We have assumed that $r > 2$, so ξ is at least C^1. Thus for every $p \in U_0 \times I$ there is a local flow line through p. Of course, the curve $\gamma : I \to U \times I$, $t \mapsto (0,0,t)$, is a flow line through every point of $\{0\} \times I$. Thus we have local solutions of ξ through every point of $U \times I$. Although ξ itself is not differentiable on the t-axis we will see that 4.4 is sufficient for ξ to have a continuous flow near the t-axis. In fact we have:

Lemma 4.4. *There is an open neighbourhood $U' \subset U$ of the origin in $\mathbb{R}^2$ and an injective continuous map $\phi : U' \times I \to \mathbb{R}^3$ such that $\forall\,(p,t) \in U' \times I$,*

$$\phi(p,t) = (h_t(p),t), \qquad \phi(p,0) = (p,0) \qquad \textit{and} \qquad \frac{\partial}{\partial t}\phi(p,t) = \xi(\phi(p,t)).$$

Proof. Equation 4.4 and the differentiability of ξ in $U_0 \times I$ imply that the Lipschitz condition of Theorem 2 in [8] is satisfied by ξ. Thus we can find the flow φ of Theorem 2 in [8]. From 4.4 and the fact that the t-component of ξ is 1, it clear that if U' is small enough the flow line through each $(p,0)$, $p \in U'$ must reach every t-level in I before it reaches the boundary of $U \times I$. Putting $\phi(p,t) = \varphi(t,(p,0))$ we get the desired map ϕ. Since ξ has t-component equal 1, ϕ can be written as $\phi(p,t) = (h_t(p),t)$ for some level-map h_t. □

Since ξ is tangent to $\Sigma(F)$, we get a map $\Sigma(\omega) \times \{0\} \to \Sigma(\omega_t) \times \{t\}$ given by $(p,0) \mapsto \phi(p,t) = (h_t(p),t)$. This is a homeomorphism of $\Sigma(\omega) \times \{0\}$ onto its image. The map ϕ therefore induces homeomorphisms $h_t|_{\Sigma(\omega)} : \Sigma(\omega) \to \Sigma(\omega_t)$.

Lemma 4.5. *Let $0 < \delta < \epsilon$, then* $\displaystyle\sup_{t \in [-\delta, 1+\delta]} \|h_t(p) - p\| = o(\|p\|)$.

Proof. Suppose there is a constant $K > 0$ and a sequence $\{p_n\}$ such that $\|p_n\| \to 0$ and

$$\sup_{t \in [-\delta, 1+\delta]} \|h_t(p_n) - p_n\| > K\,\|p_n\|\,.$$

Write $\phi(p,t) = \phi_p(t) = \big(\phi_p^1(t), \phi_p^2(t), t\big)$, $\xi(v,t) = \big(\xi^1(v,t), \xi^2(v,t), 1\big)$ and $p_n = (p_n^1, p_n^2)$. Applying the Mean Value Theorem and equation (4.4), we get that

$$\begin{aligned}
K\|p_n\| &< \sup_{t\in[-\delta,1+\delta]} \|h_t(p_n) - p_n\| \\
&= \sup_{t\in[-\delta,1+\delta]} \big\|\big(\phi_{p_n}^1(t), \phi_{p_n}^2(t)\big) - (p_n^1, p_n^2)\big\| \\
&\le 2(1+2\delta) \sup_{t\in[-\delta,1+\delta]} \left\|\big(\frac{\partial}{\partial t}\phi_{p_n}^1(t), \frac{\partial}{\partial t}\phi_{p_n}^2(t)\big)\right\| \\
&= 2(1+2\delta) \sup_{t\in[-\delta,1+\delta]} \big\|\big(\xi^1(\phi_{p_n}(t)), \xi^2(\phi_{p_n}(t))\big)\big\| \\
&= 2(1+2\delta)\big\|\big(\xi^1(v_n,t_n), \xi^2(v_n,t_n)\big)\big\| = o(\|v_n\|)
\end{aligned}$$

for (v_n, t_n) on the curve ϕ_{p_n} with

$$\big\|\big(\xi^1(v_n,t_n), \xi^2(v_n,t_n)\big)\big\| = \sup_{-\delta\le s\le 1+\delta} \big\|\big(\xi^1(\phi_{p_n}(s)), \xi^2(\phi_{p_n}(s))\big)\big\|.$$

Suppose $\|v_n\| < 2\|p_n\|$. Then we get the contradiction $K\|p_n\| < o(\|p_n\|)$. If this assumption is wrong, we can find a subsequence of $\{v_n\}$ with $\|v_n\| \ge 2\|p_n\|$. Let C be the trace of $\pi \circ \phi_{p_n}$, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection onto the first two coordinates. We consider the arc length of C, and see that

$$\begin{aligned}
\frac{1}{2}\|v_n\| \le \|v_n\| - \|p_n\| &\le \int_C \big\|\big(\xi^1(\phi_{p_n}(s)), \xi^2(\phi_{p_n}(s))\big)\big\|\, ds \\
&\le (1+2\delta)\big\|\big(\xi^1(v_n,t_n), \xi^2(v_n,t_n)\big)\big\| = o(\|v_n\|)
\end{aligned}$$

which is a new contradiction. The lemma follows. ☐

Lemma 4.6. *For small $\epsilon > 0, t \in I$, $\Sigma(\omega_t) \subset H_\epsilon$ in a neighbourhood of the origin in $\mathbb{R}^2$.*

Proof. From the proof of Lemma 4.3, we get

$$J\omega(p) = J\omega_t(p) + o(\|p\|^{r-1})\,\|D\omega(p)\|.$$

If $p \in \Sigma(\omega_t)$, $\frac{|J\omega(p)|}{\|D\omega(p)\|} = o(\|p\|^{r-1})$, and the lemma follows from (3.3) of Subsection 3.1. ☐

Remark 3. Let $\hat\omega$ be a C^r-realization of ω, then we can define a family $\omega_t = \omega + t(\hat\omega - \omega)$ of C^r-realizations such that $\omega_1 = \hat\omega$. Let $C_1, \dots, C_N$ be the connected components of $\Sigma(\omega)\setminus\{0\}$. Since $\Sigma(\omega)\setminus\{0\}$ and $\Sigma(\omega_t)\setminus\{0\}$ are homeomorphic, $\Sigma(\omega_t)\setminus\{0\}$ consists of N connected components for each t. Let h_t, $t \in I$ be the family of homeomorphisms constructed above. Since $h_0(C_i) = C_i$ and the set $\{h_t(p)\,|\,p \in C_i,\, t \in I\}$ is connected, it follows from the Lemma 4.6 and Proposition 2.1 that each $\Sigma(\omega_t)\setminus\{0\}$ has exactly one connected component in each H_i. So if $\epsilon > 0$ is chosen so small that the conclusion of 2.1 holds, each such realization $\hat\omega$ of ω has exactly one connected component of $\Sigma(\hat\omega)\setminus\{0\}$ in each connected component of $H_\epsilon - \{0\}$.

The corollary below gives a sort of stability property of inequality (II) under perturbation of the jet by C^r-mappings with r-jet vanishing at 0.

Corollary 4.7. *Let the hypothesis be as in Theorem 2.2, and assume that inequality (II) holds for ω with a constant $C > 0$. Let ω_t be as above. Then there exists a neighbourhood U of $0 \in \mathbb{R}^2$ such that if $t \in [0,1]$ and $p, q \in \Sigma(\omega_t) \cap U$ are points belonging to different components of $\Sigma(\omega_t)\setminus\{0\}$, then*

$$\|\omega_t(p) - \omega_t(q)\| \ge \frac{C}{2}(\|p\|^{r-1} + \|q\|^{r-1})\,\|p - q\|.$$

The inequality also holds if either p or q is equal 0.

Proof. Write $\omega_t = \omega + t\tilde{\omega}$. Since $j^r\tilde{\omega}(0) = 0$ we have $\|D\tilde{\omega}(p)\| = o(\|p\|^{r-1})$ where this time $\|D\tilde{\omega}(p)\|$ denotes the operator norm. From this follows that

$$\begin{aligned}\|\tilde{\omega}(p) - \tilde{\omega}(q)\| &= \left\|\int_0^1 D\tilde{\omega}(sp + (1-s)q)(p-q)\,ds\right\| \\ &\leq \sup_{s\in[0,1]} \|D\tilde{\omega}(sp+(1-s)q)\|\,\|p-q\| \\ &= o(\|p\|^{r-1} + \|q\|^{r-1})\,\|p-q\|\,.\end{aligned}$$

From Remark (3) we get that, if U is sufficiently small, then there exists $i, j,\ i \neq j$ such that $p \in H_i$ and $q \in H_j$. From inequality (II) and above it follows that

$$\begin{aligned}\|\omega_t(p) - \omega_t(q)\| &= \|(\omega(p) - \omega(q)) + t(\tilde{\omega}(p) - \tilde{\omega}(q))\| \\ &\geq \|\omega(p) - \omega(q)\| - |t|\,\|\tilde{\omega}(p) - \tilde{\omega}(q)\| \\ &\geq C(\|p\|^{r-1} + \|q\|^{r-1})\,\|p-q\| - o(\|p\|^{r-1} + \|q\|^{r-1})\,\|p-q\| \\ &\geq \frac{C}{2}(\|p\|^{r-1} + \|q\|^{r-1})\,\|p-q\|\,.\end{aligned}$$

Since $\|\tilde{\omega}(p)\| = o(\|p\|^r)$, the last statement of the corollary follows easily from (II) if say, $q = 0$. □

4.3. **Consequences of (I) and (II).** The inequalities (I) and (II) from Proposition 2.1 and Theorem 2.2 have several implications which will be important to us.

Lemma 4.8. *If $\omega \in J^r(2,2)$ satisfies (I) and (II) in a neighbourhood U of the origin, then there is a constant $K > 0$ such that $\|\omega(p)\| \geq K\,\|p\|^r$ for all p in a neighbourhood of the origin.*

Proof. Let

$$A = \left\{p \mid\ \|\omega(p)\| = \min_{\|q\|=\|p\|} \|\omega(q)\|\,,\ p, q \in U_0\right\}.$$

An application of the Tarski-Seidenberg Theorem shows that A is a semi-algebraic set. Hence, we can apply the curve selection lemma to find an analytic curve $\beta : [0,\epsilon) \to \mathbb{R}^2$ with $\beta(0) = 0$ and $\beta(0,\epsilon) \subset A$. Let s be chosen such that $\|\beta(t)\| \sim t^s$ as $t \to 0$. Assume that the lemma is false. Then $\|\omega(\beta(t))\| = o(\|\beta(t)\|^r) = o(t^{rs})$, and differentiation with respect to t gives $\|D\omega(\beta(t))\beta'(t)\| = o(t^{rs-1})$, and since we have that $\|\beta'(t)\| \sim t^{s-1}$ we obtain

$$\left\|D\omega(\beta(t))\frac{\beta'(t)}{\|\beta'(t)\|}\right\| = o(t^{rs-1-s+1}) = o(\|\beta(t)\|^{r-1}).$$

Since $\beta'(t)/\|\beta'(t)\|$ is a unit vector, it follows from (3.11) of Subsection 3.3 that $d(j^1\omega(\beta(t)), \Sigma) = o(\|\beta(t)\|^{r-1})$ and we get that $\beta(t) \in H_\epsilon$. From (II) with $p = \beta(t)$ and $q = 0$, we get that $\|\omega(\beta(t))\| \geq C\,\|\beta(t)\|^r$, which is a contradiction. □

Corollary 4.9. *Suppose (I) and (II) hold. Then there is a neighbourhood U of the origin and a constant $K > 0$ such that*

$$\|\omega_t(p)\| \geq K\,\|p\|^r$$

for all $t \in I$ and $p \in U$.

Proof. This follows easily from Lemma 4.8 since $\|\omega_t(p)\| = \|\omega(p)\| + o(\|p\|^r)$. □

Remark 4. The hypothesis of Lemma 4.8 can be weakened. In fact, the lemma follows from inequality (I) alone. This can be seen as follows: If there is a sequence $p_n \to 0$ such that $\|\omega(p_n)\| = o(\|p_n\|^r)$, then we may apply a variant of the technique in the proof of Lemma 4.10 below to show that ω has a C^r- representative which is identically equal 0 along some non-constant curve starting at 0. Such a representative has singular points different from folds along this curve, and hence cannot satisfy (I). This will however contradict the conclusion of Lemma 4.1.

Lemma 4.10. *Let $r > 2$. Let $\omega = (f, g) \in J^r(2, 2)$ be as in the hypothesis of Proposition 2.1 and assume ω satisfies (I) of Proposition 2.1, then there is a neighbourhood U of 0 and a constant $C > 0$ such that for each i either*

$$\forall p \in H_i, \ \|\nabla f(p)\| \geq C \|p\|^{r-1}$$

or

$$\forall p \in H_i, \ \|\nabla g(p)\| \geq C \|p\|^{r-1}.$$

Proof. Assume the lemma is false. Then, by the technique employed in the proof of Lemma 4.8, there exist analytic curves $\beta(t)$ and $\gamma(t)$, $t \in [0, \delta)$ with $\beta(0) = \gamma(0) = (0, 0)$, $\beta(0, \delta), \gamma(0, \delta) \subset H_i$ for sufficiently small $\delta > 0$ such that

$$\|\nabla f(\beta(t))\| = o(\|\beta(t)\|^{r-1}) \tag{4.5}$$

and

$$\|\nabla g(\gamma(t))\| = o(\|\gamma(t)\|^{r-1}), \tag{4.6}$$

for $t > 0$. We claim that

$$f(\beta(t)) = o(\|\beta(t)\|^r). \tag{4.7}$$

To see this, assume $\|\beta(t)\| \sim t^s$ and let u be such that $|f(\beta(t))| \sim t^u$. Then $|\frac{d}{dt} f(\beta(t))| \sim t^{u-1}$, and also

$$|\frac{d}{dt} f(\beta(t))| = |\nabla f(\beta(t)) \cdot \beta'(t)| \leq \|\nabla f(\beta(t))\| \cdot \|\beta'(t)\| = o(t^{sr-1}).$$

It follows that $u - 1 > sr - 1$ and the claim follows from this. In the same manner we get

$$g(\gamma(t)) = o(\|\gamma(t)\|^r). \tag{4.8}$$

We consider the curve β, and follow an argument of Kuo's article [9]. By a suitable rotation of $\mathbb{R}^2$ we can make β tangent to the x-axis at 0. Assume this is the case. By a change of parameter if necessary, $\beta_1(t) = t^s$ and $|\beta_2(t)| = o(t^s)$. We make a C^1 change of coordinates: $X = x$, $Y = y - \beta_2(|x|^{\frac{1}{s}})$. In these coordinates, β is the positive X-axis.

Using the Taylor expansion of f about 0, we can write f as a polynomial in Y as follows:

$$f(x, y) = f(X, Y + \beta_2(|X|^{\frac{1}{s}})) = \tilde{f}_0(X) + \tilde{f}_1(X)Y + \tilde{f}_2(X)Y^2 + \cdots \tag{4.9}$$

Putting $Y = 0$, we get that $\tilde{f}_0(X) = f(X, \beta_2(|X|^{\frac{1}{s}}))$ and we see from (4.7) that the function $f_0(x) = \tilde{f}_0(|x|)$, is a C^r map with $j^r f_0(0) = 0$. Differentiating (4.9) with respect to Y and putting $Y = 0$, we see that $\tilde{f}_1(X) = \frac{\partial f}{\partial y}(X, \beta_2(|X|^{\frac{1}{s}}))$, and it follows from (4.5) that the function $f_1(x) = \tilde{f}_1(|x|)$ is a C^{r-1} function with $j^{r-1} f_1(0) = 0$.

Let $K = \{(x, y) \,|\, |y| \leq |x|, x \geq 0\} \cap \overline{B_r(0)}$ where $B_r(0)$ is some small open ball around 0. Define $\tilde{F}(x, y) = f_1(x)(y - \beta_2(|x|^{\frac{1}{s}}))$. $\tilde{F}$ is analytic at points (x, y) with $x \neq 0$. From (4.5) it follows that $\frac{d^m}{dx^m} f_1(x) = o(|x|^{r-1-m})$ for $m \geq 0$. Furthermore since $\beta_2(t) = o(t^s)$, we get that $\frac{\partial^m}{\partial x^m}(y - \beta_2(|x|^{\frac{1}{s}})) = o(|x|^{1-m})$ when $m > 0$. Also, $|y - \beta_2(|x|^{\frac{1}{s}})| \leq 2|x|$ when

$(x,y) \in K$. Altogether this implies that $\frac{\partial^{|m|}}{\partial x^{m_1} \partial y^{m_2}}(\tilde{F})(p) = o(|(x,y)|^{r-|m|})$ with $m = (m_1, m_2)$ when $p = (x,y) \in K - \{0\}$.

Now let Q be the r-th order Taylor field on K with values in $\mathbb{R}$ defined by $Q^m(0) = 0$ for all m and $Q^m(p) = \frac{\partial^{|m|}}{\partial x^{m_1} \partial y^{m_2}}(\tilde{F})(p)$ for all $m = (m_1, m_2)$ and all $p \in K \setminus \{0\}$. It follows from Lemma 4.11 below that Q is a C^r-Whitney field. Thus, by Whitney's Extension Theorem Q has a C^r-extension F defined on a neighbourhood of $0 \in \mathbb{R}^2$ such that $j^r F(0) = 0$ (see [10] for a statement and proof of Whitney's Extension Theorem).

Apply the same construction to g along γ to obtain g_0 and G as C^r-functions both with r-jet equal 0 at (0). Then define

$$\hat{\omega} = (\hat{f}, \hat{g}) = (f - f_0 - F, g - g_0 - G).$$

Then $\hat{\omega}$ is a C^r-realization of ω, and by construction, $\nabla \hat{f} = 0$ along $\beta(t)$ and $\nabla \hat{g} = 0$ along $\gamma(t)$. If the traces of β and γ are the same, then obviously $\hat{\omega}$ has singularities which are not folds along this curve, which contradicts Lemma 4.1. If the traces of β and γ are not intersecting in a neighbourhood of 0, then we have found a C^r-realization $\hat{\omega}$ of ω such that $\Sigma(\hat{\omega}) \setminus \{0\}$ has at least two connected components in H_i. This will however contradict Remark 3. □

Lemma 4.11. *Let $U \subset \mathbb{R}^n$ be an open set with $0 \in \overline{U}$. Let F be a C^r-function defined on U. Assume that $\frac{\partial^{|\alpha|} F}{\partial x^\alpha}(p) \to 0$ when $p \to 0$ for each multiindex α with $|\alpha| \le r$. Let $K \subset \{0\} \cup U$ be a compact, convex set with $0 \in K$. Let Q be the r-th order Taylor field on K defined by $Q^\alpha(p) = \frac{\partial^{|\alpha|} F}{\partial x^\alpha}(p)$ if $p \neq 0$, and $Q^\alpha(p) = 0$ if $p = 0$, $|\alpha| \le r$. Then Q is a C^r-Whitney field.*

Proof. Let $p, q \in K$. Let $m = (m_1, \ldots, m_n)$ be a multiindex with $|m| \le r$. Let

$$R_q Q^m(p) = Q^m(p) - \frac{\partial^{|m|}}{\partial x^m}\Big(\sum_{|\alpha| \le r} \frac{1}{\alpha_1! \ldots \alpha_n!} Q^\alpha(q)(x-q)^\alpha\Big)\Big|_{x=p}.$$

We must show that $R_q Q^m(p) = o(\|p-q\|^{r-|m|})$ for each such multiindex m. We will only show this when $m = \mathbf{0} = (0, \ldots, 0)$ since the proof is similar when $|m| > 0$. Extend F to $\{0\} \cup U$ by putting $F(0) = 0$. Let $p, q \in K$ and define $g(t) = F(tp + (1-t)q)$. Then g can be extended to a C^r function on some open interval containing $[0,1]$ (if q or p is 0 extend g to the zero-function on $(-\epsilon, 0)$ or $(1, 1+\epsilon)$ respectively). Note that

$$g^{(k)}(t) = \sum_{|\alpha|=k} \frac{k!}{\alpha_1! \ldots \alpha_n!} Q^\alpha(tp + (1-t)q)(p-q)^\alpha,$$

for $t \in [0,1]$. So, by applying an integral version of Taylor's formula with remainder, we get

$$\begin{aligned} R_q Q^{\mathbf{0}}(p) = g(1) - \sum_{k=0}^{r} \frac{1}{k!} g^{(k)}(0) &= \frac{1}{(r-1)!} \int_0^1 g^{(r)}(t)(1-t)^{r-1} dt - \frac{1}{r!} g^{(r)}(0) \\ &= \frac{1}{(r-1)!} \int_0^1 (g^{(r)}(t) - g^{(r)}(0))(1-t)^{r-1} dt \\ &= \frac{1}{(r-1)!} \sum_{|\alpha|=r} \Big(\int_0^1 \frac{r!}{\alpha_1! \ldots \alpha_n!} (Q^\alpha(tp + (1-t)q) - Q^\alpha(q))(1-t)^{r-1} dt\Big)(p-q)^\alpha. \end{aligned}$$

Now, each Q^α is continuous on the compact, convex set K and therefore uniformly continuous, and from this it follows easily that $\int_0^1 (Q^\alpha(tp + (1-t)q) - Q^\alpha(q))(1-t)^{r-1} dt \to 0$ when $\|p-q\| \to 0$. Since $|(p-q)^\alpha| \le \|p-q\|^r$ when $|\alpha| = r$, we thus get that $R_q Q^{\mathbf{0}}(p) = o(\|p-q\|^r)$. □

Lemma 4.12. *If $\omega = (f, g) \in J^r(2,2)$ satisfies the inequalities (I) and (II) in a neighbourhood U of 0, then there is a smaller neighbourhood U' of 0 such that $F|_{\Sigma(F)\cap(U'\times I)}$ is injective.*

Proof. It is enough to show that ω_t is injective when restricted to $\Sigma(\omega_t)$. Consider the component H_i for some i. By Lemma 4.10 we may assume that $\|\nabla f(p)\| \geq C\,\|p\|^{r-1}$ for all p in H_i. Then there is a smaller neighbourhood V of 0 such that $\|\nabla f_t(p)\| \geq \frac{C}{2}\,\|p\|^{r-1}$ for all t and $p \in H_i$. As before, $j^2\omega_t(p)$ is identified with the 10-tuple $(a, \dots, j)$. $\Sigma(\omega_t)$ is given by the equation $ad - bc = 0$. Suppose $f_t|_{\Sigma(\omega_t)}$ has an extremum at $p \in H_i \cap \Sigma(\omega_t)$. By the method of Lagrange multipliers, at p,

$$a = \lambda \cdot \frac{\partial J\omega_t}{\partial x} = \lambda(ai - bh - cf + de)$$

$$b = \lambda \cdot \frac{\partial J\omega_t}{\partial y} = \lambda(aj - bi - cg + df).$$

We have $(a, b) = \|\nabla f_t(p)\| \geq \frac{C}{2}\,\|p\|^{r-1} \neq 0$ which implies that $\lambda \neq 0$ and hence,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} aj - bi - cg + df \\ -ai + bh + cf - de \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This means that $(L_{\omega_t}(p), H_{\omega_t}(p)) \in \Gamma$. The conclusion must be that every such p lies outside some open neighbourhood of the origin, since ω_t by assumption satisfies (I). Hence f_t and consequently ω_t is injective when restricted to the component of $\Sigma(\omega_t)\backslash\{0\}$ lying in H_i. Together with Corollary 4.7, this proves the lemma. $\square$

Recall the definition of F_0 given above Lemma 4.3. Let $M = \Sigma(F_0)$ and $\Omega = F(M)$.

Lemma 4.13. *Let U be chosen so small that the conclusions of Lemma 4.1 and Lemma 4.12 hold. Then Ω is a two-dimensional C^{r-1} submanifold of the target.*

Proof. $F|_{\overline{M}}$ is an injective continuous map from a compact space to a Hausdorff space, so it must be a homeomorphism onto its image. So $F|_M$ is a topological embedding and by Lemma 4.1, $F|_M$ is a C^{r-1} immersion, hence a C^{r-1} embedding. Thus Ω is a C^{r-1} manifold. $\square$

5. Construction of trivializing vector fields in source and target

Let $\omega \in J^r(2,2)$ be as in the hypothesis of Proposition 2.1, assume that $r > 2$ and that ω satisfies the inequalities (I) and (II) in a neighbourhood U of 0. Let F, M and Ω be as in Section 4, and assume that the neighborhood U in the definition of M and Ω also is chosen so small that the conclusion of Corollary 4.9 holds. Clearly, Corollary 4.9 implies that $F((\overline{U}-U)\times I)\cap\{0\}\times I = \varnothing$. It follows that we can find a neighborhood V of 0 in $\mathbb{R}^2$ such that $(\Omega\cup(\{0\}\times I))\cap(V\times I)$ is closed in $V\times I$. Let us change notation and denote $\Omega\cap(V\times I)$ by Ω. Let $(q, t) = (\omega_t(p), t) = F(p, t) \in \Omega$ for $(p, t) \in M$. Since $F|_M$ has rank 2 everywhere, $DF(p, t)\mathbf{v} \in T_{F(p,t)}\Omega$ for all $\mathbf{v} \in \mathbb{R}^3$. With this in mind, we can define a tangent vector field $\mathbf{u}$ on Ω by

$$\mathbf{u}(q, t) = \mathbf{u}(F(p, t)) = DF((p, t))(0, 0, 1) = (\tilde{\omega}(p), 1).$$

Since $\omega_t(p) = \omega(p) + t\tilde{\omega}(p)$ and $\|\tilde{\omega}(p)\| = o(\|p\|^r)$, it follows from Corollary 4.9 that

$$\|\mathbf{u}(F(p, t)) - (0, 0, 1)\| = o(\|\omega_t(p)\|) = o(\|q\|).$$

This equation is similar to the conclusion of Lemma 4.3. Put $\mathbf{u}|_{\{0\}\times I} = (0, 0, 1)$. Since Ω is a C^{r-1} manifold in the target, $\mathbf{u}$ can be extended to a neighborhood $V \times I$ of $\{0\} \times I$ in a way completely analogous to the way the vector field $\mathbf{v}$, defined in Subsection 4.2, was extended to all of source. We scale this extended vector field such that the component in the t-direction becomes 1 and denote this vector field by η. By this construction, η becomes C^{r-2} outside the t-axis, and we get the following lemma which is similar to (4.4).

Lemma 5.1. $\|\eta(q,t)-(0,0,1)\| = o(\|q\|)$.

This lemma implies that η like the vector field ξ constructed in Subsection 4.2 satisfies the hypothesis of Kuo's Theorem 3 in [8]. Therefore η has a continuous flow ψ in $V \times I$. Moreover, since the component of η in the t-direction equals 1, each flow line will live until it reaches either $(\overline{V}-V)\times I$ or $V \times \{-\epsilon, 1+\epsilon\}$. An easy estimate using 5.1 shows that if $V_1 \subset V$ is sufficiently small and $(q,t) \in V_1 \times I$ then $\psi_{(q,t)}$ will stay close to $\{0\} \times I$ and therefore reach $V \times \{-\epsilon, 1+\epsilon\}$ and therefore cannot have any closure points in $(\overline{V}-V)\times I$. So when $(q,t) \in V_1 \times I$ we can define the flow $\psi_{(q,t)}(s)$ for $s \in (-\epsilon - t, 1+\epsilon-t)$, especially each flow line through points in $V_1 \times \{0\}$ can be defined on I, and we will get a map $k : V_1 \times I \to \mathbb{R}^3$ defined by $k(q,t) = k_t(q) = \psi_{(q,0)}(t)$. Each k_t is a homeomorphism which maps the 0-level of Ω to the t-level of Ω. Let us choose such a neighborhood V_1 and let $U_1 \subset U$ be a neighborhood of 0 in $\mathbb{R}^2$ such that $F(U_1 \times I) \subset V_1 \times I$. Define a tangent vector field $\mathbf{w}$ on $M \cap (U_1 \times I)$ by

$$DF((p,t))\mathbf{w}(p,t) = \mathbf{u}(F(p,t)).$$

This definition is unambiguous because we have required $\mathbf{w}$ to be tangential and $F|_{M\cap(U_1\times I)} : M \cap (U_1 \times I) \to \Omega$ is an immersion. Put $\mathbf{w}|_{\{0\}\times I} = (0,0,1)$. Outside $M \cup \{0\} \times I$, DF is invertible so we can define an extension ζ of $\mathbf{w}$ to all of source by the equation

$$DF_{(p,t)}\zeta(p,t) = \eta(F(p,t)).$$

We are now going to show that ζ has a continuous flow. To this end, we will need the lemma below.

Lemma 5.2. *If $p \in M$, then there is a neighbourhood W of p such that for all $q \in W$, $F(q) \in \Omega \Rightarrow q \in M$.*

Proof. Let $p \in M$. Then p is a fold point and if $r \geq 4$, this will follow from the standard normal form of a fold. When $r > 2$, there are (for example following the arguments in [15] Section 15), C^{r-1}-coordinates (x,y,t) around p, (u,v,t) around $F(p)$ in which $p = (0,0,0) = F(p)$ and such that in these coordinates F has the form $F(x,y,t) = (x, h(x,y,t), t)$ where $h(x,0,t) = \frac{\partial h}{\partial y}(x,0,t) = 0 \neq \frac{\partial^2 h}{\partial y^2}(0,0,0)$. In these coordinates, $\Sigma(F) = \{y=0\}$ and $F(\Sigma(F)) = \{v = 0\}$. The lemma now follows by an easy argument using Taylor's formula. □

The existence and continuity of the flow of η is given in the following lemma.

Lemma 5.3. *Let $0 < \delta < \epsilon$. Then there exists a neighborhood $U_2 \subset U_1$ such that ζ has a continuous flow $\vartheta(p,t,s) = \vartheta_{(p,t)}(s)$ in the set $\{(p,t,s) \mid (p,t) \in U_2 \times (-\delta, 1+\delta),\ s \in (-\delta - t, 1 + \delta - t)\}$.*

Proof. Again, change notation and put $M := M \cap (U_1 \times I)$. Consider $\{\{0\}\times I, M, U_1 \times I \setminus \Sigma(F)\}$ as a stratification of $U_1 \times I$. We can think of ζ as a stratified vector field whose restriction to each stratum is a C^{r-2}-vector field. These restrictions have each a C^{r-2}- flow defined on each stratum. For each $p = (x,y,t) \in U_1 \times I$, let $\vartheta(p,s) = \vartheta_p(s)$ denote the flow through p of the restriction of ζ to the stratum of p. Let ϑ_p be defined on its maximal interval of existence. Now we will prove that this flow is continuous, by using the continuous flow in the target to control the flow in the source.

To this end, consider the vector field η in the target which also can be considered as a stratified vector field with respect to the stratification $\{\{0\} \times I, \Omega, (V \setminus \{0\}) \times I \setminus \Omega\}$. Since η has a continuous flow on $V \times I$ and each flow line lives until it reaches the boundary of $V \times I$, each flow line stays in its respective stratum and no flow line can have closure points belonging to lower dimensional strata. From the equation $DF((p,t))\zeta(p,t) = \eta(F(p,t))$, we get that the flow of ζ is mapped to the flow ψ of η. Let $p \in (U_1 - \{0\}) \times I$. Then the flow line ϑ_p is mapped to the flow line $\psi_{F(p)}$ which is a flow line either in Ω or in $(V \setminus \{0\}) \times I \setminus \Omega$, and therefore cannot have a closure point in $\{0\} \times I$. It follows that ϑ_p cannot have a closure point in $\{0\} \times I$ either. By the same sort of arguments it follows that if $F(p) \in (V \setminus \{0\}) \times I \setminus \Omega$ then ϑ_p cannot have a closure point in M either. When $p \in U_1 \times I \setminus \Sigma(F)$ and $F(p) \in \Omega$, $F(\vartheta_p)$ is a flow line of η in Ω. It then follows from Lemma 5.2 that ϑ_p cannot have a closure point in M either. So, for each $p \in U \times I$, each flow line ϑ_p does not have closure points in lower dimensional strata and since the component of ζ in the t-direction equals 1, each flow line ϑ_p can be continued until it meets the boundary of $U_1 \times I$.

Let U' be a neighborhood of $0 \in \mathbb{R}^2$ such that $\overline{U}' \subset U_1$, and let $0 < \delta < \epsilon$. We will prove that there exists another neighborhood $\tilde{U} \subset \overline{\tilde{U}} \subset U'$ such that flow lines of ζ through points in $\overline{\tilde{U}} \times [-\delta, 1+\delta]$ cannot have closure points in $(\overline{U'} - U') \times [-\delta, 1+\delta]$.

Corollary 4.9 implies that there exists $\rho > 0$ such that $B(0,\rho) \times [-\delta, 1+\delta] \subset V_1 \times I$ and $F((\overline{U'} - U') \times [-\delta, 1+\delta]) \subset \mathbb{R}^3 \setminus (B(0,\rho) \times [-\delta, 1+\delta])$, where $B(0,\rho)$ is the ball around $0 \in \mathbb{R}^2$ with radius ρ. Since the flow ψ of η is continuous, we can find $\rho_1 < \rho$ such that when $(q,t) \in B(0,\rho_1) \times [-\delta, 1+\delta]$ the flow line $\psi_{(q,t)}(s)$ stays in $B(0,\rho) \times [-\delta, 1+\delta]$ for $s \in [-\delta - t, 1+\delta - t]$. By continuity of F, let $\tilde{U} \subset U'$ be such that $F(\overline{\tilde{U}} \times [-\delta, 1+\delta]) \subset B(0,\rho_1) \times [-\delta, 1+\delta]$. Let $(p,t) \in \overline{\tilde{U}} \times [-\delta, 1+\delta]$. Then the flow $\vartheta_{(p,t)}(s)$ is mapped to $\psi_{F(p,t)}(s)$ and since the latter flow stays in $B(0,\rho) \times [-\delta, 1+\delta]$ for $s \in [-\delta - t, 1+\delta - t]$ and $(\overline{U'} - U') \times [-\delta, 1+\delta]$ is mapped to the complement of $B(0,\rho) \times [-\delta, 1+\delta]$, the flow $\vartheta_{(p,t)}(s)$ can never meet $(\overline{U'} - U') \times [-\delta, 1+\delta]$ but must stay in $U' \times [-\delta, 1+\delta]$ when $s \in [-\delta - t, 1+\delta - t]$. Putting $U_2 = \tilde{U}$ the above argument shows that ζ has a flow $\vartheta(x,y,t,s)$ in

$$\hat{U} = \{(p,t,s) \,|\, (p,t) \in U_2 \times [-\delta, 1+\delta],\ s \in [-\delta - t, 1+\delta - t]\}.$$

Since U' can be chosen arbitrarily small, the argument also shows that this flow is continuous in $\{0\} \times (-\delta, 1+\delta)$.

Since ζ is a C^{r-2} vector field in the open set $U_1 \times I - \Sigma(F)$ and we have seen that the flow stays in this set until it meets the closure of $U_1 \times I$ the flow is continuous in this set. Especially the flow $\vartheta(p,t,s)$ is continuous when $(x,y,t) \in U_2 \times (-\delta, 1+\delta) - \Sigma(F)$ and $s \in (-\delta - t, 1+\delta - t)$.

We will show that by replacing U_2 with a smaller neighbourhood U_3, we will get a continuous flow at all points. To this end, let $U_3 \subset \overline{U_3} \subset U_2$ be a neighborhhood of 0 of $\mathbb{R}^2$ such that when $(p,t) \in \overline{U_3} \times [-\delta, 1+\delta]$ then $\vartheta_{(p,t)}(s) \in U_2 \times [-\delta, 1+\delta]$ for $s \in [-\delta - t, 1+\delta - t]$. (Such a neighbourhood exists since we have seen that the flow ϑ is continuous in $\{0\} \times (-\delta, 1+\delta)$.) It remains to see that ϑ is continuous at points (p,t,s) when $(p,t) \in U_3 \times (-\delta, 1+\delta) \cap M$, and $s \in (-\delta - t, 1+\delta - t)$. Assume this is not the case. Then there exist such (p,t,s) and a sequence $(p_n, t_n, s_n) \to (p,t,s)$ such that $\vartheta_{(p_n,t_n)}(s_n) \nrightarrow \vartheta_{(p,t)}(s)$. Since the restriction of ζ is C^{r-2} on M and the restriction of the flow therefore is continuous there, we must have $(p_n, t_n) \in U_3 \times (-\delta, 1+\delta) \setminus M$. Since the flow lines in $U_3 \times [-\delta, 1+\delta]$ stay in $U_2 \times [-\delta, 1+\delta]$, the sequence $\vartheta_{(p_n,t_n)}(s_n)$ is contained in the compact subset $\overline{U_2} \times [-\delta, 1+\delta]$ and we may therefore assume that it converges to some point $(\tilde{p}, t+s) \in \overline{U_2} \times [-\delta, 1+\delta]$. Since the flow in the source is mapped to the flow in the target and the flow in the target is continuous, we get that $F(\tilde{p}, t+s) = F(\vartheta_{(p,t)}(s))$. Since the flow line $\vartheta_{(p,t)}(s)$ is in M and $F|\Sigma(F)$ is 1-1, $(\tilde{p}, t+s) \in \overline{U_2} \times (-\delta, 1+\delta) \setminus \Sigma(F)$. Since the flow $\vartheta_{(p',t)}(s)$ through points (p', t) in $\overline{U_2} \times [-\delta, 1+\delta]$ stays in $U_1 \times [-\delta, 1+\delta]$ and can be defined for $s \in [-\delta - t, 1+\delta - t]$, $\vartheta_{(\tilde{p}, t+s)}(-s)$ is defined. Since the flow ϑ is continuous on $U_1 \times I \setminus \Sigma(F)$, $(p_n, t_n) = \vartheta_{\vartheta_{(p_n,t_n)}(s_n)}(-s_n) \to \vartheta_{(\tilde{p}, t+s)}(-s)$. This implies that $\vartheta_{(\tilde{p}, t+s)}(-s) = (p,t)$ which is impossible since flow lines in $U_1 \times I \setminus \Sigma(F)$ never meet M. Putting $U_2 := U_3$ we thus get continuity of the flow ϑ in $\{(p,t,s) \,|\, (p,t) \in U_2 \times (-\delta, 1+\delta),\ s \in (-\delta - t, 1+\delta - t)\}$. $\square$

When $r > 4$, we only need to check continuity of the flow ϑ at points in the t-axis, the remaining cases we treat above will follow automatically from the lemma below.

Lemma 5.4. *If $r > 4$, then $\zeta|_{U_0 \times I}$ is C^{r-4}.*

Proof. Let $p = (x_p, y_p, t_p) \in M$. Then p is a fold point of ω_{t_p}, and by Theorem 15A of [15], there are suitable centered coordinates H around p and K around $F(p)$ such that $(u, v, t) = K \circ F \circ H(x, y, z) = (x, y^2, t)$. If we look closely into the proof of this theorem we find that K can be chosen to be C^{r-1} and H to be C^{r-3}. We know that both $\zeta = (\zeta^1, \zeta^2, \zeta^3)$ and $\eta = (\eta^1, \eta^2, \eta^3)$ are tangential on M and Ω respectively, and hence, $\zeta^2(x, 0, t) = \eta^2(u, 0, t) = 0$. Thus, since K is C^{r-1} and η is C^{r-2}, we can, in the new coordinates, write $\eta^2(u, v, t) = v\eta'(u, v, t)$ for some C^{r-3} function η'. For $y \neq 0$, we get from the definition of ζ that

$$DF_{(x,y,t)}\zeta(x, y, t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta^1(x, y, t) \\ \zeta^2(x, y, t) \\ \zeta^3(x, y, t) \end{pmatrix} = \begin{pmatrix} \eta^1(x, y^2, t) \\ y^2\eta'(x, y^2, t) \\ \eta^3(x, y^2, t) \end{pmatrix}.$$

From this relation we see that $\zeta^2(x, y, t) = \frac{1}{2}y\eta'(x, y^2, t)$. Because $\zeta^2(x, 0, t) = 0$, we see that the same equation must hold also for $y = 0$. Hence we can conclude that ζ is C^{r-3} in our new coordinates around p, and since DH is C^{r-4}, ζ is C^{r-4} in $U_0 \times I$ where U has been shrinked as to be contained in $F^{-1}(V)$. $\square$

Proof of the sufficiency part of Theorem 2.2. Consider the neighborhood V_1 of $0 \in \mathbb{R}^2$ and the homeomorphisms k_t defined on V_1, in the beginning of this section. Let $0 < \delta < \epsilon$ and let U_2 be the neighborhood of 0 given in Lemma 5.3. Since the flow $\vartheta(p, t, s)$ of the vector field ζ can be defined and is continuous for $p \in U_2$, $t \in (-\delta, 1 + \delta)$ and $s \in (-\delta - t, 1 + \delta - t)$, we can define $h_t : U_2 \to \mathbb{R}^2$ by the equation $(h_t(p), t), = \vartheta(p, 0)(t)$. Since the flow is continuous it is clear that each h_t is a homeomorphism onto its image. Since the flow ϑ is mapped by $F = (\omega_t, t)$ to the flow ξ in the target, it follows from the definition of k_t that $\omega_t(h_t(p)) = k_t(\omega(p))$. For $t = 1$, this means precisely that ω and $\omega_1 = \omega + \tilde{\omega}$ are $\mathcal{A}_0$-equivalent. Since $\tilde{\omega}$ was arbitrarily chosen, this means that ω is $\mathcal{A}_0$-sufficient. $\square$

6. Realizations of r-jets

Every r-jet has a quite well behaved realization in the sense to be made precise below. If an r-jet fails to satisfy (I) or (II), then it has another realization with different topological properties. We use this to prove the necessity part of Theorem 2.2.

6.1. **A nice realization of an r-jet.**

In this section we show that every r-jet has a realization which has no singular double points and only fold points and regular points outside the origin. Let $\omega : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be an r-jet.

Lemma 6.1. *There is some finite determined smooth germ f with $j^r f = \omega$.*

Proof. This is true because for smooth germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, finite determinacy holds in general. See [6] for details. $\square$

Since f is finitely determined, we can assume that f is a polynomial map. Also, the germ f is stable outside the origin. From the classification of stable germs we conclude that every singular point of $f|_{U_0}$ is either a fold or a cusp. Moreover the only singular double points occuring are double points of folds in general positions, which are isolated.

Lemma 6.2. *f has no singular double points and only fold points and regular points outside the origin.*

Proof. Let
$$C_1 = \{p \in \mathbb{R}^2 \mid p \text{ is a cusp point}\}$$
and let
$$C_2 = \{p \in \Sigma(f) \mid \exists\, q \in \Sigma(f),\, q \neq p \text{ with } f(p) = f(q)\}.$$
From the Tarski-Seidenberg Theorem it will follow that both C_1 and C_2 are semialgebraic sets. Since C_1 and C_2 also consists of isolated points, they cannot have 0 in their closure. □

6.2. **Bad realizations.**

Lemma 6.3. *If (I) fails for an r-jet $\omega \in J^r(2,2)$, then there is a a C^r-germ g with $j^r g(0) = \omega$ having a sequence of distinct cusp points converging to the origin.*

Proof. If (I) fails for ω, then there is a sequence (x_n) converging to 0 and a sequence $(L_n, H_n) \in \Gamma$ such that
$$\|L_n - L_\omega(x_n)\| + \|H_n - H_\omega(x_n)\|\, \|x_n\| = o(\|x_n\|^{r-1}).$$

Define a Taylor field Q on $S = \{0\} \cup (\bigcup_n \{x_n\})$ with values in $\mathbb{R}^2$ by $Q^m(0) = 0$ for all m and
$$Q^m(x_n) = \begin{cases} 0, & m = 0, \\ L_n^m - L_\omega^m(x_n), & |m| = 1, \\ H_n^m - H_\omega^m(x_n), & |m| = 2, \\ 0, & |m| \geq 3. \end{cases}$$
This notation requires some explanation. For instance, let $\omega = (f, g)$ and
$$(L_n, H_n) = (a_n, b_n, \dots, j_n).$$
Then
$$Q^{(1,0)}(x_n) = \big(a_n - \frac{\partial f}{\partial x}(x_n), c_n - \frac{\partial g}{\partial x}(x_n)\big)$$
$$Q^{(0,1)}(x_n) = \big(b_n - \frac{\partial f}{\partial y}(x_n), d_n - \frac{\partial g}{\partial y}(x_n)\big)$$
$$Q^{(2,0)}(x_n) = \big(2e_n - \frac{\partial^2 f}{\partial x^2}(x_n), 2h_n - \frac{\partial^2 g}{\partial x^2}(x_n)\big)$$
etc. Assuming $\|x_{n+1}\| < \frac{1}{2}\|x_n\|$, it is straight forward to verify that Q is a C^r-Whitney field. Therefore we can find a C^r map extending Q. Let h be one such map. Then $j^r h(0) = 0$ and also
$$(L_{\omega+h}(x_n), H_{\omega+h}(x_n)) = (L_n, H_n). \tag{6.1}$$
By construction, $j^2(\omega + h)(x_n) \in \Gamma$.

Now it is not hard to see that the set of 2-jets in Γ which are transverse to Σ^1 is a dense subset. Recall that whether or not a point is a cusp point is determined by the 3-jet at that point. It is not hard to see that in the set of 3-jets with a given 2-jet in Γ transverse to Σ^1 the subset of 3-jets which are cusps is a dense subset. Therefore we can always suppose that $j^2(\omega + h)(x_n) \in \Gamma$ is transverse to Σ^1, and by perturbing the 3-jet if necessary, we can suppose the $j^3(\omega + h)(x_n)$ are cusps for all n.

(If $\omega + h$ has singularities appearing along (x_n) besides simple cusps, then one can define a new Whitney field providing a C^r perturbation h' (in fact h' can be taken to be smooth) with $j^r h'(0) = 0$ and $j^2 h'(x_n) = 0$ such that $\omega + h + h'$ has only cusps along (x_n). Then $g = \omega + h + h'$ is the desired realization of ω.) □

Lemma 6.4. *Assume $\omega \in J^r(2,2)$ satisfies condition (I), but assume condition (II) fails. Also assume that $\Sigma(\omega) - \{0\}$ has N connected components $C_1, \ldots, C_N$ with 0 in their closure all with distinct oriented tangent directions at 0. Then there is a C^r-germ g with $j^r g(0) = \omega$ having a sequence of singular double points converging to the origin.*

Proof. Assume that (II) fails for ω and let the sets H_i and H_ϵ be defined as before. Then we can find a sequence $\epsilon_n \to 0$ and sequences of points (x_n) and (y_n), both converging to $0 \in \mathbb{R}^2$, with each $x_n \in H_i \cup \{0\}$ and $y_n \in H_j \cup \{0\}$ $i \neq j$ where H_i and H_j are components of $H_{\epsilon_n} - \{0\}$such that (II) fails for $\{x_n, y_n\}$. Assume first that $x_n \neq 0$ and $y_n \neq 0$. Then

$$d(j^1\omega(w_n), \Sigma) = o(\|w_n\|^{r-1}), \tag{6.2}$$

for $w_n = x_n, y_n$ and

$$\|\omega(x_n) - \omega(y_n)\| = o\big((\|x_n\|^{r-1} + \|y_n\|^{r-1})\,\|x_n - y_n\|\big). \tag{6.3}$$

Then, since $d(j^1\omega(x_n), \Sigma) = o(\|x_n\|^{r-1})$, using an argument with The Whitney Extension Theorem (similar to the one given in the proof of Lemma 6.3), we can find a representative $\hat{\omega}$ such that $\hat{\omega}$ has singular points along the sequence $\{x_n\}$. By the results of Subsection 4.2, we can find a homeomorphism h mapping $\Sigma(\omega)$ to $\Sigma(\hat{\omega})$ and therefore points $p_n \in C_i$ such that $h(p_n) = x_n$, and by Lemma 4.5, we get that $\|p_n - x_n\| = o(\|p_n\|)$. By the same sort of argument, there is a point $q_n \in C_j$ with $\|q_n - y_n\| = o(\|q_n\|)$. We may also assume that $\|x_n\| \geq \|y_n\|$ and $\|x_{n+1}\| < \frac{1}{2}\|y_n\|$. Notice that because our assumption of the tangent directions of the components $C_1, \ldots C_N$ and the estimates above, there exists $\delta > 0$ such that for all n, $\|x_n - y_n\| > \delta \|x_n\|$.

Let $K = \{0\} \bigcup_n \{x_n, y_n\}$. For each $p \in K$, let $S(p)$ be the singular matrix closest to $D\omega(p)$ in $J^1(2,2)$ and let $M(p) = S(p) - D\omega(p)$. It follows from equation (6.2) that $\|M(w_n)\| = o(\|w_n\|^{r-1})$ for $w_n = x_n, y_n$. Define a r-th order Taylor field Q on K with values in $\mathbb{R}^2$ by

$$Q^m(p) = \begin{cases} 0, & p = 0 \\ 0, & p = y_n, m = 0 \\ \omega(y_n) - \omega(x_n), & p = x_n, m = 0 \\ M^m(p), & |m| = 1 \\ 0, & |m| \geq 2 \end{cases}.$$

Arguments similar to the arguments in [4] show that Q is a Whitney field on K.

Let h be a C^r extension of Q to $\mathbb{R}^2$ and let $g = \omega + h$. Since $j^r h(0) = 0$, g is a realization of ω. For $p \in K$, $Dg(p) = D\omega(p) + Dh(p) = S(p)$, so all points of K are singular points. Also $g(y_n) = \omega(y_n) + h(y_n) = \omega(y_n)$ and $g(x_n) = \omega(x_n) + h(x_n) = \omega(x_n) + \omega(y_n) - \omega(x_n) = \omega(y_n) = g(y_n)$, so (x_n) and (y_n) are sequences of singular double points converging to zero. If say $y_n = 0$ for all n we can use the same Whitney field and we obtain a representative of ω with singular zero-points all along the sequence $\{x_n\}$. □

Lemma 6.5. *If f and g has only regular points and folds outside the origin and there are homeomorhpisms H and K such that $g = K \circ f \circ H$ then $\Sigma(g) = H^{-1}(\Sigma(f))$.*

Proof. This is clear since regular points and fold points are topologically distinct. □

Lemma 6.6. *If $r > 2$ and $f \in \mathcal{E}_{[r]}(2,2)$ is a cusp, then f is topologically different from regular germs and fold germs.*

Proof. Since f has fold singularities close to the origin, f is clearly topologically different from regular germs. To see that f is topologically different from fold germs, notice that the normal form of a fold implies that the image of a neighbourhood of a fold is not a neighbourhood of its target point. We prove that f maps every neighbourhood of 0 to a neighbourhood of 0. This is easily seen from the normal form of a cusp, but to be able to write f in this form, f has to have a considerable degree of differentiability (see [15]). Consider $j^3f(0)$ as a polynomial map $P(x,y)$. Then $f(x,y) = P(x,y) + o(\|(x,y)\|^3)$. Since $P(x,y)$ is a cusp, we may change smooth coordinates and write $f(x,y) = (x, xy + y^3) + o(\|(x,y)\|^3)$. Example 7.2 in Section 7 below shows that $(x, xy+y^3)$ is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[3]}(2,2)$, and hence, f is topologically equivalent to $(x, xy+y^3)$. The conclusion follows. $\square$

Proof of the necessity part of Theorem 2.2. Assume that $\omega \in J^r(2,2)$ does not satisfy (I). Let (x_n) be the sequence in the proof of Lemma 6.3. Let f be a nice realization of ω in the sense of Section 6.1 , and let g be the bad realization of Lemma 6.3.

Suppose the germs at 0 of f and g are $\mathcal{A}_0$-equivalent germs. Then we can find germs at 0 of homeomorphisms H and K such that $g = K \circ f \circ H$. Let U be a neighbourhood of 0 in which g and $K \circ f \circ H$ coincide and choose U so small that f has only fold points and regular points in U_0. Choose N so large that $x_N \in U$. Then the germ of g at x_N and the germ of f at $H(x_N)$ are topologically equivalent. This will however contradict the conclusion of Lemma 6.6, since x_N is a cusp point of g and $H(x_N)$ is either a fold point of f or a regular point of f.

Next, assume that (I) holds and (II) fails for ω, and assume that the oriented tangent directions of the components $C_1, \dots C_N$ of $\Sigma(\omega) \setminus \{0\}$ are all distinct. Let f be as above, but let g be the realization of Lemma 6.4. Suppose there exist germs of homeomorphisms $H : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $K : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f \circ H = K \circ g$ and let U be a neighbourhood of 0 where representatives of the germs are equal. If necessary, choose a smaller U such that f has no singular double points in U. We can find n large enough to ensure that both x_n and y_n are contained in U. According to Lemma 6.5, H maps $\Sigma(g)$ into $\Sigma(f)$. We have that $K \circ g(x_n) = K \circ g(y_n)$ but $f \circ H(x_n) \neq f \circ H(y_n)$ because otherwise $H(x_n)$ and $H(y_n)$ would be singular double points. This contradiction finishes the proof. $\square$

7. Examples

Before we give examples of the use of Theorem 2.2, we prove a proposition which is helpful when trying to verify that a jet is sufficient. To understand where the inequality in the next proposition comes from, recall the expression from Section 3.2 measuring the distance from $(L, H) \in J^2(2,2)$ with L singular to the set $\{ (J,K) \in J^2(2,2) \mid J = L, (J,K) \in \Gamma \}$.

Proposition 7.1. *Let $\omega \in J^r(2,2)$. Then the Lojasiewicz inequality (I) is implied by the following Lojasiewicz inequality:*

There is a neighbourhood U of 0 in $\mathbb{R}^2$ and a real number $C > 0$ such that for all $p \in U$,

$$\left\| D\omega(p) \begin{pmatrix} \frac{\partial}{\partial y} J\omega(p) \\ -\frac{\partial}{\partial x} J\omega(p) \end{pmatrix} \right\| \geq C \|p\|^{r-2}. \tag{III}$$

Proof. Assume that (III) holds for an r-jet ω and that there is a sequence $(L_n, H_n) \in \Gamma$ and a sequence (p_n) of points converging to zero in $\mathbb{R}^2$ such that (I) does not hold, that is

$$\|L_\omega(p_n) - L_n\| + \|H_\omega(p_n) - H_n\| \, \|p_n\| = o(\|p_n\|^{r-1}). \tag{7.1}$$

Let us introduce some notation. Let

$$(L_\omega(p_n), H_\omega(p_n)) = (a(p_n), b(p_n), \dots, j(p_n))$$

and let

$$(L_n, H_n) = (a_n, b_n, \dots, j_n).$$

Finally define $(a'_n, \dots, j'_n)$ by $a'_n = a(p_n) - a_n$, $b'_n = b(p_n) - p_n$, etc. It is easily seen from (7.1) that

$$\|(a'_n, \dots, j'_n)\| = o(\|p_n\|^{r-2}).$$

Now, because $(L_n, H_n) \in \Gamma$,

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} a_n j_n - b_n i_n - c_n g_n + f_n d_n \\ -a_n i_n + b_n h_n + c_n f_n - d_n e_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By writing $a(p_n) = a_n + a'_n$ etc., it is clear that

$$\begin{aligned} &\left\| \begin{pmatrix} a(p_n) & b(p_n) \\ c(p_n) & d(p_n) \end{pmatrix} \begin{pmatrix} a(p_n)j(p_n) - b(p_n)i(p_n) - c(p_n)g(p_n) + f(p_n)d(p_n) \\ -a(p_n)i(p_n) + b(p_n)h(p_n) + c(p_n)f(p_n) - d(p_n)e(p_n) \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} a_n j_n - b_n i_n - c_n g_n + f_n d_n \\ -a_n i_n + b_n h_n + c_n f_n - d_n e_n \end{pmatrix} + \begin{pmatrix} A \\ B \end{pmatrix} \right\| = o(\|p_n\|^{r-2}) \end{aligned}$$

because each term of A and B contains at least one primed factor. But (III) implies that

$$\begin{aligned} &\left\| \begin{pmatrix} a(p_n) & b(p_n) \\ c(p_n) & d(p_n) \end{pmatrix} \begin{pmatrix} a(p_n)j(p_n) - b(p_n)i(p_n) - c(p_n)g(p_n) + f(p_n)d(p_n) \\ -a(p_n)i(p_n) + b(p_n)h(p_n) + c(p_n)f(p_n) - d(p_n)e(p_n) \end{pmatrix} \right\| \\ &\quad \geq C\,\|p_n\|^{r-2} \end{aligned}$$

so we arrive at a contradiction. Thus (I) must hold and the proof is finished. □

Example 7.2. Let $\omega(x, y) = (\, x\,,\, xy + y^k\,)$ for some integer $k > 2$. For $k = 3$ this is the normal form of a cusp. We want to show that ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[k]}(2, 2)$. A computation gives the following.

$$D\omega(x, y) = \begin{pmatrix} 1 & 0 \\ y & x + ky^{k-1} \end{pmatrix}$$

and

$$J\omega(x, y) = x + ky^{k-1}.$$

It is clear that the singular set is a single curve tangent to the y-axis at the origin. So, $C_1 = \{x + ky^{k-1} = 0 \,|\, y > 0\}$ and $C_2 = \{x + ky^{k-1} = 0 \,|\, y < 0\}$ are the two components of $\Sigma(\omega) - \{0\}$. After some computation, we get that close to the origin we have

$$\left\| D\omega(x, y) \begin{pmatrix} \frac{\partial}{\partial y} J\omega(x, y) \\ -\frac{\partial}{\partial x} J\omega(x, y) \end{pmatrix} \right\| = \left\| \begin{pmatrix} k(k-1)y^{k-2} \\ -x + (k^2 - 2k)y^{k-1} \end{pmatrix} \right\| \geq \|(x, y)\|^{k-2}.$$

Hence, ω satisfies (III). By proposition 7.1, ω satisfies (I).

It is more cumbersome to verify (II). Notice that if (x, y) is close enough to the origin and $\epsilon > 0$ is sufficiently small, then by (3.2) of Subsection 3.1

$$\begin{aligned} H_\epsilon &= \{\, (x, y) \mid d(j^1\omega(x, y), \Sigma) \leq \epsilon\, \|(x, y)\|^{k-1} \,\} \\ &\subset \{\, (x, y) \mid |J\omega(x, y)| \leq \frac{(2 + \sqrt{2})\epsilon}{2}\, \|D\omega(x, y)\|\, \|(x, y)\|^{k-1} \,\} \\ &\subset \{\, (x, y) \mid |x + ky^{k-1}| \leq (2 + \sqrt{2})\epsilon |y|^{k-1} \,\} =: H^*_\epsilon. \end{aligned}$$

Let

$$H_\pm = H^*_\epsilon \cap \{\, (x, y) \,|\, y \gtrless 0 \,\}$$

be the two components of $H^*_\epsilon \setminus \{0\}$. It is enough to verify (II) for pairs of points in $H_\pm \cup \{0\}$. It is clear from above that if (x_n, y_n) is a sequence in H^*_ϵ converging to 0, then $2k|y_n|^{k-1} > |x_n| > \frac{k}{2}|y_n|^{k-1}$ provided $\epsilon > 0$ is sufficiently small.

If k is an even number and $p = (x, y) \in H_+$ and $q = (x', y') \in H_-$, then x and x' have opposite signs and the first component of ω becomes dominating and

$$\begin{aligned}\|\omega(p) - \omega(q)\| &= \big\|(\, x - x' \,,\, xy + y^k - x'y' - y'^k \,)\big\| \\ &\geq |x - x'| \\ &= |x| + |x'| \\ &\geq \frac{k}{2}(|y|^{k-1} + |y'|^{k-1}) \\ &\geq (\|p\|^{k-1} + \|q\|^{k-1}) \\ &\geq (\|p\|^{k-1} + \|q\|^{k-1})\,\|p - q\|\end{aligned}$$

as long as $\|p\|$, $\|q\|$ and ϵ are chosen small enough. The same estimate is valid if either $p = (0, 0)$ or $q = (0, 0)$.

If k is odd, then $p = (x, y) \in H_+$ and $q = (x', y') \in H_-$ may have nearly equal first components, but in this case ω separates these points in the second component if the first components are getting very close. We have

$$\begin{aligned}\|\omega(p) - \omega(q)\| &= \big\|(\, x - x' \,,\, xy + y^k - x'y' - y'^k \,)\big\| \\ &\geq |xy - x'y'| - |y^k - y'^k| = |xy| + |x'y'| - |y|^k - |y'|^k \\ &\geq (\frac{k}{2} - 1)(|y|^k + |y'|^k) \\ &\geq C(\|p\|^k + \|q\|^k)\end{aligned}$$

for some $C > 0$ as long as $\|p\|$, $\|q\|$ and ϵ are chosen small enough. If $\|p\| \geq \|q\| \geq 1/2\,\|p\|$, then

$$\|p\|^k + \|q\|^k \geq (\|p\|^{k-1} + \|q\|^{k-1})\,\|q\| \geq \frac{1}{4}(\|p\|^{k-1} + \|q\|^{k-1})\,\|p - q\|\,,$$

and if $\|q\| \leq 1/2\,\|p\|$, then

$$\|p\|^k + \|q\|^k \geq \|p\|^k \geq \frac{1}{2}\,\|p\|^{k-1}\,\|p - q\| \geq \frac{1}{4}(\|p\|^{k-1} + \|q\|^{k-1})\,\|p - q\|\,.$$

This shows that ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[k]}(2, 2)$ for every integer $k \geq 3$.

Example 7.3. Let $\omega(x, y) = (\, xy^2 - \frac{1}{3}x^3 \,,\, y^2 \,)$. We find

$$D\omega(x, y) = \begin{pmatrix} y^2 - x^2 & 2xy \\ 0 & 2y \end{pmatrix},$$

and

$$J\omega(x, y) = 2y(y^2 - x^2).$$

Since $\Sigma(\omega)$ consists of the lines $y = 0$, $y = x$ and $y = -x$, the 6 components of $\Sigma(\omega) - \{0\}$ has different tangent directions. But since $\omega(x, x) = \omega(x, -x)$, (II) of Theorem 2.2 does not hold for any r. So $\omega \in J^r(2, 2)$ is not sufficient in $\mathcal{E}_{[r]}(2, 2)$ for any r.

Example 7.4. Let $\omega(x, y) = (\, xy^2 - \frac{1}{3}x^3 \,,\, y^2 + y^3 \,)$. We find

$$D\omega(x, y) = \begin{pmatrix} y^2 - x^2 & 2xy \\ 0 & 2y + 3y^2 \end{pmatrix},$$

and

$$J\omega(x,y) = y(2+3y)(y^2-x^2).$$

Since the germ of $\Sigma(\omega)$ at 0 consists of the lines $y=0$, $y=x$ and $y=-x$, $\Sigma(\omega)-\{0\}$ has 6 components which have different tangent directions at the origin. Consider $p=p(t)=(t,t)$ and $q=q(t)=(t+t^2,-t-t^2)$. p and q are singular points from different components of $\Sigma(\omega)-\{0\}$. We find $\|\omega(p(t))-\omega(q(t))\| \sim |t|^4 = o(|t|^3) = o(\|p(t)-q(t)\|\,(\|p(t)\|^2+\|q(t)\|^2))$. This shows that (II) of Theorem 2.2 does not hold when $r=3$, so ω is not sufficient in $\mathcal{E}_{[3]}(2,2)$. However, regarding ω as a jet in $J^4(2,2)$, we will show that (I) of Proposition 2.1 and (II) of Theorem 2.2 will hold when $r=4$, so ω will be sufficient as a 4-jet among C^4-realizations.

Let $p_n=(x_n,y_n)$ be a sequence converging to $(0,0)$, and assume that $p_n \in H_\epsilon$ for any $\epsilon>0$ when n is large. Then it follows from (3.3) of Subsection 3.1 that $\frac{|J\omega(p_n)|}{\|D\omega(p_n)\|} = o(\|p_n\|^3)$. It is enough to consider the following two cases.

Case 1; $y_n = o(|x_n|)$. Then $\|p_n\| \sim |x_n|$, $|J\omega(p_n)| \sim |y_n| x_n^2$, so $\frac{|y_n|}{\|D\omega(p_n)\|} = o(|x_n|)$. Since $\|D\omega(p_n)\| \sim \max\{x_n^2,\, |y_n|\}$, we must have $\|D\omega(p_n)\| \sim x_n^2$ and therefore $y_n = o(\,|x_n|^3)$.

Case 2; There exists ϵ such that $|y_n| \ge \epsilon |x_n|$ for all n. Then we get that $\|D\omega(p_n)\| \sim |y_n|$ and therefore

$$\frac{|J\omega(p_n)|}{\|D\omega(p_n)\|} \sim \frac{|y_n|\,|y_n^2-x_n^2|}{|y_n|} = |y_n^2-x_n^2| = o(\|p_n\|^3).$$

This will imply that $|y_n| \sim |x_n|$, $\|p_n\| \sim |x_n|$ and $y_n = \pm\, x_n + o(|x_n|^2)$.

We will now prove that (I) of Proposition 2.1 will hold when $r=4$. Assume this is not the case. Then there exist a sequence (p_n) in $\mathbb{R}^2$, $p_n \to 0$, and a sequence $(L_n,H_n) \in \Gamma$ such that $\|L_\omega(p_n)-L_n\| + \|H_\omega(p_n)-H_n\|\,\|p_n\| = o(\|p_n\|^3)$. Since

$$\|L_\omega(p_n)-L_n\| \ge d(D\omega(p_n),\Sigma) \sim \frac{J\omega(p_n)}{\|D\omega(p_n)\|},$$

we must have $\frac{|J\omega(p_n)|}{\|D\omega(p_n)\|} = o(\|p_n\|^3)$, and from above it follows that we can assume that either $y_n = o(|x_n|^3)$ or $y_n = \pm\, x_n + o(|x_n|^2)$. Let $L_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, and put $\tilde{L}_n = L_n - L_\omega(p_n) = L_n - D\omega(p_n)$. Then $\left\|\tilde{L}_n\right\| = o(\|p_n\|^3)$. Write

$$\begin{aligned} H_\omega(p_n) &= (e(p_n), f(p_n), g(p_n), h(p_n), i(p_n), j(p_n)) \\ &= (-x_n, y_n, x_n, 0, 0, 1+3y_n). \end{aligned}$$

Moreover, let $C_n = \frac{1}{2}\begin{pmatrix} \frac{\partial}{\partial y} J\omega(p_n) \\ -\frac{\partial}{\partial x} J\omega(p_n) \end{pmatrix}$ and let

$$\bar{C}_n = \begin{pmatrix} a_n\, j(p_n) - b_n\, i(p_n) - c_n\, g(p_n) + d_n\, f(p_n) \\ -a_n\, i(p_n) + b_n\, h(p_n) + c_n\, f(p_n) - d_n\, e(p_n) \end{pmatrix}.$$

Let $\tilde{C}_n = \bar{C}_n - C_n$. Let $z_n = (L_n, H_\omega(p_n)) \in J^2(2,2)$ and let $E_n = E_{z_n}$ be the linear subspace of $J^2(2,2)$ defined in Subsection 3.2. By the estimate (3.9) in Subsection (3.2) we get

$$\begin{aligned} o(\|p_n\|^2) &= \|H_\omega(p_n)-H_n\| = \|(L_n,H_\omega(p_n)) - (L_n,H_n)\| \\ &\ge d((L_n,H_\omega(p_n)),E_n) \sim \frac{\|L_n(\bar{C}_n)\|}{\|L_n\|^2}. \end{aligned} \tag{7.2}$$

We can write

$$L_n(\bar{C}_n) = D\omega(p_n)(C_n) + \tilde{L}_n(C_n) + D\omega(p_n)(\tilde{C}_n) + \tilde{L}_n(\tilde{C}_n).$$

Assume first that $y_n = o(|x_n|^3)$. Then $\|D\omega(p_n)(C_n)\| \sim x_n^4$ and $\left\|\tilde{L}_n(C_n)\right\| = o(|x_n|^5)$. Moreover $\tilde{C}_n = \begin{pmatrix} o(|x_n|^3) \\ o(x_n^4) \end{pmatrix}$ and this implies that $\left\|D_\omega(p_n)(\tilde{C}_n)\right\| = o(|x_n|^5)$ and $\left\|\tilde{L}_n(\tilde{C}_n)\right\| = o(x_n^6)$. Altogether this implies that $\left\|L_n(\bar{C}_n)\right\| \sim x_n^4$. Moreover $\|L_n\| \sim \|D_\omega(p_n)\| \sim x_n^2$, so we get that $\frac{\|L_n(\bar{C}_n)\|}{\|L_n\|^2} \sim |x_n|^0$ which contradicts (7.2).

Assume now that $y_n = \pm x_n + o(x_n^2)$. In this case $\|D\omega(p_n)(C_n)\| \sim |x_n|^3$, $\left\|\tilde{L}_n(C_n)\right\| = o(|x_n|^5)$, $\left\|D_\omega(p_n)(\tilde{C}_n)\right\| = o(|x_n|^5)$ and $\left\|\tilde{L}_n(\tilde{C}_n)\right\| = o(x_n^6)$. From this we get $\left\|L_n(\bar{C}_n)\right\| \sim |x_n|^3$. Furthermore $\|L_n\| \sim \|D_\omega(p_n)\| \sim |x_n|$ so we get that $\frac{\|L_n(\bar{C}_n)\|}{\|L_n\|^2} \sim |x_n|$. Since $\|p_n\| \sim |x_n|$ this again contradicts (7.2). Therefore we cannot find a sequence (L_n, H_n) contradicting inequality (I), and (I) must therefore hold when $r = 4$.

Now let us assume that inequality (II) does not hold . Then there must exist sequences $p_n = (x_n, y_n)$ and $q_n = (u_n, v_n)$ such that p_n, $q_n \in H_\epsilon$ for any $\epsilon > 0$ when n is large, $p_n \in H_i$ and $q_n \in H_j$ with $i \neq j$ and $\|\omega(p_n) - \omega(q_n)\| = o(\|p_n - q_n\| (\|p_n\|^3 + \|q_n\|^3)) = o(\|p_n\|^4 + \|q_n\|^4)$. (Note that it will follow from what we have shown above that $\|p_n - q_n\| \sim (\|p_n\| + \|q_n\|)$ when $p_n \in H_i$, $q_n \in H_j$ and $i \neq j$. This also follows from Lemma 4.5 and the proof of Lemma 6.4). Since we may assume that p_n and q_n satisfy Case 1 or Case 2 above, we have to consider several subcases. Assume first $y_n = \pm x_n + o(|x_n|^2)$ and $v_n = \pm u_n + o(|u_n|^2)$ and x_n and u_n have different signs. Then

$$|(x_n y_n^2 - \frac{1}{3}\, x_n^3) - (u_n v_n^2 - \frac{1}{3}\, u_n^3)| = \frac{2}{3}\, |x_n|^3 + \frac{2}{3}\, |u_n|^3 + o(|x_n|^4) + o(|u_n|^4) \sim \|p_n\|^3 + \|q_n\|^3 .$$

So we cannot have such a pair of sequences violating (II). The case $y_n = \pm x_n + o(|x_n|^2)$ and $v_n = o(|u_n|^3)$ where x_n and u_n have the same sign, can be treated in a similar manner and we get the same conclusion. Consider the case $y_n = \pm x_n + o(|x_n|^2)$ and $v_n = o(|u_n|^3)$ where x_n and u_n have opposite signs. Then

$$|(x_n y_n^2 - \frac{1}{3}\, x_n^3) - (u_n v_n^2 - \frac{1}{3}\, u_n^3)| = |\frac{2}{3}\, x_n^3 + \frac{1}{3}\, u_n^3| + o(|x_n|^4) + o(|u_n|^7).$$

If $|u_n| > 2|x_n|$ the right hand side of the equation above is dominated by the term $|\frac{2}{3}\, x_n^3 + \frac{1}{3}\, u_n^3| \sim \frac{1}{3}\, |u_n|^3 \sim \|p_n\|^3 + \|q_n\|^3$. If $|u_n| \leq 2|x_n|$ then $v_n = o(|x_n|^3) = o(|y_n|^3)$. This implies that.

$$|(y_n^2 + y_n^3) - (v_n^2 + v_n^3)| \sim\ |y_n|^2 \sim \|p_n\|^2 + \|q_n\|^2 .$$

Therefore we cannot find sequences contradicting (II) in this case either.

The next case is $|y_n| = o(|x_n|^3)$ and $|v_n| = o(|u_n|^3)$. Then it is clear that such p_n, q_n must belong to components H_i and H_j containing the positive and negative part of the x-axis respectively, and x_n and u_n must consequently have different signs. Then

$$|(x_n y_n^2 - \frac{1}{3}\, x_n^3) - (u_n v_n^2 - \frac{1}{3}\, u_n^3)| = \frac{1}{3}|x_n|^3 + \frac{1}{3}|u_n|^3 + o(|x_n|^7) + o(|u_n|^7) \sim \|p_n\|^3 + \|q_n\|^3 .$$

Thus, (II) cannot fail along such sequences.

The only case left is when $y_n = \pm x_n + o(|x_n|^2)$ and $v_n = \pm u_n + o(|u_n|^2)$ and x_n and u_n have the same sign. Since p_n and q_n belong to different H_i's, y_n and v_n must have opposite signs. We may assume that x_n, u_n, $y_n > 0$ and $v_n < 0$. So $x_n = y_n + o(|y_n|^2)$ and $u_n = -v_n + o(|v_n|^2)$. Assume that

$$\|\omega(p_n) - \omega(q_n)\| = o(\|p_n\|^4 + \|q_n\|^4) = o(|y_n|^4 + |v_n|^4).$$

Let $\tilde{p}_n = (y_n, y_n)$ and $\tilde{q}_n = (-v_n, v_n)$. Then

$$\|\omega(p_n) - \omega(\tilde{p}_n)\| = o(|y_n|^4) \text{ and } \ \|\omega(q_n) - \omega(\tilde{q}_n)\| = o(|v_n|^4).$$

This implies that,

$$\|\omega(\tilde{p}_n) - \omega(\tilde{q}_n)\| \sim |\frac{2}{3}y_n^3 + \frac{2}{3}v_n^3| + |y_n^2 + y_n^3 - v_n^2 - v_n^3| = o(|y_n|^4 + |v_n|^4).$$

So we get that

$$|y_n^2 + y_n^3 - v_n^2 - v_n^3| = |y_n - v_n|\,|y_n + v_n + y_n^2 + y_n\,v_n + v_n^2| = o(|y_n|^4 + |v_n|^4).$$

Then since y_n and v_n have opposite signs, $|y_n - v_n| \sim |y_n| + |v_n|$, and we get

$$|y_n + v_n + y_n^2 + y_n\,v_n + v_n^2| = o(|y_n|^3 + |v_n|^3).$$

But since $|y_n^2 + y_n\,v_n + v_n^2| \sim |y_n|^2 + |v_n|^2$ we must then have that $|y_n + v_n| \sim |y_n|^2 + |v_n|^2$. This will imply that

$$|\frac{2}{3}y_n^3 + \frac{2}{3}v_n^3| = \frac{2}{3}\,|y_n + v_n|\,|y_n^2 - y_n v_n + v_n^2| \sim (|y_n|^4 + |v_n|^4)$$

which gives a contradiction. This proves that in any case, we cannot find a pair of sequences (p_n) and (q_n) violating (II). So (II) must hold when $r = 4$. We conclude that ω satisfies the hypothesis of Theorem 2.2 and hence is sufficient for $r = 4$.

8. Topological trivialization of 1-parameter families of germs

So far we have studied the perturbation of an r-jet z by an arbitrary C^r mapping h with $j^r h(0) = 0$. In particular, we have studied the 1-parameter family of C^r map-germs $z + th$. In this section we deal with a somewhat different problem. We are going to consider C^r 1-parameter families $\alpha_t = (f_t, g_t)$ of C^r map-germs $\alpha_t : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. By this we mean that there exists a C^r map $F : U \times I \to \mathbb{R}^3$ given by $F(p, t) = (\beta_t(p), t)$ such that each β_t is a representative of the germ α_t. (We call such F a representative of the family.) The techniques we have developed in the earlier sections can be used to give some sufficient conditions to decide that such a 1-parameter family of germs can be topological trivialized, i.e. there are 1-parameter families of homeomorphisms H_t and K_t such that $\alpha_t \circ H_t = K_t \circ \alpha_0$.

Proposition 8.1. *Let $r > 2$ and let $\alpha_t = (f_t, g_t) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^r 1-parameter family of C^r germs from the $\mathbb{R}^2$ to $\mathbb{R}^2$. The following conditions are sufficient for α_t to be topologically trivializable:*

There exists a representative $F : U \times I \to \mathbb{R}^3$, $F(p, t) = (\beta_t(p), t)$ having the following properties.

(1) Each $\beta_t|_{U_0}$ has only fold singularities (recall that $U_0 = U - \{0\}$.)
(2) $F|_{\Sigma(F)}$ is 1-1.
(3) $\|\beta_t(p)\| > 0$ for $(p, t) \in U_0 \times I$.
(4) $\left\|\frac{\partial F}{\partial t}(p, t) - (0, 0, 1)\right\| = o(\|\beta_t(p)\|)$ as $p \to 0$.

Proof. The proof of this proposition is very similar to the proof of the sufficiency part of Theorem 2.2. We will therefore only sketch this proof refering to the relevant details of that proof. Property 1 above ensures that $M = \Sigma(F_0)$ is a C^{r-1} submanifold of $\mathbb{R}^3$ and that $F|_{\Sigma(F_0)}$ is an immersion. Together with property 2 this also makes $N = F(M)$ a C^{r-1} submanifold, completely analogous to Lemma 4.13. Now we can define a vector field $\mathbf{w}$ on N by

$$\mathbf{w}(F(p, t)) = DF_{(p,t)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial \alpha_t}{\partial t} \\ 1 \end{pmatrix}$$

which will be tangent to N because F has rank 2 at every point of M. Also define $\mathbf{w}(0, 0, t) = (0, 0, 1)$. This gives a vector field on all of $F(\Sigma(F))$. Property 4 guarantees us that $\mathbf{w}$ satisfies Kuo's condition. Indeed, the situation is exactly the same as for the vector field $\mathbf{u}$ on Ω in

Section 5. Recall the technique we used to extend $\mathbf{u}$ to all of the target. We can use the same technique to extend $\mathbf{w}$ to a vector field μ defined on some open neighbourhood $V \times I$ of $\{0\} \times I$ in the target, and as in Lemma 5.1 we get

$$\|\mu(q,t) - \mathbf{k}\| = o(\|q\|).$$

Thus we can integrate μ and get a continuous flow $\theta(q,t)$ defined on $V \times I$. The vector field μ is C^{r-2} outside the t-axis, just like the vector fields η and ζ of earlier sections.

The next step is to define a corresponding vector field ν on the source. This is defined to be the unique vector field whose restriction to M is a tangent vector field and which is mapped onto μ under DF. We can now use the same arguments as in the proof of Lemma 5.3 to see that ν has a continuous flow. Let U' be a neighborhood of 0 in the source such that $U' \subset \overline{U}' \subset U$. Then property 4 give us that if J is a compact interval with $J \subset I$, then $F(\overline{U}' - U') \times J$ is bounded away from $\{0\} \times I$. Using this we can use the continuous flow $\theta(q,t)$ in the target to control the flow in the source, and we can argue exactly as in Lemma 5.3 to obtain the existence and continuity of the flow of ν. The flows of μ and ν induce the required homeomorphisms, and the proof is finished. □

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