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CHOW GROUPS AND TUBULAR NEIGHBOURHOODS

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Abstract. We will prove theorems of Zariski-Lefschetz type for the analytic Chow groups of a quasi-projective variety. We will also derive an algebraic analogue, using formal instead of tubular neighbourhoods.

I. In this paper we will look at the algebraic and analytic Chow groups for complex quasi-projective varieties.

First, let X be a scheme over $\mathbb{C}$ of finite type, $k \geq 0$. Then the k-th Chow group $A_k(X)$ is defined as follows: $A_k(X) := C_k(X)/Z_k(X)$. Here $C_k(X)$ is the group of k-cycles in X, i.e. the free abelian group of formal $\mathbb{Z}$-linear combinations of k-dimensional algebraic subvarieties (i.e. closed non-empty reduced and irreducible subschemes) of X, and $Z_k(X)$ is the subspace of $\mathbb{Z}$-linear combinations of elements of the form $div\, f$, where $f \in \mathcal{M}(D)^*$, D a $(k+1)$-dimensional algebraic subvariety of X. Note that $\mathcal{M}(D)$ is the field of rational functions on D and $div\, f$ the divisor of f.

See [Fu] I.1.3, where $A_k(X)$ is called the group of k-cycles modulo rational equivalence. It is reasonable to speak of “Chow groups” because $\oplus_k A_k(X)$ is called “Chow ring” in the non-singular case where we have a ring structure indeed.

If X is everywhere of dimension n we have that $A_{n-1}(X) = Cl(X) :=$ Weil divisor class group $=$ group of Weil divisors modulo principal divisors.

We can define analytic Chow groups, too, for a complex space. However, in the analytic context $C_k(X)$ is defined using locally finite linear combinations instead of finite linear combinations, and $Z_k(X)$ consist of elements $\sum_i div\, f_i$, where $(D_i)_{i\in I}$ is a locally finite set of $(k+1)$-dimensional analytic subvarieties of X and f_i is a non-zero meromorphic function on D_i.

Note that this is not the same definition as in [V] but it is at least reasonable in the following sense: If the complex space X is everywhere of dimension n we have again that $A_{n-1}(X) = Cl(X) :=$ Weil divisor class group.

From now on let X be a closed subscheme of $\mathbb{P}_N(\mathbb{C})$, Y a Zariski-closed subspace of X, and H a hyperplane. The complex space associated to X will be denoted by X^{an}. We assume that X is reduced because this is not an essential restriction.

A Lefschetz type theorem for the Chow groups should compare those of $X \setminus Y$ and $X \cap H \setminus Y$. But looking for such a theorem seems to be very difficult. A considerable simplification is obtained in the analytic context if one replaces the hyperplane section by some neighbourhood (“Zariski-Lefschetz type theorem”). There are two possibilities: first, one can take a fundamental system of neighbourhoods V of $X^{an} \cap H^{an} \setminus Y^{an}$ in $X^{an} \setminus Y^{an}$ and compare $A_k(X^{an} \setminus Y^{an})$ with $A_k(V)$, or one can take a fundamental system of neighbourhoods U of $X^{an} \cap H^{an}$ in X^{an} and compare $A_k(X^{an} \setminus Y^{an})$ with $A_k(U \setminus Y^{an})$. Note that the neighbourhoods $U \setminus Y^{an}$ of

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$X^{an} \cap H^{an} \setminus Y^{an}$ are big compared with V.

The second alternative has already been studied in [H1] in the special case of the Weil divisor class group: If dim $X \geq 3$ everywhere we have $Cl(X^{an} \setminus Y^{an}) \simeq Cl(U \setminus Y^{an})$ for some fundamental system of neighbourhoods U of $X^{an} \cap H^{an}$ in X^{an}, see [H1] Theorem 1.2.

II. The analogue of tubular neighbourhoods in the algebraic context is given by formal completion. Let $\hat{X}$ be the formal completion of X along $X \cap H$, see [GD] I §10. Then the formal completion of $X \setminus Y$ along $X \cap H \setminus Y$ is given by $\hat{X} \setminus \hat{Y}$. This is the algebraic analogue of the neighbourhoods V above (in the limit).

This approach in the algebraic context goes back to A.Grothendieck when he studied the Picard group. In fact Grothendieck has proved in [G] a Lefschetz theorem for the Picard group $Pic(X \setminus Y)$ in the case $Y = \emptyset$. This has been generalized in [HL2]. The case where Y is arbitrary has been studied in [HL1] (smooth case) and [HL3] (general case).

Note that $Pic(X \setminus Y) \simeq Cl(X \setminus Y)$ if $X \setminus Y$ is smooth. This could be used in order to derive a Lefschetz theorem for the Weil divisor class group, see [HL1] Theorem 1.5: If dim $X \geq 4$ everywhere, codim $Sing\, X \geq 2$ and H is generic we have that $Cl(X) \simeq Cl(X \cap H)$.

When working with formal neighbourhoods we have to make precise what we mean by the dimension: If $\hat{Z}$ is a closed formal subscheme of $\mathbb{P}_N(\mathbb{C}) \setminus Y$, dim $\hat{Z} \geq k$ everywhere if for all closed points z of $\hat{Z}$ and all associated prime ideals $\mathfrak{p}$ of $\mathcal{O}_{\hat{Z},z}$ we have dim $\mathcal{O}_{\hat{Z},z}/\mathfrak{p} \geq k$.

Furthermore, a closed formal subscheme $\hat{Z}$ of $\hat{X}$ is called reducible if there are proper formal closed subschemes such that $\hat{Z} = \hat{Z}_1 \cup \hat{Z}_2$, where $\mathcal{J}_1 \cdot \mathcal{J}_2 = 0$ for the ideal sheaves $\mathcal{J}_1, \mathcal{J}_2$ of $\hat{Z}_1$, $\hat{Z}_2$ in $\hat{Z}$. Otherwise, $\hat{Z}$ is called irreducible, of course.

Note that if Z is a subscheme of $\mathbb{P}_N(\mathbb{C}) \setminus Y$ of pure dimension k, $Z \cap H \neq \emptyset$, we have dim $\hat{Z} = k$, too.

What is the algebraic analogue of neighbourhoods of the form $U \setminus Y^{an}$? It is easier to give a direct definition of the corresponding Chow group than to define an analogue of the space itself. Let us start from a different description of $A_k(X \setminus Y)$ in the algebraic case: we have $A_k(X \setminus Y) \simeq A_k(X, Y) := C_k(X)/(Z_k(X) + C_k(Y))$. The notation might be misleading: obviously we still have an arrow $A_k(X) \to A_k(X, Y)$.

Note that $A_k(X, Y) \simeq C_k(X, Y)/Z_k(X, Y)$ with $C_k(X, Y) := C_k(X)/C_k(Y)$ and $Z_k(X, Y) = Z_k(X)/Z_k(X) \cap C_k(Y) \simeq (Z_k(X) + C_k(Y))/C_k(Y)$, by the isomorphism theorems of group theory.

Then it is natural to define $A_k(\hat{X}, \hat{Y})$ with $\hat{X}, \hat{Y}$ instead of X, Y. Now $A_k(\hat{X}, \hat{Y})$ seems to be the appropriate algebraic analogue of $\varinjlim A_k(U \setminus Y^{an})$, as we will see from the results.

We have an analogous notion $A_k(X^{an}, Y^{an})$ in the analytic context which does not, however, coincide necessarily with $A_k(X^{an} \setminus Y^{an})$ in general because analytic subsets of $X^{an} \setminus Y^{an}$ do not necessarily extend to analytic subsets of X^{an}.

III. Now we have all types of Chow groups which we will use at our disposal and can phrase our theorems. As often define dim $\emptyset := -1$.

In the analytic context we have:

Theorem 1: The mapping $A_k(X^{an} \setminus Y^{an}) \to \varinjlim A_k(U \setminus Y^{an})$ is bijective if $k \geq 2$ and injective if $k \geq 1$.

Here U runs through the set of open neighbourhoods of $X^{an} \cap H^{an}$ in X^{an}.

Theorem 1': The mapping $A_k(X^{an}, Y^{an}) \to \varinjlim A_k(U, U \cap Y^{an})$ is bijective if $k \geq 2$ and injective if $k \geq 1$.

Again, U runs through the set of open neighbourhoods of $X^{an} \cap H^{an}$ in X^{an}.

Theorem 2: The mappings $A_k(X^{an} \setminus Y^{an}) \to \varinjlim A_k(U \setminus Y^{an}) \to \varinjlim A_k(V)$ are bijective if $k \geq \dim(Y \cap H) + 3$ and injective if $k \geq \dim(Y \cap H) + 2$.

Here U (resp. V) runs through the set of all open neighbourhoods of $X^{an} \cap H^{an}$ in X^{an} (resp. of $X^{an} \cap H^{an} \setminus Y^{an}$ in $X^{an} \setminus Y^{an}$).

Similarly, in the algebraic context we obtain:

Theorem 3: The mapping $A_k(X \setminus Y) \to A_k(\hat{X}, \hat{Y})$ is bijective if $k \geq 2$ and injective if $k \geq 1$.

Theorem 4: The mappings $A_k(X \setminus Y) \to A_k(\hat{X}, \hat{Y}) \to A_k(\hat{X} \setminus \hat{Y})$ are bijective if $k \geq \dim(Y \cap H) + 3$ and injective if $k \geq \dim(Y \cap H) + 2$.

Remark: In the case $Y = \emptyset$ Theorem 1, 1' and 2 coincide, the same holds for Theorem 3 and 4.

Finally we will compare the algebraic and analytic context, this will make it possible, in particular, to make Theorem 1' more precise. See Remark 3.1 below.

From the literature to be used it is evident that the results in the algebraic context go over to the case of an arbitrary algebraically closed field instead of $\mathbb{C}$.

1. Analytic context: Proof of Theorem 1, 1' and 2

We can identify $\mathbb{P}_N^{an}(\mathbb{C}) \setminus H^{an}$ with $\mathbb{C}^N$. For $R > 0$ let U_R be the complement of $\{z \in \mathbb{C}^N \cap X^{an} \mid \max|z_j| \leq R\}$ in X^{an}. The U_R form a fundamental system of neighbourhoods of $X^{an} \cap H^{an}$ in X^{an}. Fix R.

First let us prove

Lemma 1.1: a) If $k \geq 2$ (resp. $k \geq 1$), for every purely k-dimensional (sc. closed) analytic subset C of $U_R \setminus Y^{an}$ there is exactly (resp. at most) one purely k-dimensional analytic subset C' of $X^{an} \setminus Y^{an}$ such that $C' \cap U_R = C$.

b) The mapping $C_k(X^{an} \setminus Y^{an}) \to C_k(U_R \setminus Y^{an})$ is bijective if $k \geq 2$ and injective if $k \geq 1$.

Proof: a) see Theorem 3.2 in [H1].

b) follows from a).

Lemma 1.2: a) If D is a purely k-dimensional analytic subvariety of $X^{an} \setminus Y^{an}$, $k \geq 2$, every meromorphic function on $D \cap U_R$ extends to a unique meromorphic function on D.

b) $Z_k(X^{an} \setminus Y^{an}) \to Z_k(U_R \setminus Y^{an})$ is bijective if $k \geq 1$.

Proof: a) We modify (and correct) the proof of [H1] Theorem 3.4 which covers the special case where D can be extended to a subvariety of X^{an}:

Let f be a meromorphic function on $U_R \cap D$, and let $p : \tilde{D} \to D$ be the normalization. Let $\tilde{D}_{sing}$ be the singular locus of $\tilde{D}$, $D^* := D \setminus p(\tilde{D}_{sing})$, $\tilde{D}^* := p^{-1}(D^*)$. Let $I_{f\circ p}$ be the set of points of $p^{-1}(U_R \cap D^*)$ where $f \circ p$ is indeterminate. Put $D_R^{**} := U_R \cap D^* \setminus p(I_{f\circ p})$, $\tilde{D}_R^{**} := p^{-1}(D_R^{**})$ and $p_R := p|\tilde{D}_R^{**} : \tilde{D}_R^{**} \to D_R^{**}$. Let $\tilde{W}$ be a sufficiently small neighbourhood of a point in $\tilde{D}_R^{**}$. On $\tilde{W}$, $f\circ p$ can be written in the form g/h where g, h are holomorphic functions on $\tilde{W}$ whose germs are relatively prime. Then (g,h) defines a section of $\mathcal{O}^2_{\tilde{D}}|\tilde{W}$; it generates an invertible $\mathcal{O}_{\tilde{D}}|\tilde{W}$-module which depends only on f. Patching together we obtain an invertible $\mathcal{O}_{\tilde{D}_R^{**}}$-submodule $\mathcal{S}$ of $\mathcal{O}^2_{\tilde{D}_R^{**}}$. Then $(p_R)_*\mathcal{S}$ is an invertible $p_*\mathcal{O}_{\tilde{D}}|D_R^{**}$-submodule of $p_*\mathcal{O}^2_{\tilde{D}}|D_R^{**}$, at the same time we can consider these two sheaves as coherent $\mathcal{O}_{D_R^{**}}$-modules, too.

It is easy to see that $(p_R)_*\mathcal{S}$ coincides with its $(k-1)$-st gap sheaf relative to $p_*\mathcal{O}^2_{\tilde{D}}|D_R^{**}$ (see [S] p. 132): Let W be an open set in D_R^{**} and A an analytic subset of W of dimension $\leq k-1$. Let s be a section of $p_*\mathcal{O}^2_{\tilde{D}}|W$ such that $s|W \setminus A$ is a section of $(p_R)_*\mathcal{S}$. Then s can be considered as an element of $\Gamma(p^{-1}(W), \mathcal{O}^2_{\tilde{D}})$ whose restriction to $p^{-1}(W \setminus A)$ is a section in $\mathcal{S}$. The latter can be uniquely extended to an element of $\Gamma(p^{-1}(W), \mathcal{S})$ which has to coincide with $s \in \Gamma(p^{-1}(W), \mathcal{O}^2_{\tilde{D}})$.

Therefore $(p_R)_*\mathcal{S}$ can be extended to a coherent $\mathcal{O}_{U_R\cap D}$-submodule of $p_*\mathcal{O}^2_{\tilde{D}}|U_R \cap D$ with analogous properties, by the subsheaf extension theorem, see [ST], first part of the proof of Theorem 1b. Note that the resulting sheaf can be considered after trivial extension as a coherent $\mathcal{O}_{U_R\setminus Y}$-module, too.

By Theorem 3.3 of [H1] the subsheaf above can be uniquely extended to a coherent $\mathcal{O}_{X\setminus Y}$-submodule of $p_*\mathcal{O}^2_{\tilde{D}}$ which coincides with its $(k-1)$-st relative gap sheaf; note that $k-1 \geq 1$ because $k \geq 2$. Of course, it must be the trivial extension of a coherent $\mathcal{O}_D$-submodule $\mathcal{T}$ of $p_*\mathcal{O}^2_{\tilde{D}}$.

There is a discrete subset Σ of D such that $\mathcal{T}|D \setminus \Sigma$ is even a $p_*\mathcal{O}_{\tilde{D}}|D \setminus \Sigma$-module: note that we have a multiplication mapping $p_*\mathcal{O}_{\tilde{D}} \otimes_{\mathcal{O}_D} \mathcal{T} \to p_*\mathcal{O}^2_{\tilde{D}}$ whose image is contained in $\mathcal{T}$ if we restrict to $U_R \cap D$. Then use Lemma 3.1 of [H1]. (Note that $X \subset Y$ should be replaced by $X \setminus Y$ there.)

Now $\mathcal{T}|D \setminus \Sigma$ is finite as a $\mathcal{O}_{D\setminus\Sigma}$-module, hence as a $p_*\mathcal{O}_{\tilde{D}}|D \setminus \Sigma$-module. As such it is coherent, and its restriction to $U_R \cap D$ is invertible outside some analytic subset of codimension ≥ 2. Therefore $\mathcal{T}|D \setminus \Sigma$ is an invertible $p_*\mathcal{O}_{\tilde{D}}|D \setminus \Sigma$-module, too, outside some analytic subset of codimension ≥ 2, after enlarging Σ if necessary: Otherwise there would be an irreducible analytic subset of $D \setminus \Sigma$ of dimension $\geq k-1 > 0$ where $\mathcal{T}$ is not invertible. Note that this irreducible subset could be continued to an analytic subset of D, by the theorem of Remmert-Stein ([GR] Theorem V D 5). Then use Lemma 3.1 of [H1] again.

Let D^{**} be the subset of $D^* \setminus \Sigma$ where $\mathcal{T}$ is invertible. If (g,h) is a local generator we obtain using g/h a meromorphic function on D^{**} which can be uniquely continued to a meromorphic function on $D^* \setminus \Sigma$, hence on $D \setminus \Sigma$ and finally on D, by the Kontinuitätssatz [KK] 53.A.9. This gives the desired extension of f.

b) Suppose that f is a meromorphic function on D, where D is a purely $(k+1)$-dimensional analytic subset of $U_R \setminus Y^{an}$. By Lemma 1.1a), there is exactly one purely $(k+1)$-dimensional analytic subset D' of $X^{an} \setminus Y^{an}$ such that $D' \cap U_R = D$. By a) we may extend f to exactly one meromorphic function on D'. The rest is clear.

Proof of Theorem 1: First assume that $k \geq 2$. By Lemma 1.1b), the mapping $C_k(X^{an} \setminus Y^{an}) \to C_k(U_R \setminus Y^{an})$ is bijective. By Lemma 1.2b), $Z_k(X^{an} \setminus Y^{an}) \to Z_k(U_R \setminus Y^{an})$ is bijective. This implies that $A_k(X^{an} \setminus Y^{an}) \to A_k(U_R \setminus Y^{an})$ is bijective, hence $A_k(X^{an} \setminus Y^{an}) \to \lim\limits_{\to} A_k(U \setminus Y^{an})$, too.

Now assume only $k \geq 1$. Then we know that $C_k(X^{an} \setminus Y^{an}) \to C_k(U_R \setminus Y^{an})$ is injective, whereas $Z_k(X^{an} \setminus Y^{an}) \to Z_k(U_R \setminus Y^{an})$ is bijective. This implies that $A_k(X^{an} \setminus Y^{an}) \to A_k(U_R \setminus Y^{an})$ is injective, hence $A_k(X^{an} \setminus Y^{an}) \to \lim\limits_{\to} A_k(U \setminus Y^{an})$, too.

Proof of Theorem 1': We apply Lemma 1.1 and Lemma 1.2 in the case $Y = \emptyset$. According to Lemma 1.1 we have that for every purely k-dimensional analytic subset C of U_R there is exactly (resp. at most) one purely k-dimensional analytic subset C' of X^{an} such that $C' \cap U_R = C$. If no irreducible component of C is contained in Y^{an} we know that the same holds for C', too. So we obtain that $C_k(X^{an}, Y^{an}) \to C_k(U_R, Y^{an} \cap U_R)$ is bijective (resp. injective).

Similarly, if $k \geq 1$ and D is an analytic subvariety of X of dimension $k+1$, we can extend D to exactly one analytic subvariety of X^{an} of dimension $k+1$, and if f is meromorphic on D we can extend f to D'. Again, if D is not contained in Y^{an}, D' is not contained in Y^{an}, too. Therefore $Z_k(X^{an}, Y^{an}) \simeq Z_k(U_R, U_R \cap Y^{an})$. Altogether we obtain Theorem 1'.

Now let us turn to the proof of Theorem 2. Suppose that $k \geq \dim (Y \cap H) + 2$, so $k \geq \dim Y + 1$, and that U is an open neighbourhood of $X^{an} \cap H^{an}$ in X^{an}. As we will see in Proposition 3.2, $A_k(X^{an}) \simeq A_k(X^{an} \setminus Y^{an})$; with the same techniques we have $A_k(U) \simeq A_k(U \setminus Y^{an})$.

Therefore we can suppose in the proof of Theorem 2 that $Y \subset H$. Furthermore we can assume $Y \neq \emptyset$ because otherwise Theorem 2 coincides with Theorem 1 and 1'.

Let $\mathcal{A}_k$ be the sheaf of purely k-dimensional analytic subsets on X^{an}: if W is open in X^{an} let $\Gamma(W, \mathcal{A}_k)$ be the set of all closed purely k-dimensional analytic subsets of W. If A is a locally closed subset of X^{an} we have $\Gamma(A, \mathcal{A}_k) = \lim\limits_{\to} \Gamma(W, \mathcal{A}_k)$ where W runs through the set of all open neighbourhoods of A in X^{an}: this follows from [Go] II 3.3 Corollaire 1.

Lemma 1.3: The mapping $\Gamma(X^{an} \cap H^{an}, \mathcal{A}_k) \to \Gamma(X^{an} \cap H^{an} \setminus Y^{an}, \mathcal{A}_k)$ is bijective if $k \geq \dim Y + 3$ and injective if $k \geq \dim Y + 2$.

Proof: We may suppose $X = \mathbb{P}_N$. It is sufficient to show that the mapping

$$\Gamma(X^{an} \cap H^{an} \setminus (Y')^{an}, \mathcal{A}_k) \to \Gamma(X^{an} \cap H^{an} \setminus (Y'')^{an}, \mathcal{A}_k)$$

is bijective resp. injective if $Y'' \subset Y' \subset Y$ and $Y' \setminus Y''$ is smooth of dimension $l \leq \dim Y$.

Let $j : X^{an} \cap H^{an} \setminus (Y')^{an} \to X^{an} \cap H^{an} \setminus (Y'')^{an}$ be the inclusion. Then it suffices to show that the mapping

$$j_*(\mathcal{A}_k | X^{an} \cap H^{an} \setminus (Y')^{an}) \to \mathcal{A}_k | X^{an} \cap H^{an} \setminus (Y'')^{an})$$

is bijective resp. injective.

We have to show this at every point of $(Y')^{an} \cap H^{an} \setminus (Y'')^{an}$. Choose local coordinates $z_1, \ldots, z_N$ centered at this point such that Y'^{an} is locally described by $z_{l+1} = \ldots = z_N = 0$ and H^{an} by $z_N = 0$. Fix $\epsilon_0 = \delta_0 > 0$ sufficiently small. For $0 < \epsilon, \delta < \epsilon_0$ put $W_{\epsilon,\delta} := \{z \,|\, |z_j| < \epsilon_0, j = 1, \ldots, l, \epsilon < \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, |z_N| < \delta\}$. Let $(\epsilon_\nu)_{\nu \geq 1}, (\delta_\nu)_{\nu \geq 1}$ be strictly monotonously decreasing sequences of positive real numbers which converge to 0, where $\epsilon_1 \leq \epsilon_0, \delta_1 \leq \delta_0$, and put $W := \bigcup\limits_{\nu=1}^{\infty} W_{\epsilon_\nu, \delta_\nu}$. Note that the closures of the sets W obtained in this way form a fundamental system of neighbourhoods of $\{z\,|\,|z_j| \leq \epsilon_0, j = 1, \ldots, N, (z_{l+1}, \ldots, z_{N-1}) \neq 0, z_N = 0\}$ in $\{z\,|\,|z_j| \leq \epsilon_0, j = 1, \ldots, N, (z_{l+1}, \ldots, z_N) \neq 0\}$.

Now it is sufficient to show: Every purely k-dimensional closed analytic subset of W admits exactly (resp. at most) one extension to a closed analytic subset of $\{z\,|\,|z_j| < \epsilon, j = 1, \ldots, N-1, |z_N| < \delta_1\}$. (*)

Here we proceed similarly as in the proof of Lemma 9 in [H2]. The essential point is the following: Every purely k-dimensional analytic subset of $\{z\,|\,|z_j| < \epsilon_0, j = 1, \ldots, l, \epsilon_\nu < \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$ admits exactly (resp. at most) one extension to a purely k-dimensional analytic subset of $\{z\,|\,|z_j| < \epsilon_0, j = 1, \ldots, l, \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$.

But this is just a consequence of [S] Theorem 2.18 resp. Lemma 2.17.

By induction, this makes it possible to extend every purely k-dimensional analytic subset of W to exactly (resp. at most) one purely k-dimensional analytic subset of $W \cup \{z\,|\,|z_j| < \epsilon_0, j = 1, \ldots, N-1, \delta_\nu < |z_N| < \delta_1\} \cup \{z\,|\,|z_j| < \epsilon_0, j = 1, \ldots, N-1, \max(|z_{j+1}|, \ldots, |z_{N-1}| > \epsilon_\nu, |z_N| < \delta_1\}$, hence of $\{z\,|\,|z_j| < \epsilon_0, j = 1, \ldots, N-1, |z_N| < \delta_1\} \setminus Y'$, or of $\{z\,|\,|z_j| < \epsilon_0, j = 1, \ldots, N-1, |z_N| < \delta_1\}$, by the extension theorem of Remmert-Stein ([GR] Theorem V D 5). This implies (*).

As a consequence we obtain the following Lefschetz type theorem:

Theorem 1.4: The mapping $\Gamma(X^{an} \setminus Y^{an}, \mathcal{A}_k) \to \Gamma(X^{an} \cap H^{an} \setminus Y^{an}, \mathcal{A}_k)$ is bijective if $k \geq \dim Y + 3$ and injective if $k \geq \dim Y + 2$.

Proof: By Lemma 1.1, $\Gamma(X^{an}, \mathcal{A}_k) \simeq \Gamma(X^{an} \cap H^{an}, \mathcal{A}_k)$. Using Lemma 1.3 we conclude that $\Gamma(X^{an}, \mathcal{A}_k) \to \Gamma(X^{an} \cap H^{an} \setminus Y^{an}, \mathcal{A}_k)$ is bijective (resp. injective). By the theorem of Remmert-Stein ([GR] Theorem V D 5), $\Gamma(X^{an}, \mathcal{A}_k) \simeq \Gamma(X^{an} \setminus Y^{an}, \mathcal{A}_k)$.

Now let us look at meromorphic functions:

Lemma 1.5: If $k \geq \dim Y + 3$ and D is a k-dimensional subvariety of X we have that $\Gamma(D^{an} \cap H^{an}, \mathcal{M}_{D^{an}}) \simeq \Gamma(D^{an} \cap H^{an} \setminus Y^{an}, \mathcal{M}_{D^{an}})$.

Proof: Replacing Y by $Y \cap D$ we may assume that $Y \subset D$.

Let us take up the notations of the proof of Lemma 1.3. Then it is sufficient to show:

$$j_*(\mathcal{M}_k | D^{an} \cap H^{an} \setminus (Y')^{an}) \simeq \mathcal{M}_k | D^{an} \cap H^{an} \setminus (Y'')^{an}$$

Again, it suffices to show that every meromorphic function on $W \cap D^{an}$ extends (uniquely) to a meromorphic function on $D^{an} \cap \{z\,|\,|z_j| < \epsilon_0, j = 1, \ldots, N-1, |z_N| < \delta_1\}$. The essential point is to show that every meromorphic function on $D^{an} \cap \{z\,|\,|z_j| < \epsilon_0, j = 1, \ldots, l, \epsilon_\nu < \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$ admits exactly one meromorphic extension on $D^{an} \cap \{z\,|\,|z_j| < \epsilon_0, j = 1, \ldots, l, \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_\nu\}$. (**)

If we have this we proceed as in the proof of Lemma 1.3: Every meromorphic function on $D^{an} \cap W$ admits exactly one meromorphic extension to $D^{an} \cap \{z\,|\,|z_j| < \epsilon_0, j = 1, \ldots, N-1, |z_N| < \delta_1\} \setminus Y'$, hence to $D^{an} \cap \{z\,|\,|z_j| < \epsilon_0, j = 1, \ldots, N-1, |z_N| < \delta_1\}$, by the Kontinuitätssatz, see [KK] 53.A.9.

In order to prove (**) we proceed as in the proof of Lemma 1.2, case $k \geq 3$. The essential point is to show the following lemma:

Lemma 1.6: Let $\mathcal{G}$ be a coherent analytic sheaf on $\{z \,|\, |z_j| < \epsilon_0, j = 1, \ldots, l, \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$ and $\mathcal{F}$ a coherent analytic subsheaf of $\mathcal{G}|\{z \,|\, |z_j| < \epsilon_0, j = 1, \ldots, l, \epsilon_\nu < \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$. Assume that for all open subsets W of $\{z \,|\, |z_j| < \epsilon_0, j = 1, \ldots, l, \epsilon_\nu < \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$ and all analytic subsets A of W with $\dim\ A \leq l+1$ the following holds:

Every section of $\mathcal{G}|W$ whose restriction to $W \setminus A$ belongs to $\mathcal{F}|W \setminus A$ is a section of $\mathcal{F}|W$.

Then $\mathcal{F}$ extends uniquely to a coherent analytic subsheaf of $\mathcal{G}$ with the analogous property.

Proof: Apply [S] Theorem 4.5, p. 156, with $n = l + 1$.

Therefore we get the following Lefschetz theorem for meromorphic functions:

Theorem 1.7: If $k \geq \dim Y + 3$ and D is a k-dimensional subvariety of X not contained in Y we have that $\Gamma(D^{an} \setminus Y^{an}, \mathcal{M}_{D^{an}}) \simeq \Gamma(D^{an} \cap H^{an} \setminus Y^{an}, \mathcal{M}_{D^{an}})$.

Proof: By the theorem of Remmert-Stein, we have $\Gamma(D^{an}, \mathcal{M}_{D^{an}}) \simeq \Gamma(D^{an} \setminus Y^{an}, \mathcal{M}_{D^{an}})$. The rest follows from Lemma 1.2 and 1.5.

Proof of Theorem 2: By Theorem 1.4, $C_k(X^{an} \setminus Y^{an}) \to \lim\limits_{\rightarrow} C_k(V)$ is bijective (resp. injective).

Furthermore, Theorem 1.7 implies that $Z_k(X^{an} \setminus Y^{an}) \simeq \lim\limits_{\rightarrow} Z_k(V)$.

This implies that the mapping $A_k(X^{an} \setminus Y^{an}) \to \lim\limits_{\rightarrow} A_k(V)$ is bijective (resp. injective). By Theorem 1 we have $A_k(X^{an} \setminus Y^{an}) \simeq \lim\limits_{\rightarrow} A_k(U \setminus Y^{an})$. Note that we have assumed $Y \subset H, Y \neq \emptyset$.

2. Algebraic context: Proof of Theorem 3 and 4

Here we need the following two lemmas:

Lemma 2.1: If $k \geq \dim Y + 3$, for every k-dimensional formal subvariety (i.e. non-empty closed irreducible reduced formal subscheme) C of $\hat{X} \setminus \hat{Y}$ there is exactly one subvariety C' of $X \setminus Y$ such that $\hat{C}' = C$.

Proof: Existence: C is also a formal subvariety of $\hat{\mathbb{P}}_N(\mathbb{C}) \setminus \hat{Y}$. Then apply Corollary 6 of [F1] with Y instead of Z: there is an extension of C to a closed subscheme C' of $\mathbb{P}_N(\mathbb{C}) \setminus Y$, "extension" means that $\hat{C}' = C$. Replacing C' by $C' \cap X$ if necessary we may suppose that C' is a closed subscheme of $X \setminus Y$. We may take C' to be reduced. If we take an irreducible component C'_0 with $\hat{C}'_0 \neq \emptyset$ we get that $\hat{C}'_0 = C$, so there is an extension to a subvariety of $X \setminus Y$. The uniqueness is clear.

Lemma 2.2: If D is a $(k+1)$-dimensional subvariety of $X \setminus Y$, $k \geq \dim Y + 2$, every rational function on $\hat{D}$ extends to a (unique) rational function on D.

Proof: This follows from [F1] Corollary 3 with Y instead of Z.

Proof of Theorem 3: Apply Lemma 2.1 and 2.2 with $Y := \emptyset$.

First suppose that $k \geq 1$. If C' is a k-dimensional subvariety of X not contained in Y we have that $C' \cap H \neq \emptyset$, so $\hat{C}' \neq \emptyset$, end $\hat{C}' \not\subset \hat{Y}$ because otherwise $C' \subset Y$. This implies that

$C_k(X\setminus Y) \to C_k(\hat{X},\hat{Y})$ is injective. By Lemma 2.1 and 2.2 we obtain that $Z_k(X\setminus Y) \to Z_k(\hat{X},\hat{Y})$ is bijective. So we obtain injectivity.

Now suppose $k \geq 2$. By Lemma 2.1 and 2.2, for every k-dimensional formal subvariety C of $\hat{X}$ not contained in $\hat{Y}$ there is exactly one subvariety C' of X such that $\hat{C}' = C$; in fact, we have $C' \subset X \setminus Y$. Similarly, if f is a rational function on a $(k+1)$-dimensional formal subvariety C of $\hat{X} \setminus \hat{Y}$, we have a unique subvariety D' of $X \setminus Y$ with $\hat{D}' = D$ and a unique rational function on D' which induces f. In total we obtain bijectivity.

Proof of Theorem 4: Using Lemma 2.1 and 2.2 we get that $C_k(X \setminus Y) \to C_k(\hat{X} \setminus \hat{Y})$ is bijective (resp. injective) and that $Z_k(X \setminus Y) \to Z_k(\hat{X} \setminus \hat{Y})$ is bijective. Note that a subvariety C' of $X \setminus Y$ of dimension $\geq \dim Y + 2$ must intersect $H \setminus Y$, so $\hat{C}' \neq \emptyset$ in $\hat{X} \setminus \hat{Y}$. We conclude that $A_k(X \setminus Y) \to A_k(\hat{X} \setminus \hat{Y})$ is bijective (resp. injective).

Furthermore, $A_k(X \setminus Y) \simeq A_k(\hat{X},\hat{Y})$ by Theorem 3. So we obtain Theorem 4.

3. Remarks on the comparison of the analytic and algebraic context

The comparison is especially simple in the case of $A_k(X,Y)$ and the corresponding analytic object. If we pass to the formal context it seems that the following assertion (*) is considered as a consequence of GAGA theory, see [F2] p. 737 resp. [B] §10, p. 115:

a) For every formal analytic subvariety C of $\hat{X}^{an}$ there is exactly one formal subvariety C' of $\hat{X}$ such that $(C')^{an} = C$.
b) Let D be a formal subvariety of $\hat{X}$. Every formal meromorphic function f on D^{an} is rational, i.e. there is a (unique) formal rational function g on D such that $g^{an} = f$. (*)

Remark 3.1: Adopting (*) we have a commutative diagram

$$\begin{array}{ccccc} A_k(X,Y) & & \longrightarrow & & A_k(\hat{X},\hat{Y}) \\ \downarrow\simeq & & & & \downarrow\simeq \\ A_k(X^{an},Y^{an}) & \to & \lim\limits_{\to} A_k(U,U\cap Y^{an}) & \to & A_k(\hat{X}^{an},\hat{Y}^{an}) \end{array}$$

where all arrows are bijective if $k \geq 2$ resp. injective if $k \geq 1$.

Here U runs through the set of open neighbourhoods of $X^{an} \cap H^{an}$ in X^{an}.

Proof: By Chow's theorem ([GR] Theorem V D 7), analytic subvarieties of X^{an} are algebraic. Therefore it is easy to see that $C_k(X,Y) \simeq C_k(X^{an},Y^{an})$. Now let D be a subvarity of X. By Hurwitz' theorem, see [Fi] 4.7, every meromorphic function on D^{an} is rational, i.e. comes from a (unique) rational function on D. Therefore $Z_k(X,Y) \simeq Z_k(X^{an},Y^{an})$. Altogether, the left vertical arrow is bijective.

By (*) it is easy to see that $C_k(\hat{X},\hat{Y}) \simeq C_k(\hat{X}^{an},\hat{Y}^{an})$ and $Z_k(\hat{X},\hat{Y}) \simeq Z_k(\hat{X}^{an},\hat{Y}^{an})$, so the right vertical is bijective, too.

The upper arrow is bijective (resp. injective) by Theorem 3.

So the composition of the lower horizontal mappings is bijective (resp. injective).

By Theorem 1', the first lower horizontal arrow is bijective (resp. injective). If $k \geq 2$ we obtain our statement. But in order to treat the case $k = 1$ we need that the second lower horizontal arrow is injective in this case, too. This can easily be proved: Let $k \geq 1$. Every purely k-dimensional analytic subvariety C' of U_R is uniquely determined by its completion $\hat{C}'$, so $C_k(U_R,U_R\cap Y^{an}) \to C_k(\hat{X}^{an},\hat{Y}^{an})$ is injective, and $Z_k(U_R,U_R\cap Y^{an}) \simeq Z_k(\hat{X}^{an},\hat{Y}^{an})$: the injectivity is clear, the surjectivity comes from that of $Z_k(X^{an},Y^{an}) \to Z_k(\hat{X}^{an},\hat{Y}^{an})$: we have

$Z_k(X^{an}, Y^{an}) \simeq Z_k(X,Y) \simeq Z_k(\hat{X},\hat{Y}) \simeq Z_k(\hat{X}^{an},\hat{Y}^{an})$. This makes the proof of Theorem 1' superfluous!

It is plausible that we should have a connection between the algebraic and analytic case with respect to Theorem 2 and 4, too. First notice:

Proposition 3.2: If $k \geq \dim Y + 1$ we have a commutative diagram

$$\begin{array}{ccc} A_k(X) & \simeq & A_k(X \setminus Y) \\ \downarrow\simeq & & \downarrow\simeq \\ A_k(X^{an}) & \simeq & A_k(X^{an} \setminus Y^{an}) \end{array}$$

Proof: By the theorem of Remmert-Stein ([GR] Theorem V D 5), irreducible analytic subsets of $X^{an} \setminus Y^{an}$ of dimension $\geq \dim Y + 1$ extend to X^{an}. By Chow ([GR] V D 7), analytic subsets of X^{an} are algebraic.

Of course, Zariski-closed subsets of $X \setminus Y$ extend to X.

On the other hand, if D is an irreducible subvariety of X of dimension $\geq \dim Y + 2$, every meromorphic function on $D^{an} \setminus Y^{an}$ is meromorphic on D^{an} by the Kontinuitätssatz [KK] 53.A.9. Meromorphic functions on X^{an} are rational by Hurwitz' Theorem, see [Fi] 4.7. Note that rational functions on $X \setminus Y$ coincide wth those on X.

Now let us state the following conjecture:

Conjecture 3.3: The mapping $A_k(\hat{X} \setminus \hat{Y}) \to A_k(\hat{X}^{an} \setminus \hat{Y}^{an})$ is bijective if $k \geq \dim(Y \cap H) + 3$ and injective if $k \geq \dim(Y \cap H) + 2$.

Remark 3.4: Suppose that Conjecture 3.3 holds. Then we have a commutative diagram

$$\begin{array}{ccccc} A_k(X \setminus Y) & & \longrightarrow & & A_k(\hat{X} \setminus \hat{Y}) \\ \downarrow & & & & \downarrow \\ A_k(X^{an} \setminus Y^{an}) & \to & \varinjlim A_k(V) & \to & A_k(\hat{X}^{an} \setminus \hat{Y}^{an}) \end{array}$$

where all arrows are bijective if $k \geq \dim(Y \cap H) + 3$ resp. injective if $k \geq \dim(Y \cap H) + 2$.

Here V runs through the set of all open neighbourhoods of $X^{an} \cap H^{an} \setminus Y^{an}$ in $X^{an} \setminus Y^{an}$.

Proof: By Proposition 3.2, the left vertical is bijective. Now Conjecture 3.3 yields that the right vertical is bijective (resp. injective).

The upper horizontal is bijective (resp. injective) because of Theorem 4.

So the composition of the lower horizontal mappings is bijective (resp. injective).

Now suppose $k \geq \dim Y \cap H + 3$. Then the first mapping in the lower horizontal is bijective, by Theorem 2. Altogether this implies that all arrows are bijective.

However we can argue in a simpler way which would lead (if Conjecture 3.3 holds) to a new proof of Theorem 2 and allows to treat the case $k = \dim Y \cap H + 2$, too: It is easy to see that the second arrow in the lower horizontal is injective for $k \geq \dim Y \cap H + 2$.

Every purely k-dimensional analytic subset C of V is uniquely determined by its completion $\hat{C}$, so $\varinjlim C_k(V) \to C_k(\hat{X}^{an} \setminus \hat{Y}^{an})$ is injective. Also, $\varinjlim Z_k(V) \to Z_k(\hat{X}^{an} \setminus \hat{Y}^{an})$ is surjective: this follows from $Z_k(X^{an} \setminus Y^{an} \simeq Z_k(X \setminus Y) \simeq Z_k(\hat{X} \setminus \hat{Y}) \simeq Z_k(\hat{X}^{an} \setminus \hat{Y}^{an})$. This yields the desired injectivity.

References:

[B] L. Bădescu: Projective Geometry and Formal Geometry. Monografie Mat. **65**. Birkhäuser: Basel 2004.

[F1] G. Faltings: A contribution to the theory of formal meromorphic functions. Nagoya Math. J. **77**, 99-106 (1980).

[F2] G. Faltings: Some theorems about formal functions. Publ. Res. Inst. Math. Sci. **16**, 721-737 (1980). DOI: 10.2977/prims/1195186927

[Fi] G. Fischer: Complex Analytic Geometry. Lecture Notes in Math. **538**. Springer-Verlag: Heidelberg 1976.

[Fu] W. Fulton: Intersection Theory. Springer-Verlag: Berlin 1984.

[G] A. Grothendieck: Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA II). North-Holland: Amsterdam 1968.

[Go] R. Godement: Topologie algébrique et théorie des faisceaux. Hermann: Paris 1964.

[GD] A. Grothendieck, J.A. Dieudonné: Éléments de Géométrie Algébrique I. Springer-Verlag: Berlin 1971.

[GR] R. C. Gunning, H. Rossi: Analytic Functions of Several Complex Variables. Prentice Hall: Englewood Cliffs, N.J. 1965.

[H1] H. A. Hamm: On theorems of Zariski-Lefschetz type. Contemp. Math. **475**, 69-78 (2008).

[H2] H. A. Hamm: On the local Picard group. Proc. Steklov Inst. Math. **267**, 1-8 (2009). DOI: 10.1134/S0081543809040117

[HL1] H.A.Hamm, Lê D. T.: On the Picard group for non-complete algebraic varieties, Singularités Franco-Japonaises, Sém. Congr. **10**, 71–86, Soc. Math. Fr., Paris 2005.

[HL2] H.A.Hamm, Lê D. T.: A Lefschetz theorem on the Picard group of complex projective varieties. In: Singularities in Geometry and Topology, Proc. Sing. Summer School and Workshop (Trieste 2005), ed. J.-P. Brasselet et al., pp. 640–660. World Sc. Pub., Singapore 2007. DOI: 10.1142/9789812706812_0021

[HL3] H.A. Hamm, Lê D.T.: Théorèmes d'annulation et groupes de Picard. Journal of Singularities **1**, 13-36 (2010).

[KK] L. Kaup, B. Kaup: Holomorphic functions of several variables. De Gruyter: Berlin 1983.

[S] Y.-T. Siu: Techniques of extension of analytic objects. Marcel Dekker: N.Y. 1974.

[ST] Y.-T. Siu, G. Trautmann: Extension of coherent analytic subsheaves. Math. Ann. **188**, 128-142 (1970). DOI: 10.1007/BF01350816

[V] J.-L. Verdier: Classe d'homologie associée à un cycle. Astérisque **36-37**, 101-151 (1976).