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SINGULARITIES OF ONE-PARAMETER PEDAL UNFOLDINGS OF SPHERICAL PEDAL CURVES

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Abstract. In this paper, we present the concept of one-parameter pedal unfoldings of a pedal curve in the unit sphere S^2, and we classify their generic singularities with respect to $\mathcal{A}$-equivalence.

1. Introduction

Let I be an open interval containing zero, and let S^2 be the unit sphere in Euclidean space $\mathbb{R}^3$. A C^∞ map $\mathbf{r} : I \to S^2$ is called a *spherical unit speed curve* if $\left\|\frac{d\mathbf{r}}{ds}(s)\right\|$ is 1 for any $s \in I$. For a given spherical unit speed curve $\mathbf{r} : I \to S^2$, we put

$$\mathbf{t}(s) = \frac{d\mathbf{r}}{ds}(s),\ \mathbf{n}(s) = \mathbf{r}(s) \times \mathbf{t}(s),$$

where $\mathbf{r}(s) \times \mathbf{t}(s)$ denotes the vector product of $\mathbf{r}(s)$ and $\mathbf{t}(s)$. The construction clearly shows that the vector $\mathbf{t}(s)$ is perpendicular to the vector $\mathbf{r}(s)$ and that the vector $\mathbf{n}(s)$ is perpendicular to both $\mathbf{r}(s)$ and $\mathbf{t}(s)$. The map $\mathbf{n} : I \to S^2$ is called the *spherical dual* of $\mathbf{r}$; the singularities of spherical dual curves are Legendrian singularities that are relatively well investigated [1, 2, 3, 4, 5, 21].

For a point $P \in S^2$, let E_P denote the set $\{X \in S^2 \mid P \cdot X = 0\}$, where $P \cdot X$ denotes the scalar product of P and X. For a given spherical unit speed curve $\mathbf{r} : I \to S^2$, consider a point P of $S^2 - \{\pm\mathbf{n}(s) \mid s \in I\}$, where $\mathbf{n}$ is the spherical dual of $\mathbf{r}$. The *spherical pedal curve relative to the point* P for a given spherical unit speed curve $\mathbf{r} : I \to S^2$ is a curve obtained by mapping $s \in I$ to the nearest point in $E_{\mathbf{n}(s)}$ from P. The pedal curve relative to P for $\mathbf{r}$ is denoted by $ped_{\mathbf{r},P}$, and the point P is called the *pedal point of* the pedal curve $ped_{\mathbf{r},P}$. Note that all points in $E_{\mathbf{n}(s)}$ are equidistant from $\pm\mathbf{n}(s)$; hence, the point P must lie outside $\{\pm\mathbf{n}(s) \mid s \in I\}$ to satisfy the definition of $ped_{\mathbf{r},P}$. The classification of singularities of spherical pedal curves can be found in literature [17, 18, 19].

Suppose that the location of the pedal point P moves smoothly, depending on one-parameter $\lambda \in J$, where J is an open interval containing zero in $\mathbb{R}$. In other words, suppose that there exist an open interval J containing zero and a C^∞ immersion $P : J \to S^2$. Then, the *pedal unfolding of the pedal curve* $ped_{\mathbf{r},P(0)}$ can be defined as the map $Un\text{-}ped_{\mathbf{r},P} : I \times J \to S^2 \times J$, given by

$$Un\text{-}ped_{\mathbf{r},P}(s, \lambda) = (ped_{\mathbf{r},P(\lambda)}(s), \lambda).$$

Two C^∞ map-germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are said to be $\mathcal{A}$-*equivalent* if there exist germs of C^∞-diffeomorphisms $h_1 : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $h_2 : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that $f \circ h_1 = h_2 \circ g$. For a spherical unit speed curve germ $\mathbf{r} : (I, 0) \to S^2$, we put $\kappa(s) = \mathbf{n}(s) \cdot \mathbf{t}'(s)$, where $\mathbf{t}'$ denotes

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TABLE 1. Normal forms of $\mathcal{A}$-simple monogerms $(\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ ([15])

Germ	*Name*
$f(s,\lambda) = (s, s^2, \lambda)$	*Immersion*
$f(s,\lambda) = (s^3 + \lambda s, s^2, \lambda)$	*Cross-cap* (S_0)
$f(s,\lambda) = (s^3 \pm \lambda^{k+1} s, s^2, \lambda),\ (k \geq 1)$	$S_k^{\pm}$
$f(s,\lambda) = (\lambda^2 s \pm s^{2k+1}, s^2, \lambda),\ (k \geq 2)$	$B_k^{\pm}$
$f(s,\lambda) = (\lambda s^3 \pm \lambda^k s, s^2, \lambda),\ (k \geq 3)$	$C_k^{\pm}$
$f(s,\lambda) = (\lambda^3 s + s^5, s^2, \lambda)$	F_4
$f(s,\lambda) = (\lambda s + s^{3k-1}, s^3, \lambda),\ (k \geq 2)$	H_k

the derivative of $\mathbf{t}$. Then, the point $\mathbf{r}(0)$ is called the *inflection point* (resp., *ordinary inflection point*) if $\kappa(0) = 0$ holds (resp., $\kappa(0) = 0$ and $\kappa'(0) \neq 0$ hold). For any $k \geq 0$, a C^∞ immersed curve germ $P : (J,0) \to S^2$ is said to *have* $(k+1)$*-point contact with* $\mathbf{r} : (I,0) \to S^2$ at $P(0) = \mathbf{r}(0)$ if $P(0) = \mathbf{r}(0)$, $F \circ P(0) = (F \circ P)'(0) = \cdots = (F \circ P)^{(k)}(0) = 0$, and $(F \circ P)^{(k+1)}(0) \neq 0$ hold for any neighbourhood U of $\mathbf{r}(0)$ and any non-singular C^∞ function $F : U \to \mathbb{R}$ such that $F \circ \mathbf{r}(s) = 0$ (for details on $(k+1)$-point contact, see [5]). It can be clearly seen that a C^∞ immersed curve germ $P : (J,0) \to S^2$ has 1-point contact with $\mathbf{r} : (I,0) \to S^2$ at $P(0) = \mathbf{r}(0)$ if and only if P and $\mathbf{r}$ are transverse at $P(0) = \mathbf{r}(0)$.

Theorem 1. *Let I, J be open intervals containing $0 \in \mathbb{R}$, and let $\mathbf{r} : I \to S^2$ be a spherical unit speed curve such that $\mathbf{r}(0)$ is not an inflection point. Furthermore, let $P : J \to S^2$ be a C^∞ immersion. Then, the following hold:*

1. *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is immersive if and only if $P(0) \neq \mathbf{r}(0)$.*
2. *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cross-cap in Table 1 if und only if $P(0) = \mathbf{r}(0)$ and $P, \mathbf{r}$ are transverse at $P(0) = \mathbf{r}(0)$.*
3. *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to $S_k^{\pm}$ in Table 1 if and only if $P(0) = \mathbf{r}(0)$ and P has $(k+1)$-point contact with $\mathbf{r}$ at $0 \in J$ $(k \geq 1)$.*
4. *The $\mathcal{A}$-equivalence classes of map-germs $B_k^{\pm}, C_k^{\pm}, F_4$, and H_k in Table 1 can never be realized as singularities of the pedal unfolding $Un\text{-}ped_{\mathbf{r},P}$.*
5. *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal edge in Table 2 if and only if $P(0) = \mathbf{r}(0)$ and $(P(J), P(0))$ coincides with $(\mathbf{r}(I), \mathbf{r}(0))$ as set-germs.*

If k is even, then it can be clearly seen that S_k^+ is $\mathcal{A}$-equivalent to S_k^- [15]. On the other hand, S_k^+ is not $\mathcal{A}$-equivalent to S_k^- if k is odd. Figure 2 shows that the curvature of $\mathbf{r}$ at zero is greater than the curvature of P at zero if and only if the pedal unfolding $Un\text{-}ped_{\mathbf{r},P}$ is $\mathcal{A}$-equivalent to S_k^-. Since $S_1^{\pm}$ has been investigated independently in [6], it is reasonable to classify the $\mathcal{A}$-equivalence class of $S_1^{\pm}$ as *Chen-Matumoto-Mond singularity.*

Theorem 2. *Let I, J be open intervals containing $0 \in \mathbb{R}$, and let $\mathbf{r} : I \to S^2$ be a spherical unit speed curve such that $\mathbf{r}(0)$ is an ordinary inflection point. Furthermore, let $P : J \to S^2$ be a C^∞ immersion. Then, the following hold:*

1. *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal edge in Table 2 if and only if $P(0) \notin E_{\mathbf{n}(0)}$.*

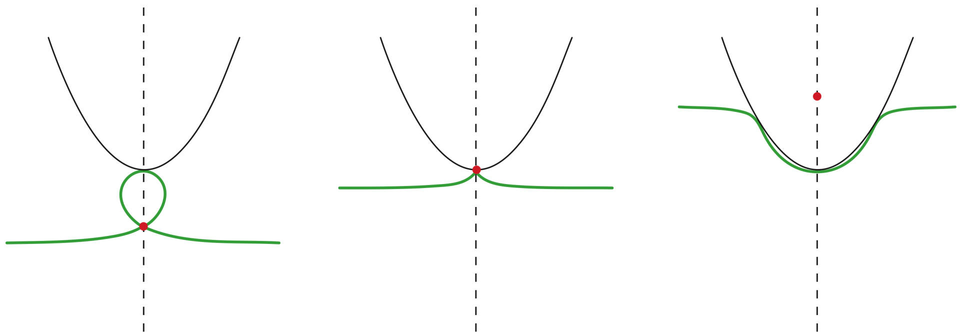

FIGURE 1. Cross-cap. Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.

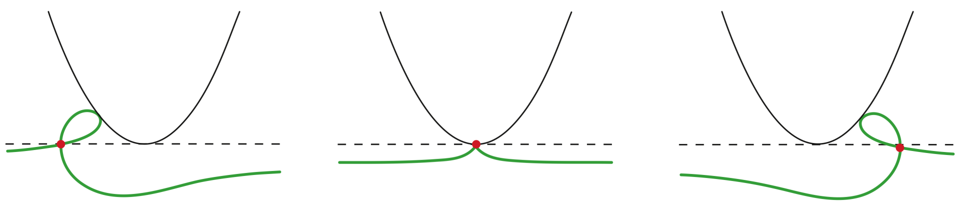

FIGURE 2. S_1^-. Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.

TABLE 2.

Germ	*Name*
$g(s, \lambda) = (s^3, s^2, \lambda)$	*Cuspidal edge*
$g_0^+(s, \lambda) = (s^5 + \lambda s^3, s^2, \lambda)$	*Cuspidal cross-cap (Cuspidal S_0)*
$g_k^\pm(s, \lambda) = (s^5 \pm \lambda^{k+1} s^3, s^2, \lambda),\ (k \geq 1)$	*Cuspidal $S_k^\pm$*

(2) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal cross-cap in Table 2 if und only if $P(0) \in E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$ and P is transverse to $E_{\mathbf{n}(0)}$ at $P(0)$.*

(3) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to cuspidal $S_k^\pm$ $(k \geq 1)$ in Table 2 if and only if $P(0) \in E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$ and P has $(k+1)$-point contact with $E_{\mathbf{n}(0)}$ $(k \geq 1)$.*

As in the case of $S_k^\pm$ singularities, it can be clearly seen that cuspidal S_k^+ singularity is $\mathcal{A}$-equivalent to cuspidal S_k^- singularity if k is even. On the other hand, cuspidal S_k^+ singularity is not $\mathcal{A}$-equivalent to cuspidal S_k^- singularity if k is odd. Figure 4 shows that for a sufficiently small positive real number ε, there exists a positive real number δ such that the union of tangent lines $\cup_{s \in (-\varepsilon, \varepsilon)} E_{\mathbf{n}(s)}$ contains the images $P((-\delta, \delta))$ if and only if the map-germ $Un\text{-}ped_{\mathbf{r},P}$: $(I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to cuspidal S_k^- singularity. Since map-germ g_0^+ singularity

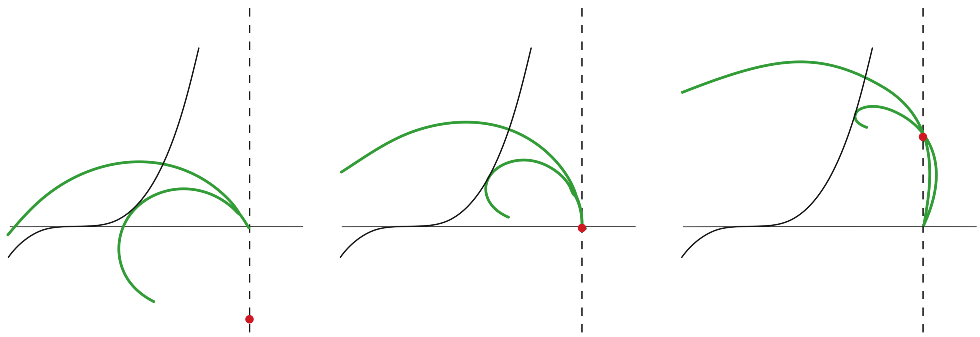

FIGURE 3. Cuspidal cross-cap. Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.

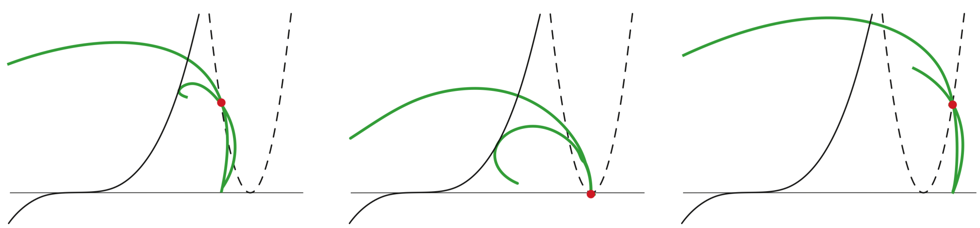

FIGURE 4. Cuspidal S_1^-. Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.

is known as the normal form of the *cuspidal cross-cap* (see [11]), it is reasonable to classify the $\mathcal{A}$-equivalence class of the map-germ $g_{k,\pm}$ (resp., $g_{1,\pm}$) as *cuspidal* $S_k^\pm$ *singularity* (resp., *cuspidal Chen-Matumoto-Mond singularity*).

It can be clearly seen that the cuspidal edge, cuspidal cross-cap, and cuspidal $S_k^\pm$ are not finitely $\mathcal{A}$-determined (but finitely $\mathcal{K}$-determined) by the Mather-Gaffney geometric characterization of finite determinacy, even though $S_k^\pm$ singularity is $(k+2)$-$\mathcal{A}$-determined [15] (for the definition of finite determinacy and Mather-Gaffney geometric characterization, see [23]). Thus, in order to prove Theorems 1 and 2 in a unified manner, it is difficult to directly use the standard techniques of the finite determinacy theory developed in [8, 9, 10, 13, 14, 15, 20, 23].

On the other hand, Saji succeeded in obtaining simple criteria for Chen-Matumoto-Mond singularity and cuspidal $S_k^\pm$-singularities [22]. Although Saji's criteria are useful, the criteria for $S_k^\pm$ singularities ($k \geq 2$) have not been provided by him; therefore, Saji's criteria are not suited to our purpose. In this study, we plan to develop a unified method for proving Theorems 1 and 2; hence, we adopt a recognition criterion for map-germs that appear as singularities of pedal unfoldings. It is important to note that this criterion has already been presented in a suitable form in [15].

The preliminary work required to prove Theorems 1 and 2 is presented in Section 2. Theorems 1 and 2 are proved in Sections 3 and 4, respectively.

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2. Preliminaries

2.1. **Spherical pedal curves.** Let I, S^2, and $\mathbf{r} : I \to S^2$ be an interval containing zero, the unit sphere in $\mathbb{R}^3$, and a spherical unit speed curve respectively. Furthermore, let $\mathbf{t} : I \to S^2$, $\mathbf{n} : I \to S^2$ be map-germs, as described in Section 1. Then, we have the following Serret-Frenet type formula.

Lemma 2.1 ([17])**.**

$$\begin{pmatrix} \mathbf{r}'(s) \\ \mathbf{t}'(s) \\ \mathbf{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa(s) \\ 0 & -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}(s) \\ \mathbf{t}(s) \\ \mathbf{n}(s) \end{pmatrix}.$$

By Lemma 2.1, the dual curve germ $\mathbf{n} : (I, 0) \to S^2$ is non-singular at 0 if and only if $\kappa(0) \neq 0$. By using Lemma 2.1 recursively, we obtain the following:

Lemma 2.2. (1) *Suppose that $\kappa(0) \neq 0$. Then, the properties $\mathbf{r}(0)\cdot\mathbf{n}'(0) = 0$, $\mathbf{r}(0)\cdot\mathbf{n}''(0) \neq 0$, and $\mathbf{t}(0)\cdot\mathbf{n}'(0) \neq 0$ hold.*

(2) *Suppose that $\kappa(0) = 0$ and $\kappa'(0) \neq 0$. Then, the properties $\mathbf{r}(0)\cdot\mathbf{n}'(0) = \mathbf{r}(0)\cdot\mathbf{n}''(0) = 0$, $\mathbf{r}(0)\cdot\mathbf{n}^{(3)}(0) \neq 0$, $\mathbf{t}(0)\cdot\mathbf{n}'(0) = 0$, and $\mathbf{t}(0)\cdot\mathbf{n}''(0) \neq 0$ hold.*

Let P be a point of $S^2 - \{\pm\mathbf{n}(s) \mid s \in I\}$.

Lemma 2.3 ([17])**.** *The pedal curve of $\mathbf{r}$ relative to the pedal point P is given by the following expression:*

$$ped_{\mathbf{r},P}(s) = \frac{1}{\sqrt{1-(P\cdot\mathbf{n}(s))^2}}(P - (P\cdot\mathbf{n}(s))\mathbf{n}(s)).$$

Let Ψ_P be the C^∞ map from $S^2 - \{\pm P\}$ to S^2, given by

$$\Psi_P(X) = \frac{1}{\sqrt{1-(P\cdot X)^2}}(P - (P\cdot X)X).$$

The map Ψ_P, which has been introduced and used in [17, 18, 19] (the hyperbolic version of Ψ_P has been introduced and investigated independently in [12]), has the following distinctive properties :

(1) $X \cdot \Psi_P(X) = 0$ for any $X \in S^2 - \{\pm P\}$.
(2) $\Psi_P(X) \in \mathbb{R}P + \mathbb{R}X$ for any $X \in S^2 - \{\pm P\}$.
(3) $P \cdot \Psi_P(X) > 0$ for any $X \in S^2 - \{\pm P\}$.

By property 3, $\Psi_P(S^2 - \{\pm P\})$ lies inside the open hemisphere centered at P. By properties 1 and 2, $\Psi_P(E_P) = P$. Let the open hemisphere centered at P be denoted by H_P, and put $B_P = \pi(S^2 - \{\pm P\})$, where $\pi : S^2 \to P^2(\mathbb{R})$ is the canonical projection. Since $\Psi_P(X) = \Psi_P(-X)$, the map Ψ_P canonically induces the map $\widetilde{\Psi}_P : B_P \to H_P$. Then, by Lemma 2.3, $ped_{\mathbf{r},P}$ is factored into three maps as follows:

$$ped_{\mathbf{r},P}(s) = \widetilde{\Psi}_P \circ \pi \circ \mathbf{n}(s).$$

Let $p : B \to \mathbb{R}^2$ be the blow up centered at the origin in $\mathbb{R}^2$.

Lemma 2.4 ([17])**.** *Let P be a point of S^2. Then, there exist C^∞ diffeomorphisms $h_1 : B_P \to B$ and $h_2 : H_P \to \mathbb{R}^2$ such that the equality $h_2 \circ \widetilde{\Psi}_P = p \circ h_1$ holds, and the set $\pi(E_P)$ is mapped to the exceptional set of p by h_1.*

2.2. **Criterion for recognition problem due to Mond.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the form $T(s, \lambda) = (-s, \lambda)$. Two C^∞ function germs $p_1, p_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ are said to be $\mathcal{K}^T$-*equivalent* if there exist a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of the form $h \circ T = T \circ h$ and a C^∞ function-germ $M : (\mathbb{R}^2, (0, 0)) \to \mathbb{R}$ of the form $M \circ T = M$, $M(0, 0) \neq 0$ such that $p_1 \circ h(s, \lambda) = M(s, \lambda)p_2(s, \lambda)$ ([15]).

Theorem 3 ([15])**.** *Two C^∞ map-germs of the following form*

$$f_i(s, \lambda) = (sp_i(s^2, \lambda), s^2, \lambda) \quad \textit{where } p_i(s^2, \lambda) \notin m_2^\infty, \quad (i = 1,\ 2)$$

are $\mathcal{A}$-equivalent if and only if the function-germs $p_i(s^2, \lambda)$ are $\mathcal{K}^T$-equivalent.

Note that Theorem 3 provides a criterion for the $\mathcal{A}$-equivalence of C^∞ map-germs of the forms $(s, \lambda) \mapsto (\varphi(s, \lambda), s^2, \lambda)$ ($\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is a C^∞ function-germ) on the basis of the Malgrange preparation theorem (for the Malgrange preparation theorem, see [4, 23]).

3. Proof of Theorem 1

Since $\mathbf{r}(0)$ is not an inflection point, the dual germ $\mathbf{n} : (I, 0) \to S^2$ is a C^∞ immersive germ.

Proof of assertion 1 of Theorem 1.

Suppose that $P(0)$ does not belong to $E_{\mathbf{n}(0)}$. Then, by Lemma 2.4, the restriction $\Psi_{P(0)}|_{S^2 - \{\pm P(0)\} - E_{P(0)}}$ is C^∞ immersive. Thus, by Lemma 2.3, the map-germ $ped_{\mathbf{r},P(0)} : (I, 0) \to S^2$ is also C^∞ immersive. Therefore, the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is also C^∞ immersive. Next, suppose that $P(0) \in E_{\mathbf{n}(0)} - \mathbf{r}(0)$. Then, the image of the dual $\mathbf{n}$ and $E_{P(0)}$ intersect transeversely at $\mathbf{n}(0)$. Thus, by Lemmata 2.3 and 2.4, the map-germ $ped_{\mathbf{r},P(0)} : (I, 0) \to S^2$ is C^∞ immersive. Therefore, the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is also C^∞ immersive.

Conversely, suppose that the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is C^∞ immersive. Then, in particular, the map-germ $ped_{\mathbf{r},P(0)} : (I, 0) \to S^2$ is also C^∞ immersive. In order to conclude the proof of assertion 1 of Theorem 1, it is sufficient to show that the assumption $P(0) = \mathbf{r}(0)$ implies a contradiction. The assumption $P(0) = \mathbf{r}(0)$ implies that the image of $\mathbf{n}$ is tangent to $E_{P(0)}$ at $\mathbf{n}(0)$. By Lemma 2.4, the map-germ $ped_{\mathbf{r},P(0)} : (I, 0) \to S^2$ must be singular; this is a contradiction. □

Proof of assertion 5 of Theorem 1.

Suppose that both $P(0) = \mathbf{r}(0)$ and $(P(J), P(0)) = (\mathbf{r}(I), \mathbf{r}(0))$ as set-germs hold. Then, for any $\lambda \in J$, $ped_{\mathbf{r},P(\lambda)} : (I, 0) \to S^2$ is $\mathcal{A}$-equivalent to the ordinary cusp $s \mapsto (s^3, s^2)$ by [17] (also, see [19]). Thus, by using the Malgrange preparation theorem and Theorem 3, the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal edge $(s, \lambda) \mapsto (s^3, s^2, \lambda)$.

Conversely, suppose that the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal edge. Then, in particular, for any sufficiently small $\lambda_0 \in J$, there exists a sufficiently small $s_0 \in I$ such that the map-germ $ped_{\mathbf{r},P(\lambda_0)} : (I, s_0) \to S^2$ is singular. Since $\mathbf{r}(0)$ is not an inflection point, by Lemma 2.4, $E_{P(\lambda_0)} = S^2 \cap (\mathbb{R}\mathbf{t}(s_0) + \mathbb{R}\mathbf{n}(s_0))$. Therefore, $P(\lambda_0) = \mathbf{r}(s_0)$. □

Proof of assertions 2 and 3 of Theorem 1.

By composing an appropriate rotation without the loss of generality, it can be assumed that $\mathbf{r}(0) = (0, 1, 0)$, $\mathbf{t}(0) = (0, 0, 1)$, $\mathbf{n}(0) = (-1, 0, 0)$. For a point Q of S^2, put $H(Q) = \{X \in S^2 \mid Q \cdot X \geq 0\}$, and let $\alpha_{\mathbf{n}(0)} : H(\mathbf{n}(0)) - E_{\mathbf{n}(0)} \to \{-1\} \times \mathbb{R}^2$ be the central projection relative to $\mathbf{n}(0)$. Then, by Lemma 2.2, the germ of composition $\alpha_{\mathbf{n}(0)} \circ \mathbf{n}$ is of the form

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n}(s) = \big(s + \varphi_1(s), s^2 + \varphi_2(s)\big) \quad (\varphi_j(s) = o(s^j)).$$

Since $\varphi_2(s) = o(s^2)$, the map-germ given by $h\left(s\sqrt{1+\frac{\varphi_2(s)}{s^2}}\right) = s$ is a well-defined germ of local C^∞ diffeomorphism. Thus, there exists a C^∞ map-germ $\widetilde{\varphi}_1 : (I, 0) \to \mathbb{R}$ such that

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n} \circ h(s) = \left(s + \widetilde{\varphi}_1(s), s^2)\right) \quad (\widetilde{\varphi}_1(s) = o(s)).$$

Let $\alpha_{P(0)} : H(P(0)) - E_{P(0)} \to \mathbb{R} \times \{1\} \times \mathbb{R}$ be the central projection relative to $P(0)$. By the form mentioned above and Lemma 2.4, the germ of composition $\alpha_{P(0)} \circ ped_{\mathbf{r},P(0)}$ is $\mathcal{A}$-equivalent to a map-germ of the following form:

$$s \mapsto \left((s + \widetilde{\varphi}_1(s))s^2, s^2\right).$$

Next, we investigate the influence of moving the pedal points $P(\lambda)$. Suppose that $P(0) = \mathbf{r}(0)$ and P has $(k+1)$-point contact with $\mathbf{r}$ at $0 \in J$ $(k \geq 0)$. In other words, suppose that there exist a sufficiently small neighborhood U of $\mathbf{r}(0)$ in S^2 and a C^∞ function $F : U \to \mathbb{R}$ such that $F \circ \mathbf{r}(s) \equiv 0$ $(\forall s \in I \cap \mathbf{r}^{-1}(U))$, $F \circ P(0) = (F \circ P)'(0) = \cdots = (F \circ P)^{(k)}(0) = 0$, and $(F \circ P)^{(k+1)}(0) \neq 0$. Since $\mathbf{r} : I \to S^2$ is a unit speed curve, it can be assumed that F is non-singular provided that I (resp., U) is a sufficiently small neighborhood of 0 (resp., $\mathbf{r}(0)$). Then, there exists a sufficiently small neighborhood $\widetilde{U} \subset U$ of $\mathbf{r}(0)$ such that for any $X \in \widetilde{U}$, the integral curve of $-\mathrm{grad}(F)$ starting from X lies within $\widetilde{U}$ until it reaches the image of the unit speed curve $\mathbf{r}(I)$. Let this reaching point be denoted by $\gamma(X)$ and define the map $\Gamma : \widetilde{U} \to I$ as $\Gamma(X) = \mathbf{r}^{-1} \circ \gamma(X)$. Then, $(\widetilde{U}, (\Gamma, F))$ can be used as a chart at $\mathbf{r}(0)$ since the map $(\Gamma, F) : \widetilde{U} \to I \times \mathbb{R}$ is non-singular. By using the chart $(\widetilde{U}, (\Gamma, F))$ and by the proof of assertion 5 of Theorem 1, the germ of composition

$$\left(s, \lambda\right) \mapsto \left(\alpha_{P(0)} \circ ped_{\mathbf{r},P(\lambda)} \circ h(s), \lambda\right)$$

is $\mathcal{A}$-equivalent to a map-germ of the following form:

$$(a) \qquad (s, \lambda) \mapsto \left((s + \widetilde{\varphi}_1(s))\left(s^2 \pm F \circ P(\lambda)\right), s^2 \pm F \circ P(\lambda), \lambda\right).$$

Furthermore, by the Malgrange preparation theorem and Theorem 3, a map-germ of the form (a) must be $\mathcal{A}$-equivalent to the map-germ $f_k^{\pm}(s, \lambda) = \left(s\left(s^2 \pm \lambda^{k+1}\right), s^2, \lambda\right)$.

Conversely, we suppose that the germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to $S_k^{\pm}$ $(k \geq 0)$, $P(0) = \mathbf{r}(0)$ and that P does not have $(k+1)$-point contact with $\mathbf{r}$ at $0 \in J$. Then, by the proof presented above, for any positive integer ℓ, P does not have $(\ell+1)$-point contact with $\mathbf{r}$ at $0 \in J$. In particular, there exists a C^∞ immersion $\widetilde{P} : J \to S^2$ such that $\widetilde{P}$ is sufficiently near P under the Whitney C^∞ topology, and $\widetilde{P}$ has $(k+2)$-point contact with $\mathbf{r}$ at $0 \in J$. By the proof of the implication described above, it can be concluded that $S_k^{\pm}$ singularity is adjacent to $S_{k+1}^{\pm}$ singularity; however, this contradicts the adjacency diagram given in [15]. □

Proof of assertion 4 of Theorem 1.

Suppose that the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to one of $B_k^{\pm}, C_k^{\pm}, F_4$, and H_k. Then, by assertions 1, 2, and 3 of Theorem 1, the given immersion $P : J \to S^2$ must satisfy not only $P(0) = \mathbf{r}(0)$ but also the condition that for any positive integer ℓ, P does not have $(\ell+1)$-point contact with $\mathbf{r}$ at $0 \in J$. Thus, for any positive integer ℓ, there exists a C^∞ immersion $\widetilde{P} : J \to S^2$ such that $\widetilde{P}$ is sufficiently near P under the Whitney C^∞ topology, and $\widetilde{P}$ has the $(\ell+1)$-contact with $\mathbf{r}$ at $0 \in J$. Hence, it can be concluded that one of $B_k^{\pm}, C_k^{\pm}, F_4$, and H_k singularity is adjacent to $S_\ell^{\pm}$ singularity for any positive integer ℓ; however, this contradicts the adjacency diagram given in [15]. □

4. Proof of Theorem 2

Since $\mathbf{r}(0)$ is an ordinary inflection point, by Lemma 2.2 and the Malgrange preparation theorem, the dual germ $\mathbf{n} : (I, 0) \to S^2$ is $\mathcal{A}$-equivalent to the ordinary cusp $s \mapsto (s^3, s^2)$.

Proof of assertion 1 of Theorem 2.

Suppose that $P(0)$ does not belong to $E_{\mathbf{n}(0)}$. Then, for any sufficiently small $\lambda_0 \in J$, $P(\lambda_0)$ lies outside $E_{\mathbf{n}(0)}$. This implies that by Lemma 2.4, the map-germ $\Psi_{P(\lambda_0)}$ at $\mathbf{n}(0)$ is non-singular. Thus, by Lemma 2.3, the map-germ $ped_{\mathbf{r},P(\lambda_0)} : (I, 0) \to S^2$ is also $\mathcal{A}$-equivalent to the ordinary cusp. Therefore, by Theorem 3, the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal edge.

Conversely, suppose that the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the map-germ $g(s, \lambda) = (s^3, s^2, \lambda)$; we show that $P(0) \in E_{\mathbf{n}(0)}$ implies a contradiction under this assumption. The property $P(0) \in E_{\mathbf{n}(0)}$ implies that $\mathbf{n}(0) \in E_{P(0)}$. Since the dual germ $\mathbf{n} : (I, 0) \to S^2$ is $\mathcal{A}$-equivalent to the ordinary cusp $s \mapsto (s^3, s^2)$, by Lemma 2.4, $\mathbf{n}(0) \in E_{P(0)}$ implies that $j^3(Un\text{-}ped_{\mathbf{r},P})(0)$ is not $\mathcal{A}^3$-equivalent to $j^3g(0)$. This contradicts the assumption that $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the map-germ $g(s, \lambda) = (s^3, s^2, \lambda)$. □

Proof of "if" parts of assertions 2 and 3 of Theorem 1.

Since $P(0)$ belongs to $E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$, by composing an appropriate rotation without the loss of generality, it can be assumed that $\mathbf{n}(0) = (-1, 0, 0)$ and $P(0) = (0, 0, 1)$. Let $\alpha_{\mathbf{n}(0)} : H(\mathbf{n}(0)) - E_{\mathbf{n}(0)} \to \{-1\} \times \mathbb{R}^2$ be the central projection relative to $\mathbf{n}(0)$. Then, by Lemma 2.2, the germ of composition $\alpha_{\mathbf{n}(0)} \circ \mathbf{n}$ is of the form

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n}(s) = (as^2 + bs^3 + \varphi_1(s), cs^2 + ds^3 + \varphi_2(s)),$$

where $bc \neq 0$ and $\varphi_j(s) = o(s^3)$. Since $c \neq 0$, there exists a germ of C^∞ diffeomorphism $h : (I, 0) \to (I, 0)$ such that

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n} \circ h(s) = (\widetilde{a}s^2 + \widetilde{b}s^3 + \widetilde{\varphi}_1(s), s^2),$$

where $\widetilde{b} \neq 0$ and $\widetilde{\varphi}_1(s) = o(s^3)$. Let $\alpha_{P(0)} : H(P(0)) - E_{P(0)} \to \mathbb{R}^2 \times \{1\}$ be the central projection relative to $P(0)$. By the form mentioned above and Lemma 2.4, the germ of composition $\alpha_{P(0)} \circ ped_{\mathbf{r},P(0)}$ is $\mathcal{A}$-equivalent to a map-germ of the following form:

$$s \mapsto \left((\widetilde{a}s^2 + \widetilde{b}s^3 + \widetilde{\varphi}_1(s))s^2, s^2\right).$$

Next, we investigate the influence of moving the pedal points $P(\lambda)$. Suppose that $P(0) = \mathbf{r}(0)$ and P has $(k+1)$-point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$ $(k \geq 0)$. Since $E_{\mathbf{n}(0)}$ is defined by the equation $\mathbf{n}(0) \cdot X = 0$, the assumption of $(k + 1)$-point contact implies that $\mathbf{n}(0) \cdot P(0) = \mathbf{n}(0) \cdot P'(0) = \cdots = \mathbf{n}(0) \cdot P^{(k)}(0) = 0$ and $\mathbf{n}(0) \cdot P^{(k+1)}(0) \neq 0$. Then, as in the proof of assertions 2 and 3 of Theorem 1, the germ of composition

$$(s, \lambda) \mapsto (\alpha_{P(0)} \circ ped_{\mathbf{r},P(\lambda)}(s), \lambda)$$

is $\mathcal{A}$-equivalent to the germ of the following form:

$$(b) \qquad (s, \lambda) \mapsto \left(\left(\widetilde{a}s^2 + \widetilde{b}s^3 + \varphi_1(s)\right)\left(s^2 \pm \mathbf{n}(0) \cdot P(\lambda)\right), s^2 \pm \mathbf{n}(0) \cdot P(\lambda), \lambda\right).$$

Furthermore, by the Malgrange preparation theorem and Theorem 3, a map-germ of the form mentioned in (b) must be $\mathcal{A}$-equivalent to the map-germ $g_k^{\pm}(s, \lambda) = \left(s^3(s^2 \pm \lambda^{k+1}), s^2, \lambda\right)$. □

The "only if" parts of assertions 2 and 3 of Theorem 2 can be proved as follows. Put $\widetilde{g}(s,\lambda) = s^2$, $\widetilde{g}_0(s,\lambda) = s^4 + \lambda s^2$, $\widetilde{g}^{\pm}_{2i+1}(s,\lambda) = s^4 \pm \lambda^{2i+2}s^2$, and $\widetilde{g}^{+}_{2i}(s,\lambda) = s^4 + \lambda^{2i+1}s^2$. Then, it can be clearly seen that any two distinct elements of the following set are not $\mathcal{K}^T$-equivalent.

$$\left\{\widetilde{g}, \widetilde{g}_0, \widetilde{g}^{+}_1, \widetilde{g}^{-}_1, \widetilde{g}^{+}_2, \widetilde{g}^{+}_3, \widetilde{g}^{-}_3, \cdots\right\}.$$

Hence, by Theorem 3, any two distinct elements of the set of the cuspidal edge, cuspidal cross-cap, cuspidal S^+_1, cuspidal S^-_1, cuspidal S^+_2, cuspidal S^+_3, cuspidal S^-_3 $\cdots$ are not $\mathcal{A}$-equivalent. Furthermore, by Theorem 3 and the form of $g_0, g^{\pm}_1, g^{\pm}_2, \cdots$ in Table 2, the following adjacency diagram is obtained.

$$(c) \qquad \cdots \longrightarrow \text{cuspidal } S_k \longrightarrow \cdots \longrightarrow \text{cuspidal } S_1 \longrightarrow \text{cuspidal } S_0.$$

Proof of "only if" parts of the assertions 2, 3 of Theorem 2.

As in the proof of the "only if" parts of assertions 2 and 3 of Theorem 1, we suppose that $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to cuspidal $S^{\pm}_k$ ($k \geq 0$), $P(0) \in E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$, and P does not have $(k+1)$-point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$. Then, by the "if" parts of assertions 2, 3 of Theorem 2, for any non-negative integer ℓ, P does not have $(\ell+1)$-point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$. In particular, for any non-negative integer ℓ, there exists a C^∞ immersion $\widetilde{P} : J \to S^2$ such that $\widetilde{P}$ is sufficiently near P under the Whitney C^∞ topology, and $\widetilde{P}$ has $(\ell+1)$-point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$. Hence, it can be concluded that cuspidal $S^{\pm}_k$ singularity is adjacent to cuspidal $S^{\pm}_\ell$ singularity for any positive integer ℓ; however, this contradicts diagram (c).

Next, suppose that $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to cuspidal $S^{\pm}_k$ ($k \geq 0$) and $P(0) = \{\mathbf{r}(0)\}$. In this case, the tangent cone of $\mathbf{n}(I)$ at $\mathbf{n}(0)$ coincides with E_P. Thus, by Lemma 2.4, $j^2(Un\text{-}ped_{\mathbf{r},P})(0)$ is not $\mathcal{A}^2$-equivalent to $j^2 g^{\pm}_k(0)$; this contradicts the assumption that $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the map-germ $g^{\pm}_k(s,\lambda) = (s^5 \pm \lambda^{k+1}s^3, s^2, \lambda)$. □

Remarks

It is possible to adopt the criteria given in [16] or an argument similar to that given in [7] to prove Theorems 1 and 2. However, the criteria in [16] are too general to be directly applied to our study, and the argument in [7] seems to be somewhat ad hoc. Thus, in order to apply them to our study, considerable preliminary work is required, the proofs of which are time-consuming and complicated. On the other hand, Theorem 3 is the most suitable criterion for our study. Moreover, the calculations with respect to $\mathcal{K}^T$-equivalence are relatively straightforward; hence, by using Theorem 3, we can prove both Theorem 1 and Theorem 2 in a coherent and unified manner.

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