# The mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and "flat" spacelike surfaces

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**Abstract**

Using the Legendrian dualities between surfaces in pseudo-spheres in Lorentz-Minkowski 4-space, we study various kind of flat surfaces in pseudo-spheres. We consider a surface in the pseudo-sphere and its dual surface. Flatness of a surface is defined by the degeneracy of the dual surface similar to the case for the Gauss map of a flat surface in the Euclidean space. We study singularities of these flat surfaces and dualities of singularities.

## 1 Introduction

It has been shown in [25] that a theorem of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space which gives a commutative diagram between contact manifolds defined by the dual relations. This theorem has been generalized into pseudo-spheres in semi-Euclidean space with general index in [10]. Such a commutative diagram is called a *mandala of Legendrian dualities* now [10, 26]. The mandala of Legendrian dualities is very useful for the study of the differential geometry on submanifolds in pseudo-spheres. Especially, it works well even for spacelike hypersurfaces in the lightcone where the induced metric is degenerate[25].

In this paper we consider various kinds of flatness of surfaces in pseudo-spheres in Lorentz-Minkowski space. In Euclidean space, a flat surface is characterized by the degeneracy of the Gauss map. For example, a surface is a part of a plane if the Gauss map is constant. Moreover, a surface is a developable surface if the image of the Gauss map is a point or a curve (i.e., all points of the surface are singularities of the Gauss map). We remark that the dual surface of a surface plays similar roles to those of the Gauss map of the surface [24, 31]. According to these facts on the Euclidean case, the Legendrian dual of a surface in pseudo-sphere is considered to be a kind of the Gauss map of the surface. In this sense a surface in a pseudo-sphere is "flat"if the Legendrian dual is singular at any point of the surface. Especially, we consider the case when the Legendrian dual is a curve in a pseudo-sphere. In [22] we have studied a surface in hyperbolic space whose lightcone dual is a curve. In this case the surface is called a *horo-flat surface*. Moreover, such surfaces are one-parameter families of horo-cycles. Therefore, we call it a *horo-flat horocyclic surface*. Horo-flat surfaces are "flat"surfaces in the sense of a new geometry in hyperbolic space[5, 6, 17, 18, 19, 22] which is

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called "Horospherical Geometry". In this paper we consider surfaces with similar properties as horo-flat horo-cyclic surfaces in other pseudo-spheres. These surfaces can be obtained by the aid of the mandala of Legendrian dualities. One of the main results in this paper is to give classifications of the singularities of these surfaces and show dualities among singularities. Therefore, the mandala of Legendrian dualities still remains on the singularities level. As a consequence, these surfaces are *frontals* which are the projection images of isotropic maps into the total contact manifold of a Legendrian fibration. If the isotropic map is a Legendrian immersion, the frontal is called a *wave front* (or, simply a *front*).

Singularities of wave fronts have been originally investigated by Zakalyukin[34, 35]. See [2] for the detail. He has shown that generic singularities of wave front surfaces are the cuspidal edge and the swallowtail. It is known that generic singularities of frontal surfaces are the cuspidal cross cap in addition to the above two fronts [14, 15].

Here, the *cuspidal edge* is a map germ $((\mathbb{R}^2;u,v),\mathbf{0}) \to (\mathbb{R}^3,\mathbf{0})$ defined by $(u,v) \mapsto (u,v^2,v^3)$ at the origin, the *swallowtail* is a map germ $((\mathbb{R}^2;u,v),\mathbf{0}) \to (\mathbb{R}^3,\mathbf{0})$ defined by $(u,v) \mapsto (u,3v^4+u^2v,4v^3+2uv)$ and the *cuspidal cross cap* is a map germ $((\mathbb{R}^2;u,v),\mathbf{0}) \to (\mathbb{R}^3,\mathbf{0})$ defined by $(u,v) \mapsto (u,v^2,uv^3)$ at the origin. Furthermore, the dual surfaces have the more degenerate singularities which called the cuspidal lips or the cuspidal beaks and the cuspidal butterfly. The *cuspidal lips* (resp. *cuspidal beaks*) is a map germ $((\mathbb{R}^2;u,v),\mathbf{0}) \to (\mathbb{R}^3,\mathbf{0})$ defined by $(u,v) \mapsto (u,-2v^3+u^2v,3v^4-u^2v^2)$ (resp. $(u,v) \mapsto (u,-2v^3-u^2v,3v^4-u^2v^2)$). The *cuspidal butterfly* is a map germ $((\mathbb{R}^2;u,v),\mathbf{0}) \to (\mathbb{R}^3,\mathbf{0})$ defined by $(u,v) \mapsto (u,5v^4+2uv,4v^5+uv^2-u^2)$. We can draw the pictures of these singularities here.

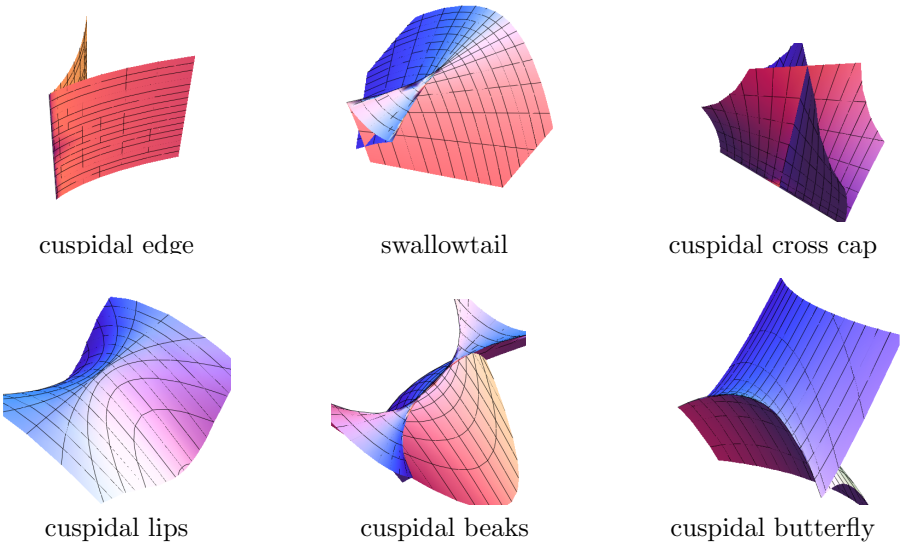



Figure 1.

We study singularities of maps up to $\mathcal{A}$-equivalence among map germs. Here, map germs $f_1, f_2 : (\mathbb{R}^2,\mathbf{0}) \to (\mathbb{R}^3,\mathbf{0})$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\phi_1 : (\mathbb{R}^2,\mathbf{0}) \to (\mathbb{R}^2,\mathbf{0})$ and $\phi_2 : (\mathbb{R}^3,\mathbf{0}) \to (\mathbb{R}^3,\mathbf{0})$ such that $\phi_2 \circ f_1 = f_2 \circ \phi_1$ holds. In Section 8 we give

criteria to detect the map-germs in the above list of frontals. In order to give classifications of "flat"surfaces we construct a basic Lorentzian invariant in Section 6. We give characterizations of the above singularities of our surfaces by using such invariants (cf., Theorems 8.6, 8.8, 8.9, 8.10, 8.11, 8.13 and 8.14).

On the other hand, there are many investigations on linear Weingarten surfaces in pseudo-spheres ([1, 8, 11, 12, 27]). The mandala of Legendrian duality is deeply related to linear Weingarten surfaces. By using the mandala of Legendrian dualities, we can unify the notion of linear Weingarten surfaces in different pseudo-spheres. (cf. Theorem 5.2)

We assume throughout the whole paper that all the maps and manifolds are $C^\infty$ unless the contrary is explicitly stated.

## 2 Basic concepts and notations

In this section we prepare basic notions on Minkowski space. For detailed properties, see [29]. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) | x_i \in \mathbb{R},\ i = 0, 1, \ldots, n\}$ be an $(n+1)$-dimensional vector space. For any vectors $\boldsymbol{x} = (x_0, \ldots, x_n)$, $\boldsymbol{y} = (y_0, \ldots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Minkowski* $(n+1)$*-space* and denoted by $\mathbb{R}^{n+1}_1$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1} \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or $< 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. For a non-zero vector $\boldsymbol{n} \in \mathbb{R}^{n+1}_1$ and a real number $c$, the hyperplane with pseudo normal $\boldsymbol{n}$ is given by

$$HP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We say that $HP(\boldsymbol{n}, c)$ is a *spacelike* , *timelike* or *lightlike hyperplane* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively.

We have the following three kinds of pseudo-spheres in $\mathbb{R}^{n+1}_1$: The *hyperbolic* $n$*-space* is defined by

$$H^n(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 |\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

the *de Sitter* $n$*-space* by

$$S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\ \}$$

and the (*open*) *lightcone* by

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\ \}.$$

For any $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n \in \mathbb{R}^{n+1}_1$, we define a vector $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n$ by

$$\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ x^1_0 & x^1_1 & \cdots & x^1_n \\ x^2_0 & x^2_1 & \cdots & x^2_n \\ \vdots & \vdots & \cdots & \vdots \\ x^n_0 & x^n_1 & \cdots & x^n_n \end{vmatrix}, \tag{2.1}$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{x}^i = (x^i_0, x^i_1, \ldots, x^i_n)$. We can easily check that

$$\langle \boldsymbol{x}, \boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n \rangle = \det(\boldsymbol{x}, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^n), \tag{2.2}$$

so that $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n$ is pseudo orthogonal to any $\boldsymbol{x}^i$ $(i = 1, \ldots, n)$.

## 3 A mandala of Legendrian dualities for pseudo-spheres

We now review some properties of contact manifolds and Legendrian submanifolds. Let $N$ be a $(2n+1)$-dimensional smooth manifold and $K$ be a tangent hyperplane field on $N$. Locally such a field is defined as the field of zeros of a 1-form $\alpha$. The tangent hyperplane field $K$ is *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of $N$. We say that $(N, K)$ is a *contact manifold* if $K$ is a non-degenerate hyperplane field. In this case $K$ is called a *contact structure* and $\alpha$ is a *contact form*. Let $\phi : N \longrightarrow N'$ be a diffeomorphism between contact manifolds $(N, K)$ and $(N', K')$. We say that $\phi$ is a *contact diffeomorphism* if $d\phi(K) = K'$. Two contact manifolds $(N, K)$ and $(N', K')$ are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold $(N, K)$ is said to be *Legendrian* if $\dim L = n$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \longrightarrow M$ is called a *Legendrian fibration* if its total space $E$ is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \longrightarrow M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \longrightarrow M$ is called a *Legendrian map*. The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of $i$ which is denoted by $W(L)$. For any $z \in E$, it is known that there is a local coordinate system $(x, p, y) = (x_1, \ldots, x_m, p_1, \ldots, p_m, y)$ around $z$ such that $\pi(x, p, y) = (x, y)$ and the contact structure is given by the 1-form $\alpha = dy - \sum_{i=1}^{m} p_i dx_i$ (cf. [2], 20.3).

In [25] we have shown the basic duality theorem which is a fundamental tool for the study of spacelike hypersurfaces in Minkowski pseudo-spheres. We consider the following four double fibrations:

(1) (a) $H^n(-1) \times S_1^n \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \}$,
(b) $\pi_{11} : \Delta_1 \longrightarrow H^n(-1)$,$\pi_{12} : \Delta_1 \longrightarrow S_1^n$,
(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_1$.

(2) (a) $H^n(-1) \times LC^* \supset \Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -1 \}$,
(b) $\pi_{21} : \Delta_2 \longrightarrow H^n(-1)$,$\pi_{22} : \Delta_2 \longrightarrow LC^*$,
(c) $\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_2$, $\theta_{22} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_2$.

(3) (a) $LC^* \times S_1^n \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 1 \}$,
(b) $\pi_{31} : \Delta_3 \longrightarrow LC^*$,$\pi_{32} : \Delta_3 \longrightarrow S_1^n$,
(c) $\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_3$, $\theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_3$.

(4) (a) $LC^* \times LC^* \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -2 \}$,
(b) $\pi_{41} : \Delta_4 \longrightarrow LC^*$,$\pi_{42} : \Delta_4 \longrightarrow LC^*$,
(c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_4$, $\theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_4$.

Here, $\pi_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$, $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0 dv_0 + \sum_{i=1}^{n} w_i dv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_0 dw_0 + \sum_{i=1}^{n} v_i dw_i$.

We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over $\Delta_i$ which is denoted by $K_i$. The basic duality theorem is the following theorem:

**Theorem 3.1.** *Under the same notations as the previous paragraph, each* $(\Delta_i, K_i)$ $(i = 1, 2, 3, 4)$ *is a contact manifold and both of* $\pi_{ij}$ $(i = 1, 2, 3, 4,\ j = 1, 2)$ *are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic to each other.*

Since the proof of the theorem was given in [25], we do not give the detailed proof here. We only remark that $(\Delta_1, K_1)$ can be canonically identified with the unit tangent bundle $S(TH^n(-1))$ over $H^n(-1)$ with the canonical contact structure ([7, 9]). Moreover, the contact structure $K_i$ $(i = 2, 3, 4)$ can be canonically induced by the following constructions. We consider smooth mappings $(i \neq j\ ;\ (i, j = 1, 2, 3, 4))$ $\Psi_{ij} : \Delta_i \longrightarrow \Delta_j$ defined by

$$
\begin{aligned}
&\Psi_{12}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, \boldsymbol{v} + \boldsymbol{w}),\ \Psi_{21} = (\boldsymbol{v}, \boldsymbol{w} - \boldsymbol{v}),\\
&\Psi_{13}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v} + \boldsymbol{w}, \boldsymbol{w}),\ \Psi_{31}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v} - \boldsymbol{w}, \boldsymbol{w})\\
&\Psi_{14}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v} - \boldsymbol{w}, \boldsymbol{v} + \boldsymbol{w}),\ \Psi_{41}(\boldsymbol{v}, \boldsymbol{w}) = \left(\frac{\boldsymbol{v} + \boldsymbol{w}}{2}, \frac{\boldsymbol{w} - \boldsymbol{v}}{2}\right),\\
&\Psi_{23}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{w}, \boldsymbol{w} - \boldsymbol{v}),\ \Psi_{32}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v} - \boldsymbol{w}, \boldsymbol{v}),\\
&\Psi_{24}(\boldsymbol{v}, \boldsymbol{w}) = (2\boldsymbol{v} - \boldsymbol{w}, \boldsymbol{w}),\ \Psi_{42}(\boldsymbol{v}, \boldsymbol{w}) = \left(\frac{\boldsymbol{v} + \boldsymbol{w}}{2}, \boldsymbol{w}\right),\\
&\Psi_{34}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v} - 2\boldsymbol{w}, \boldsymbol{v}),\ \Psi_{43}(\boldsymbol{v}, \boldsymbol{w}) = \left(\boldsymbol{w}, -\frac{\boldsymbol{v} - \boldsymbol{w}}{2}\right).
\end{aligned}
$$

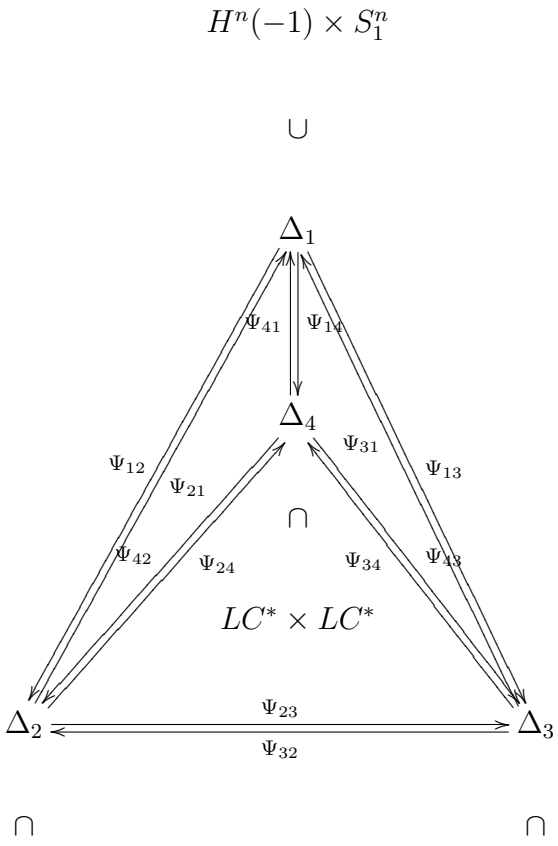
We can easily show that $\Psi_{ij}$ are contact diffeomorphisms such that $\Psi_{ij}^{-1} = \Psi_{ji}$ for any $i, j = 1, 2, 3, 4$. For example, we have

$$\Psi_{12}^*\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{v} + \boldsymbol{w}\rangle|\Delta_1 = (\langle d\boldsymbol{v}, \boldsymbol{v}\rangle + \langle d\boldsymbol{v}, \boldsymbol{w}\rangle)|\Delta_1 = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_1 = \theta_{11}$$

and

$$
\begin{aligned}
\Psi_{41}^*\theta_{11} &= \left\langle d\left(\frac{\boldsymbol{v} + \boldsymbol{w}}{2}\right), \frac{\boldsymbol{v} - \boldsymbol{w}}{2}\right\rangle|\Delta_4\\
&= \frac{1}{4}(\langle d\boldsymbol{v}, \boldsymbol{v}\rangle - \langle d\boldsymbol{v}, \boldsymbol{w}\rangle + \langle d\boldsymbol{w}, \boldsymbol{v}\rangle - \langle d\boldsymbol{w}, \boldsymbol{w}\rangle)|\Delta_4\\
&= \frac{1}{4}(-2\langle d\boldsymbol{v}, \boldsymbol{w}\rangle)|\Delta_4 = -\frac{1}{2}\langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_4 = -\frac{1}{2}\theta_{41}.
\end{aligned}
$$

Therefore $\Psi_{12} : (\Delta_1, K_1) \longrightarrow (\Delta_2, K_2)$ and $\Psi_{41} : (\Delta_4, K_4) \longrightarrow (\Delta_1, K_1)$ are contact diffeomorphisms. By the similar calculations, we can show that the other $\Psi_{ij}$ are also contact diffeomorphisms. We call these Legendrian dualities a *mandala of Legendrian dualities* (cf.,[10, 26]) because we can explain the situation as the following diagram:




The mandala of Legendrian dualities

The above mandala has the similar structure as the real mandala of Buddhism which is a religious picture of the universe. In the real mandala, the central Buddha is the symbol of the sun (the light). In the above diagram the central contact manifold is corresponding to the light, so that the analogous structure exists. This is the reason why we call the above diagram the mandala of Legendrian dualities. The mandala was generalized into the case for pseudo-spheres in general semi-Euclidean space[10]. Moreover, it can be extended into infinitely many Legendrian dualities[26].

# 4 Local differential geometry of spacelike hypersurfaces in pseudo-spheres

In this section we consider differential geometry of hypersurfaces in pseudo-spheres as an application of the mandala of Legendrian dualities. We remark that it is deeply related to

the previous theory on the differential geometry of submanifold in the hyperbolic space[17]. We now give a quick review on the theory. Let $\boldsymbol{X} : U \longrightarrow H^n(-1)$ be an embedding from an open region $U \subset \mathbb{R}^{n-1}$ and denote that $M = \boldsymbol{X}(U)$. We define the unit normal vector field $\boldsymbol{e} : U \longrightarrow S^n_1$ along $M$ in $H^n(-1)$ by

$$\boldsymbol{e}(u) = \frac{\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)}{\|\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\|}.$$

Therefore it satisfies that

$$\langle \boldsymbol{X}(u), \boldsymbol{e}(u) \rangle = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{e}(u) \rangle = \langle \boldsymbol{X}(u), \boldsymbol{e}_{u_i}(u) \rangle = 0,$$

where $i = 1, \ldots, n-1$ and $\boldsymbol{X}_{u_i} = \partial \boldsymbol{X}/\partial u_i$. Since $\langle \boldsymbol{e}(u), \boldsymbol{e}_{u_i}(u) \rangle = 0$, the above relations mean that $\boldsymbol{e}_{u_i}(u)$ is tangent to $M$ at $p = \boldsymbol{X}(u)$. Therefore $d\boldsymbol{e}(u)$ can be considered as a linear transformation on $T_pM$. We call the linear transformation $A_p = -d\boldsymbol{e}(u) : T_pM \longrightarrow T_pM$ the *de Sitter shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. Moreover, if we consider $\mathbb{L}^{\pm}(u) = \boldsymbol{X}(u) \pm \boldsymbol{e}(u)$, then $\mathbb{L}^{\pm}(u)$ are lightlike vectors. By the identification of $M$ with $U$ through $\boldsymbol{X}$, $d\boldsymbol{X}(u)$ can be identified with $1_{T_pM}$. Therefore we have a linear transformation $d\mathbb{L}^{\pm}(u) : T_pM \longrightarrow T_pM$ with $d\mathbb{L}^{\pm}(u) = 1_{T_pM} \pm d\boldsymbol{e}(u)$. We call the linear transformation $S^{\pm}_p = -d\mathbb{L}^{\pm}(u) : T_pM \longrightarrow T_pM$ the *hyperbolic shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. The *de Sitter Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$ is defined to be $K_d(u) = \det A_p$ and the *lightcone Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$ is $K^{\pm}_{\ell}(u) = \det S^{\pm}_p$. In [17] we have investigated the geometric meanings of the lightcone Gauss-Kronecker curvature from the contact viewpoint. One of the consequences is that the lightcone Gauss-Kronecker curvature estimates the contact of hypersurfaces with hyperhorospheres. It has been also shown that the Gauss-Bonnet type theorem holds on the normalized lightcone Gauss-Kronecker curvature [18].

On the other hand, we can interpret the above construction by using the Legendrian duality theorem (Theorem 3.1). For any regular hypersurface $\boldsymbol{X} : U \longrightarrow H^n(-1)$, we have $\langle \boldsymbol{X}(u), \mathbb{L}^{\pm}(u) \rangle = -1$. Therefore, we can define embeddings $\mathcal{L}^{\pm}_2 : U \longrightarrow \Delta_2$ by $\mathcal{L}^{\pm}_2(u) = (\boldsymbol{X}(u), \mathbb{L}^{\pm}(u))$. Since $\langle \boldsymbol{X}_{u_i}(u), \mathbb{L}^{\pm}(u) \rangle = 0$, each of $\mathcal{L}^{\pm}_2$ is a Legendrian embedding.

It has been shown that $\pi_{21} : \Delta_2 \longrightarrow H^n(-1)$ is a Legendrian fibration. The fiber is the intersection of $LC^*$ with a spacelike hyperplane (i.e., an elliptic hyperquadric). Therefore the intersection of the fiber with the pseudo-normal plane (i.e., a timelike plane) in $\mathbb{R}^{n+1}_1$ of $M$ consists of two points at each point of $M$. This is the reason why we have such two Legendrian embeddings. However, one of the results in the theory of Legendrian singularities (cf., the appendix) asserts that the Legendrian submanifold is uniquely determined by the wave front set at least locally. Here, $M = \boldsymbol{X}(U) = \pi_{21} \circ \mathcal{L}^{\pm}_4(U)$ are the wave front sets of $\mathcal{L}^{\pm}_2(U)$ through the Legendrian fibration $\pi_{21}$. Therefore each of the Legendrian embeddings $\mathcal{L}^{\pm}_2$ is uniquely determined with respect to $M = \boldsymbol{X}(U)$. It follows that we have a unique pair of lightcone Gauss images $\mathbb{L}^{\pm} = \pi_{22} \circ \mathcal{L}^{\pm}_2$. Moreover, we have a Legendrian embedding $\mathcal{L}_1 : U \longrightarrow \Delta_1$ defined by $\mathcal{L}_1(u) = (\boldsymbol{X}(u), \boldsymbol{e}(u))$. It follows from the mandala of Legendrian dualities that we have

$$\mathcal{L}_3(u) = \Psi_{13} \circ \mathcal{L}_1(u) = (\mathbb{L}^+(u), \boldsymbol{e}(u)), \ \mathcal{L}_4(u) = \Psi_{14} \circ \mathcal{L}_1(u) = (\mathbb{L}^-(u), \mathbb{L}^+(u)).$$

We write $\mathcal{L}_2(u) = \mathcal{L}^+_2(u)$. Eventually, we have Legendrian embeddings $\mathcal{L}_i : U \longrightarrow \Delta_i$ $(i = 1, 2, 3, 4)$ such that $\Psi_{ij} \circ \mathcal{L}_i = \mathcal{L}_j$. In this case we started the embedding $\boldsymbol{X} : U \longrightarrow H^n(-1)$. However, we have no reasons why we do not start a spacelike embedding into $S^n_1$ or $LC^*$.

According to the above arguments, we consider the following situations. Let $\mathcal{L}_1 : U \longrightarrow \Delta_1$ be a Legendrian embedding and denote that $\mathcal{L}_1(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^d(u))$. By using the contact diffeomorphism $\Psi_{14}$, we have a Legendrian embedding $\mathcal{L}_4 : U \longrightarrow \Delta_4$ defined by $\mathcal{L}_4(u) = \Psi_{14} \circ \mathcal{L}_1(u)$. We denote that $\mathcal{L}_4(u) = (\boldsymbol{X}^\ell_-(u), \boldsymbol{X}^\ell_+(u))$, so that we have the following relations:

$$\begin{aligned} &\boldsymbol{X}^\ell_-(u) = \boldsymbol{X}^h(u) - \boldsymbol{X}^d(u), \quad \boldsymbol{X}^\ell_+(u) = \boldsymbol{X}^h(u) + \boldsymbol{X}^d(u), \\ &\boldsymbol{X}^h(u) = \frac{\boldsymbol{X}^\ell_+(u) + \boldsymbol{X}^\ell_-(u)}{2}, \quad \boldsymbol{X}^d(u) = \frac{\boldsymbol{X}^\ell_+(u) - \boldsymbol{X}^\ell_-(u)}{2}. \end{aligned} \tag{4.1}$$

We also denote that $\mathcal{L}_2 = \Psi_{12} \circ \mathcal{L}_1 : U \longrightarrow \Delta_2$ and $\mathcal{L}_3 = \Psi_{13} \circ \mathcal{L}_1 : U \longrightarrow \Delta_3$, so that we have

$$\mathcal{L}_2(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^\ell_+(u)), \ \mathcal{L}_3(u) = (\boldsymbol{X}^\ell_+(u), \boldsymbol{X}^d(u)). \tag{4.2}$$

Since $\Psi_{ij}$ $(i, j = 1, 2, 3, 4)$ are contact diffeomorphisms, $\mathcal{L}_i(U)$ $(i = 1, 2, 3, 4)$ are Legendrian submanifolds. By definition, $\mathcal{L}_1(U)$ is a Legendrian submanifold in $\Delta_1$ if and only if

$$\langle \boldsymbol{X}^h(u), \boldsymbol{X}^d(u) \rangle = \langle \boldsymbol{X}^h(u), \boldsymbol{X}^d_{u_i}(u) \rangle = \langle \boldsymbol{X}^h_{u_i}(u), \boldsymbol{X}^d(u) \rangle = 0$$

for $i = 1, \ldots, n-1$. Therefore if we suppose that $\boldsymbol{X}^h$ is an embedding, then $\boldsymbol{X}^d$ can be considered as the Gauss map of $M^h = \boldsymbol{X}^h(U)$ and $-d\boldsymbol{X}^d(u)$ is the corresponding Weingarten map. If $\boldsymbol{X}^d$ is an embedding, then $\boldsymbol{X}^h$ can be considered as the Gauss map of $M^d = \boldsymbol{X}^d(U)$ and $-d\boldsymbol{X}^h(u)$ is the corresponding Weingarten map. It follows that we can define the corresponding curvatures. The situations are the same as for the other $\mathcal{L}_i(U)$. We now summarize the situations. We denote that $M^H = \boldsymbol{X}^h(U)$ and $M^D = \boldsymbol{X}^d(U)$. If $\boldsymbol{X}^h$ is an embedding, we call $\boldsymbol{X}^d$ the *de Sitter Gauss image* of hypersurface $M^H$ in the hyperbolic space $H^n(-1)$. Moreover, we define $(S^H_d)_p = -d\boldsymbol{X}^d(u) : T_pM^H \longrightarrow T_pM^H$ where $p = \boldsymbol{X}^h(u)$. We also call $(S^H_d)_p$ the *de Sitter Weingarten map* of hypersurface $M^H$ in the hyperbolic space $H^n(-1)$ at $p = \boldsymbol{X}^h(u)$. Then we have *de Sitter principal curvatures* $\kappa^H_{d,i}(u)$ $(i = 1, \ldots, n-1)$ defined as the eigenvalues of $(S^H_d)_p$ and the *de Sitter Gauss-Kronecker curvature* $K^H_d(u) = \det(S^H_d)_p$ of $M^H$ at $p = \boldsymbol{X}^h(u)$.

On the other hand, if $\boldsymbol{X}^d$ is an embedding, we call $\boldsymbol{X}^h$ the *hyperbolic Gauss image* of spacelike hypersurface $M^D$ in the de Sitter space $S^n_1$. Moreover, we define $(S^D_h)_p = -d\boldsymbol{X}^h(u) : T_pM^D \longrightarrow T_pM^D$ where $p = \boldsymbol{X}^d(u)$. We also call $(S^D_h)_p$ the *hyperbolic Weingarten map* of spacelike hypersurface $M^D$ in the de Sitter space $S^n_1$ at $p = \boldsymbol{X}^d(u)$. Then we have *hyperbolic principal curvatures* $\kappa^D_{h,i}(u)$ $(i = 1, \ldots, n-1)$ defined as the eigenvalues of $(S^D_h)_p$ and the *hyperbolic Gauss-Kronecker curvature* $K^D_h(u) = \det(S^D_h)_p$ of $M^D$ at $p = \boldsymbol{X}^d(u)$. If both the mappings $\boldsymbol{X}^h, \boldsymbol{X}^d$ are embeddings, then we define $g^D_{ij}(u) = \langle \boldsymbol{X}^h_i(u), \boldsymbol{X}^h_j(u) \rangle$, $g^H_{ij}(u) = \langle \boldsymbol{X}^d_i(u), \boldsymbol{X}^d_j(u) \rangle$ and $h^{\Delta_1}_{ij}(u) = -\langle \boldsymbol{X}^d_i(u), \boldsymbol{X}^h_j(u) \rangle = \langle \boldsymbol{X}^d_{ij}(u), \boldsymbol{X}^h(u) \rangle = \langle \boldsymbol{X}^d(u), \boldsymbol{X}^h_{ij}(u) \rangle$. We respectively call $g^H_{ij}, g^D_{ij}$ and $h^{\Delta_1}_{ij}$ a *hyperbolic first fundamental invariant* of $M^D$, a *de Sitter first fundamental invariant* of $M^H$ and a $\Delta_1$*-second fundamental invariant*. In this case we can identify $T_pM^H$ with $T_{p'}M^D$ for $p = \boldsymbol{X}^h(u)$ and $p' = \boldsymbol{X}^d(u)$. By definition, the principal directions of $(S^H_d)_p$ and $(S^D_h)'_p$ are the same. We have the following Weingarten type formulae.

**Proposition 4.1.** *Let $\mathcal{L}_1 : U \longrightarrow \Delta_1$ be a Legendrian embedding with $\mathcal{L}_1(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^d(u))$.*

(1) *Suppose that* $\boldsymbol{X}^h : U \longrightarrow H^n(-1)$ *is an embedding. Then we have*

$$\boldsymbol{X}^d_{u_i} = -\sum_{j=1}^{n-1} \left(h^D\right)_i^j \boldsymbol{X}^h_{u_j}.$$

(2) *Suppose that* $\boldsymbol{X}^d : U \longrightarrow H^n(-1)$ *is an embedding. Then we have*

$$\boldsymbol{X}^h_{u_i} = -\sum_{j=1}^{n-1} \left(h^H\right)_i^j \boldsymbol{X}^d_{u_j}.$$

*Here,* $\left((h^H)_i^j\right) = (h^{\Delta_1}_{ij})(g^H_{ij})^{-1}$ *and* $\left((h^D)_i^j\right) = (h^{\Delta_1}_{ij})(g^D_{ij})^{-1}$.

The proof of the above formulae is given by the same arguments as those for the Weingarten type formula in [17], so that we omit it. We remark that $\kappa^H_{d,i}(u)$ and $\kappa^D_{h,i}(u)$ are the eigenvalues of $\left((h^H)_i^j\right)$ and $\left((h^D)_i^j\right)$ respectively. We have the following relation between $\kappa^H_{d,i}(u)$ and $\kappa^D_{h,i}(u)$.

**Corollary 4.2.** *Suppose that both the mappings* $\boldsymbol{X}^h, \boldsymbol{X}^d$ *are embeddings. In this case we have the relation* $\kappa^H_{d,i}(u)\kappa^D_{h,i}(u) = 1$ $(i = 1, \ldots n-1)$. *Here* $\kappa^H_{d,i}(u)$ *and* $\kappa^D_{h,i}(u)$ *are principal curvatures corresponding to the same principal direction.*

*Proof.* Since both the mappings $\boldsymbol{X}^h, \boldsymbol{X}^d$ are embeddings, $K^H_d(u) \neq 0$ and $K^D_h(u) \neq 0$. By the Weingarten type formulae, $\left((h^D)_i^j\right)$ is the inverse matrix of $\left((h^H)_i^j\right)$, so that the eigenvalues have the above relations. □

We say that $\pi_{i1} \circ \mathcal{L}_i$ and $\pi_{i2} \circ \mathcal{L}_i$ are $\Delta_i$-*dual* each other if $\mathcal{L}_i : U \longrightarrow \Delta_i$ is an isotropic mapping with respect to $K_i$.

# 5 Linear Weingarten surfaces

Galvez, Martinez and Milan has investigated the linear Weingarten surfaces using the Weierstrass type representation formula [12]. In this section, we discuss linear Weingarten surfaces and their hyperbolic Gauss maps from our point of view. In this section, we identify the Minkowski 4-space with the $2 \times 2$ Hermitian matrices. For the detailed description, see [12, Section 2]. A surface $f : U \to H^3_+(-1)$ is called a *linear Weingarten surface* if the mean curvature $H^H_d = (\kappa^H_1 + \kappa^H_2)/2$ and the de Sitter Gauss-Kronecker curvature $K^H_d$ satisfies

$$2a(H^H_d - 1) + b(K^H_d - 1) = 0, \quad a,\ b \in \mathbb{R}, \quad a + b \neq 0.$$

If $a + b \neq 0$ holds, it is called a *linear Weingarten surface of Bryant type*. In [22], we have investigated "horo-flat" horospherical surfaces in $H^3_+(-1)$. It is linear Weingarten surfaces of non-Bryant type, we have considered them as surfaces whose hyperbolic Gauss map degenerates to a curve in the de Sitter space (see [22, Section 4]). This means that a horo-flat horospherical surface is the dual surface of a curve in the de Sitter space. In [12], Galvez, Martinez and Milan showed the following representation formula for linear Weingarten surfaces of Bryant type.

**Theorem 5.1.** [12, Theorem 2] *Let $V \subset \boldsymbol{C}$ be a simply connected domain. Fix a meromorphic map $A : V \longrightarrow SL(2, \boldsymbol{C})$ satisfying*

$$A^{-1}dA = \begin{pmatrix} 0 & \omega \\ dh & 0 \end{pmatrix},$$

*where $h$ is a meromorphic function and $\omega$ a holomorphic one-form. If*

$$\sigma = (a+b)\left((1+\varepsilon|h|^2)^2|\omega|^2 - \frac{(1-\varepsilon)^2|dh|^2}{(1+\varepsilon|h|^2)^2}\right)$$

*is positive definite then $f = A(\Omega_+)A^*$ is a linear Weingarten surface. Moreover, the hyperbolic Gauss map $\nu$ of $f$ is given by $\nu = A(\Omega_-)A^*$ where*

$$\Omega_\pm = \begin{pmatrix} \dfrac{1 \pm \varepsilon^2|h|^2}{1+\varepsilon|h|^2} & \mp\varepsilon\bar{h} \\ \mp\varepsilon h & \pm(1+\varepsilon|h|^2) \end{pmatrix}, \text{ respectively, } \quad \varepsilon = a/(a+b), \quad \text{and} \quad 1+\varepsilon|h|^2 > 0.$$

By the construction of Legendrian dualities (4.1) and (4.2), we can obtain the dual surfaces in $S^3_1$ and $LC^*$ by taking $\nu : U \to S^3_1$ and $f \pm \nu : U \to LC^*$:

$$f+\nu = A\begin{pmatrix} 2\dfrac{1}{1+\varepsilon|h|^2} & 0 \\ 0 & 0 \end{pmatrix}A^*, \quad f-\nu = A\begin{pmatrix} 2\dfrac{\varepsilon^2|h|^2}{1+\varepsilon|h|^2} & -2\varepsilon\bar{h} \\ -2\varepsilon h & 2(1+\varepsilon|h|^2) \end{pmatrix}A^*. \tag{5.1}$$

In [3], Aledo and Espinar showed a Weierstrass type representation formula for linear Weingarten surfaces of Bianchi type. A spacelike surface $f : U \to S^3_1$ is a *linear Weingarten surface* if the mean curvature $H^D_h$ and the hyperbolic Gauss-Kronecker curvature $K^D_h$ satisfy

$$2A(H^D_h - 1) + B(K^D_h - 1) = 0, \quad A,\ B \in \mathbb{R}.$$

If $A + B \neq 0$ holds, it is called *Bianchi type*. As a consequence of the duality theorem, we can interpret the relationship between linear Weingarten surfaces in $H^3_+(-1)$ and $S^3_1$.

**Theorem 5.2.** *Let $\mathcal{L}_1 : U \to \Delta_1$ be a Legendrian immersion. Suppose that both of $\pi_{11} \circ \mathcal{L}_1 : U \to H^3_+(-1)$ and $\pi_{12} \circ \mathcal{L}_1 : U \to S^3_1$ are immersions. Then $\pi_{11} \circ \mathcal{L}_1 = \boldsymbol{X}^h$ is a linear Weingarten surface of Bryant type if and only if $\pi_{12} \circ \mathcal{L}_1 = \boldsymbol{X}^d$ is a linear Weingarten surface of Bianchi type.*

*Proof.* Let $\kappa^H_{d,i}$ $(i = 1, 2)$ be the de Sitter principal curvatures of $M^H = \boldsymbol{X}^h(U)$ at $p = \boldsymbol{X}^h(u)$. and $\kappa^D_{h,i}$ $(i = 1, 2)$ the hyperbolic principal curvatures of $M^D = \boldsymbol{X}^d(U)$ at $p' = \boldsymbol{X}^d(u)$. By Corollary 4.2, we have the relations $\kappa^H_{d,i}\kappa^D_{h,i} = 1$. Since $K^H_d(u) = \kappa^H_{d,1}\kappa^H_{d,2}$ and $2H^H_d = \kappa^H_{d,1} + \kappa^H_{d,2}$, we have

$$2a(H^H_d - 1) + b(K^H_d - 1) = a(\kappa^H_{d,1} + \kappa^H_{d,2} - 2) + b(\kappa^H_{d,1}\kappa^H_{d,2} - 1).$$

We also have another relation

$$2A(H^D_h - 1) + B(K^D_d - 1) = A(\kappa^D_{h,1} + \kappa^D_{h,2} - 2) + B(\kappa^D_{h,1}\kappa^D_{h,2} - 1).$$

Since $\kappa^H_{d,i} = 1/\kappa^D_{h,i}$, we have

$$\begin{aligned}
2a(H^H_d - 1) + b(K^H_d - 1) &= a\left(\frac{1}{\kappa^D_{h,1}} + \frac{1}{\kappa^D_{h,2}} - 2\right) + b\left(\frac{1}{\kappa^D_{h,1}\kappa^D_{h,2}} - 1\right) \\
&= \frac{1}{\kappa^D_{h,1}\kappa^D_{h,2}}\left(a(\kappa^D_{h,1} + \kappa^D_{h,2} - 2) + (-2a - b)(\kappa^D_{h,1}\kappa^D_{h,2} - 1\right) \\
&= \frac{1}{\kappa^D_{h,1}\kappa^D_{h,2}}\left(2a(H^D_h - 1) + (-2a - b)(K^D_h - 1)\right).
\end{aligned}$$

If we put $A = a, B = -(2a + b)$, then $2a(H^H_d - 1) + b(K^H_d - 1) = 0$ if and only if $2A(H^D_h - 1) + B(K^D_h - 1) = 0$. Moreover, $A + B = 0$ if and only if $a + b = 0$. This completes the proof. □

This theorem shows that we can bring the representation formula for a surface in $H^3_+(-1)$ to representation formulae for surfaces in $S^3_1$ and $LC^*$, and get new surfaces. Remark that we have interesting families of surfaces in the lightcone obtained by taking the dual of linear Weingarten surface of non-Bryant type. In fact, the Gauss map $\nu$ of a linear Weingarten surface $f$ given in Theorem 5.1 is a linear Weingarten surface in $S^3_1$. Furthermore, surfaces $f \pm \nu$ given in (5.1) belong to this class of surfaces. Theorem 5.2 says that Theorem 5.1 also can be considered representation formula for these families of surfaces. Kokubu and Umehara investigated the topological properties of linear Weingarten surfaces giving a variant of this representation formula [27].

# 6 The Legendrian dualities for "flat"spacelike surfaces

In this section we study general properties of spacelike surfaces in pseudo-spheres which are $\Delta_i$-duals of spacelike curves in pseudo-spheres. Let $\boldsymbol{a}_0 : I \longrightarrow H^3_+(-1)$ be a smooth mapping and $\boldsymbol{a}_i : I \longrightarrow S^3_1$ $(i = 1, 2)$ be smooth mappings from an open interval $I$ with $\langle \boldsymbol{a}_i(t), \boldsymbol{a}_j(t)\rangle = 0$ if $i \neq j$. We define a unit spacelike vector $\boldsymbol{a}_3(t) = \boldsymbol{a}_0(t) \wedge \boldsymbol{a}_1(t) \wedge \boldsymbol{a}_2(t)$, so that we have a pseudo-orthonormal frame $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ of $\mathbb{R}^4_1$. We have the following fundamental invariants:

$$\begin{aligned}
c_1(t) &= \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_1(t)\rangle = -\langle \boldsymbol{a}_0(t), \boldsymbol{a}_1'(t)\rangle, & c_4(t) &= \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_2(t)\rangle = -\langle \boldsymbol{a}_1(t), \boldsymbol{a}_2'(t)\rangle, \\
c_2(t) &= \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_2(t)\rangle = -\langle \boldsymbol{a}_0(t), \boldsymbol{a}_2'(t)\rangle, & c_5(t) &= \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_3(t)\rangle = -\langle \boldsymbol{a}_1(t), \boldsymbol{a}_3'(t)\rangle, \\
c_3(t) &= \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_3(t)\rangle = -\langle \boldsymbol{a}_0(t), \boldsymbol{a}_3'(t)\rangle, & c_6(t) &= \langle \boldsymbol{a}_2'(t), \boldsymbol{a}_3(t)\rangle = -\langle \boldsymbol{a}_2(t), \boldsymbol{a}_3'(t)\rangle.
\end{aligned}$$

It can be written in the following form:

$$\begin{pmatrix} \boldsymbol{a}_0'(t) \\ \boldsymbol{a}_1'(t) \\ \boldsymbol{a}_2'(t) \\ \boldsymbol{a}_3'(t) \end{pmatrix} = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t) \\ c_1(t) & 0 & c_4(t) & c_5(t) \\ c_2(t) & -c_4(t) & 0 & c_6(t) \\ c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_0(t) \\ \boldsymbol{a}_1(t) \\ \boldsymbol{a}_2(t) \\ \boldsymbol{a}_3(t) \end{pmatrix} =: C(t) \begin{pmatrix} \boldsymbol{a}_0(t) \\ \boldsymbol{a}_1(t) \\ \boldsymbol{a}_2(t) \\ \boldsymbol{a}_3(t) \end{pmatrix}.$$

We remark that $C(t)$ is an element of the Lie algebra $\mathfrak{so}(3, 1)$ of the Lorentzian group $SO_0(3, 1)$. If $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$ is a pseudo-orthonormal frame field as the above,

the $4\times 4$-matrix determined by the frame defines a smooth curve $A : I \longrightarrow SO_0(3,1)$. Therefore we have the relation that $A'(t) = C(t)A(t)$. For the converse, let $A : I \longrightarrow SO_0(3,1)$ be a smooth curve, then we can show that $A'(t)A(t)^{-1} \in \mathfrak{so}(3,1)$. Moreover, for any smooth curve $C : I \longrightarrow \mathfrak{so}(3,1)$, we apply the existence theorem on the linear systems of ordinary differential equations, so that there exists a unique curve $A : I \longrightarrow SO_0(3,1)$ such that $C(t) = A'(t)A(t)^{-1}$ with an initial data $A(t_0) \in SO_0(3,1)$. Therefore, a smooth curve $C : I \longrightarrow \mathfrak{so}(3,1)$ might be identified with a pseudo-orthonormal frame in $H^3_+(-1)$. Let $C : I \longrightarrow \mathfrak{so}(3,1)$ be a smooth curve with $C(t) = A'(t)A(t)^{-1}$ and $B \in SO_0(3,1)$, then we have $C(t) = (A(t)B)'(A(t)B)^{-1}$. This means that the curve $C : I \longrightarrow \mathfrak{so}(3,1)$ is a Lorentzian invariant of the pseudo-orthonormal frame $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$. In the followings of this section, we construct dual surfaces of a lightlike curve $\boldsymbol{\ell}$ satisfying $||\boldsymbol{\ell}'|| \neq 0$ by using this frame.

## 6.1 $\Delta_2$, $\Delta_3$ and $\Delta_4$-dual surfaces of $\boldsymbol{\ell}$

Let $\boldsymbol{\ell}$ be a lightlike curve satisfying $||\boldsymbol{\ell}'|| \neq 0$ and set $\boldsymbol{a}_3 := \boldsymbol{\ell}'/||\boldsymbol{\ell}'||$. Then $\boldsymbol{a}_3$ is spacelike. Since $\boldsymbol{\ell}(t) \in (\boldsymbol{a}_3(t))^{\perp}$, we have curves $\boldsymbol{a}_0$ and $\boldsymbol{a}_2$ satisfying $\langle \boldsymbol{a}_0, \boldsymbol{a}_0\rangle = -1$, $\langle \boldsymbol{a}_2, \boldsymbol{a}_2\rangle = 1$, $\boldsymbol{\ell} = \boldsymbol{a}_0 + \boldsymbol{a}_2$ and $\boldsymbol{a}_0, \boldsymbol{a}_2, \boldsymbol{a}_3$ are pseudo-orthonormal each other. If we define $\boldsymbol{a}_1 = \boldsymbol{a}_0 \wedge \boldsymbol{a}_2 \wedge \boldsymbol{a}_3$, then we have a pseudo-orthonormal frame $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ satisfying $c_2 \equiv 0$, $c_1 - c_4 \equiv 0$ and $c_{36}(t) \neq 0$ for any $t$, where $\equiv 0$ means that the function is constantly equal to zero. Thus, we may assume that $\boldsymbol{\ell} = \boldsymbol{a}_0 + \boldsymbol{a}_2$, $c_2 \equiv 0$, $c_1 - c_4 \equiv 0$ and $c_{36}(t) \neq 0$ for any $t$, this means that $\boldsymbol{\ell}' = c_{36}\boldsymbol{a}_3 \neq 0$.

**(1) $\Delta_2$-dual surface of $\boldsymbol{\ell}$:** In order to obtain the $\Delta_2$-dual surface of $\boldsymbol{\ell}$, we consider a hight function $F : H^3_+(-1) \times I \longrightarrow \mathbb{R}$ defined by $F(X,t) = \langle X, \boldsymbol{\ell}(t)\rangle + 1$. There exist $x_0, x_1, x_2, x_3 \in \mathbb{R}$ such that $X = x_0\boldsymbol{a}_0 + x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3$. Then the *discriminant set* $D_F$ of $F$ is

$$\begin{aligned} D_F &= \left\{ X \in H^3_+(-1) \;\middle|\; {}^{\exists}t \in I, F(X,t) = \frac{\partial F}{\partial t}(X,t) = 0 \right\} \\ &= \{X \in H^3_+(-1) \mid {}^{\exists}t \in I, -x_0 + x_2 + 1 = 0,\ x_3 = 0\}. \end{aligned}$$

Since $X \in H^3_+(-1)$, we have

$$X = \boldsymbol{a}_0(t) + s\boldsymbol{a}_1(t) + \frac{s^2}{2}\boldsymbol{\ell}(t)$$

for $x_1 = s$, which we write $\boldsymbol{X}^h_\ell(s,t)$.

By the above construction, $(\boldsymbol{X}^h_\ell(s,t), \boldsymbol{\ell}(t)) : I \times \mathbb{R} \to \Delta_2$ is an isotropic map with respect to the contact structure defined in Theorem 3.1, so that $\boldsymbol{X}^h_\ell(s,t)$ and $\boldsymbol{\ell}(t)$ are $\Delta_2$-dual each other.

Since $c_2 \equiv c_1 - c_4 \equiv 0$ hold, the surface $\boldsymbol{X}^h_\ell$ is horo-flat in the sense of [22]. Moreover if we assume $c_3 \equiv 0$, then the singular value of $\boldsymbol{X}^h_\ell$ is $\boldsymbol{a}_0(t)$. We also consider the $\Delta_1$ and $\Delta_2$-dual surfaces of $\boldsymbol{a}_0$. By the same computations as those of the previous paragraph for obtaining the surface $\boldsymbol{X}^h_\ell$, and assumptions $c_2 \equiv c_3 \equiv 0$ instead of $c_2 \equiv c_1 - c_4 \equiv 0$, we have the $\Delta_1$-dual surface $\boldsymbol{X}^d_h$ and the $\Delta_3$-dual surface $\boldsymbol{X}^\ell_h$ of $\boldsymbol{a}_0$ as follows:

$$\boldsymbol{X}^d_h(s,t) := \cos s\, \boldsymbol{a}_2(t) + \sin s\, \boldsymbol{a}_3(t) \quad \text{and} \quad \boldsymbol{X}^\ell_h(s,t) := \boldsymbol{a}_0(t) + \cos s\, \boldsymbol{a}_2(t) + \sin s\, \boldsymbol{a}_3(t).$$

In [22], we introduced these surfaces $\boldsymbol{X}^h_\ell$ and $\boldsymbol{X}^\ell_h$ by the same construction as the above and investigated the geometric properties and singularities of them. It has been shown in [22] that $\boldsymbol{X}^h_\ell$ is a linear Weingarten surface of non-Bryant type.

**(2) $\Delta_3$-dual surface of $\boldsymbol{\ell}$:** We consider a hight function $F : S^3_1 \times I \longrightarrow \mathbb{R}$ defined by $F(X,t) = \langle X, \boldsymbol{\ell}(t)\rangle - 1$. By the same computations as those for detecting $\boldsymbol{X}^h_\ell(s,t)$, the discriminant set is given by

$$X = \boldsymbol{a}_2(t) + s\boldsymbol{a}_1(t) - \frac{s^2}{2}\boldsymbol{\ell}(t),$$

which we write $\boldsymbol{X}^d_\ell(s,t)$. Like as in the case for $\boldsymbol{X}^h_\ell$, we consider the dual surfaces of $\boldsymbol{a}_2$ here. By exactly the same calculations as those in the previous cases, and assumptions $c_2 \equiv c_6 \equiv 0$ instead of $c_2 \equiv c_1 - c_4 \equiv 0$, the $\Delta_1$-dual surface $\boldsymbol{X}^h_d$ of $\boldsymbol{a}_2(t)$ and the $\Delta_3$-dual surface $\boldsymbol{X}^\ell_d$ of $\boldsymbol{a}_2(t)$ are parameterized by

$$\boldsymbol{X}^h_d(s,t) := \cosh s\, \boldsymbol{a}_0(t) + \sinh s\, \boldsymbol{a}_3(t) \quad \text{and} \quad \boldsymbol{X}^\ell_d(s,t) := \boldsymbol{a}_2(t) + \cosh s\, \boldsymbol{a}_0(t) + \sinh s\, \boldsymbol{a}_3(t).$$

**(3) $\Delta_4$-dual surface of $\boldsymbol{\ell}$:** We consider a hight function $F : LC^* \times I \longrightarrow \mathbb{R}$ defined by $F(X,t) = \langle X, \boldsymbol{\ell}(t)\rangle + 2$. Putting $x_1 = 2s$ and by exactly the same computations as those of the previous two cases. we have

$$X = \boldsymbol{a}_0(t) - \boldsymbol{a}_2(t) + 2s\boldsymbol{a}_1(t) + s^2\boldsymbol{\ell}(t),$$

which we write $\boldsymbol{X}^\ell_\ell(s,t)$. We study geometric properties of $\boldsymbol{X}^\ell_\ell(s,t)$ in section 7 and investigate the singularities in section 8. Like as in the case of $\boldsymbol{X}^h_\ell$ and $\boldsymbol{X}^d_\ell$, we consider the dual surfaces of $\boldsymbol{\ell}_- := \boldsymbol{a}_0 - \boldsymbol{a}_2$. Under the condition $c_2 \equiv c_1 + c_4 \equiv 0$, the $\Delta_2$-dual surface $\boldsymbol{X}^h_{\ell-}$ of $\boldsymbol{\ell}_-(t)$, the $\Delta_3$-dual surface $\boldsymbol{X}^d_{\ell-}$ of $\boldsymbol{\ell}_-(t)$ and the $\Delta_4$-dual surface $\boldsymbol{X}^\ell_{\ell-}$ of $\boldsymbol{\ell}_-(t)$ are parameterized by

$$\begin{aligned}
\boldsymbol{X}^h_{\ell-}(s,t) &= \boldsymbol{a}_0(t) + s\boldsymbol{a}_1(t) + \frac{s^2}{2}\boldsymbol{\ell}_-(t),\\
\boldsymbol{X}^d_{\ell-}(s,t) &= -\boldsymbol{a}_2(t) + s\boldsymbol{a}_1(t) - \frac{s^2}{2}\boldsymbol{\ell}(t),\\
\boldsymbol{X}^\ell_{\ell-}(s,t) &= \boldsymbol{a}_0(t) + \boldsymbol{a}_2(t) + 2s\boldsymbol{a}_1(t) + s^2\boldsymbol{\ell}_-(t).
\end{aligned}$$

Since we can obtain these surfaces $\boldsymbol{X}^h_{\ell-}$, $\boldsymbol{X}^d_{\ell-}$ and $\boldsymbol{X}^\ell_{\ell-}$ by translating $\boldsymbol{a}_2 \mapsto -\boldsymbol{a}_2$, geometric properties of these surfaces are completely the same as those of $\boldsymbol{X}^h_\ell$, $\boldsymbol{X}^d_\ell$ and $\boldsymbol{X}^\ell_\ell$. Here we explain the meanings of the superscript and the subscript. For example, $\boldsymbol{X}^h_\ell$ means this surface is the dual surface of a curve in the *lightcone* and lies in the *hyperbolic* 3-space. Since surfaces $\boldsymbol{X}^h_\ell$, $\boldsymbol{X}^d_\ell$ and $\boldsymbol{X}^\ell_\ell$ are one-parameter families of parabolas, we call these surfaces *parabollatic surfaces*. If we adopt the word "parabolic" instead of the word "parabollatic", it might be confused with other notions. Now, we summerize the correspondences between

these curves and surfaces:

$$\begin{array}{rrll}
LC^* \supset \boldsymbol{\ell}(t) \longleftrightarrow & \boldsymbol{X}^h_\ell(s,t) & = \boldsymbol{a}_0(t) + s\boldsymbol{a}_1(t) + \dfrac{s^2}{2}\boldsymbol{\ell}(t) & \subset H^3_+(-1) \\
LC^* \supset \boldsymbol{\ell}(t) \longleftrightarrow & \boldsymbol{X}^d_\ell(s,t) & = \boldsymbol{a}_2(t) + s\boldsymbol{a}_1(t) - \dfrac{s^2}{2}\boldsymbol{\ell}(t) & \subset S^3_1 \\
LC^* \supset \boldsymbol{\ell}(t) \longleftrightarrow & \boldsymbol{X}^l_\ell(s,t) & = \boldsymbol{a}_0(t) - \boldsymbol{a}_2(t) + 2s\boldsymbol{a}_1(t) + s^2\boldsymbol{\ell}(t) & \subset LC^* \\
H^3_+(-1) \supset \boldsymbol{a}_0(t) \longleftrightarrow & \boldsymbol{X}^d_h(s,t) & = \cos s\boldsymbol{a}_2(t) + \sin s\boldsymbol{a}_3(t) & \subset S^3_1 \\
H^3_+(-1) \supset \boldsymbol{a}_0(t) \longleftrightarrow & \boldsymbol{X}^\ell_h(s,t) & = \boldsymbol{a}_0(t) + \cos s\boldsymbol{a}_2(t) + \sin s\boldsymbol{a}_3(t) & \subset LC^* \\
S^3_1 \supset \boldsymbol{a}_2(t) \longleftrightarrow & \boldsymbol{X}^h_d(s,t) & = \cosh s\boldsymbol{a}_0(t) + \sinh s\boldsymbol{a}_3(t) & \subset H^3_+(-1) \\
S^3_1 \supset \boldsymbol{a}_2(t) \longleftrightarrow & \boldsymbol{X}^\ell_d(s,t) & = \boldsymbol{a}_2(t) + \cosh s\boldsymbol{a}_0(t) + \sinh s\boldsymbol{a}_3(t) & \subset LC^*.
\end{array} \tag{6.1}$$

## 6.2 Dualities of "flat"surfaces

By using the equations for the pseudo-orthonormal frame, we have

$$\begin{aligned}
(\boldsymbol{X}^h_\ell)'(s,t) &= sc_1\boldsymbol{a}_0 + c_1\boldsymbol{a}_1 + sc_4\boldsymbol{a}_2 + \left(c_3 + sc_5 + \frac{s^2}{2}c_{36}\right)\boldsymbol{a}_3 \\
(\boldsymbol{X}^h_\ell)_s(s,t) &= s\boldsymbol{a}_0 + \boldsymbol{a}_1 + s\boldsymbol{a}_2,
\end{aligned}$$

where $(\ )'$ means $\partial/\partial t$ and $(\ )_s$ means $\partial/\partial s$. It follows that we have

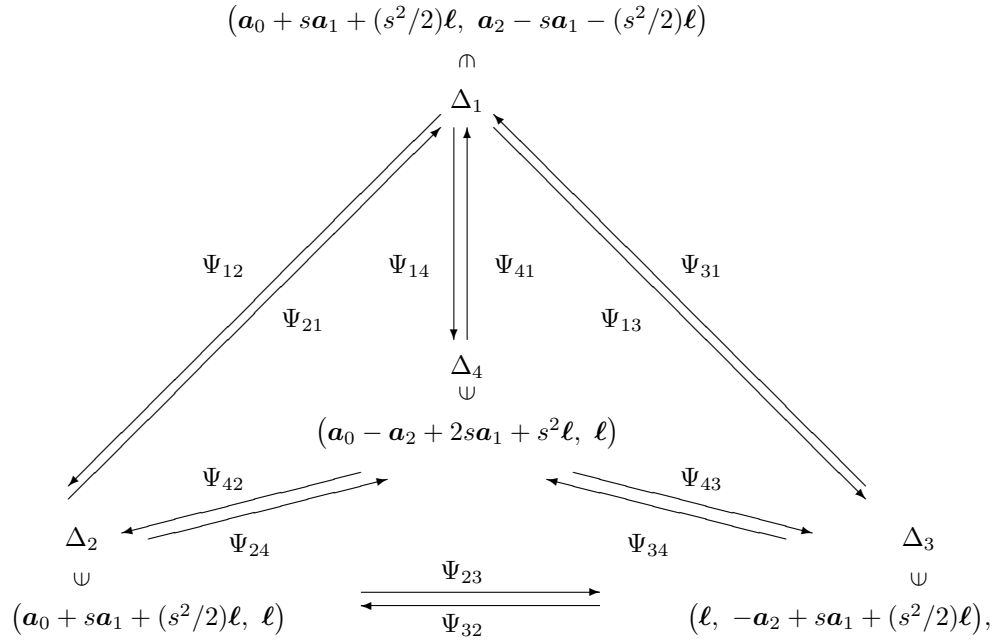
$$\left\langle \boldsymbol{X}^h_\ell(\pm s,t), \boldsymbol{X}^d_\ell(\mp s,t)\right\rangle \equiv 0,$$

$$\left\langle (\boldsymbol{X}^h_\ell)'(\pm s,t), \boldsymbol{X}^d_\ell(\mp s,t)\right\rangle \equiv 0 \text{ and } \left\langle (\boldsymbol{X}^h_\ell)_s(\pm s,t), \boldsymbol{X}^d_\ell(\mp s,t)\right\rangle \equiv 0.$$

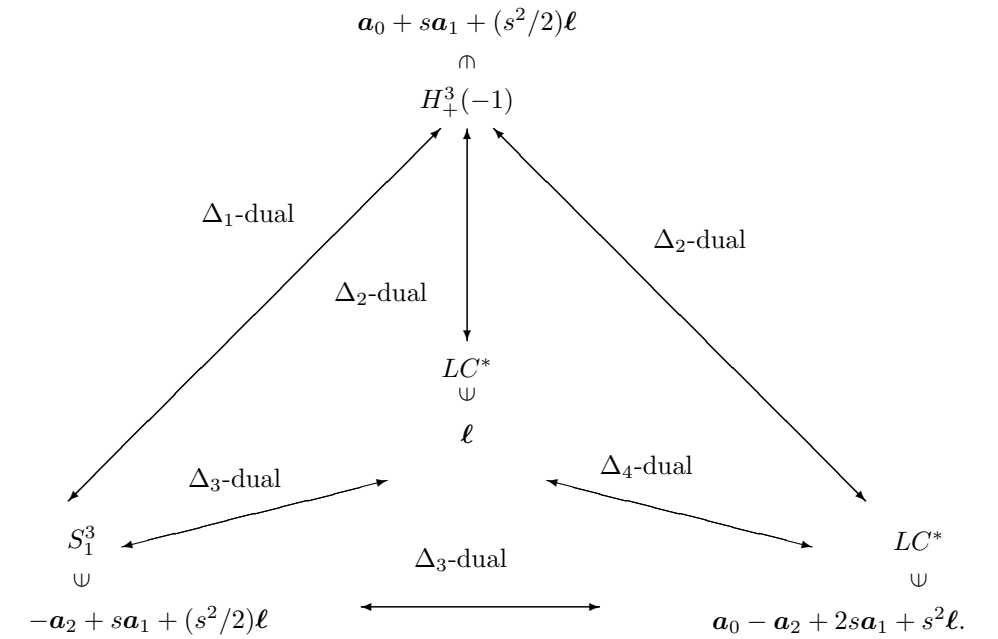
This implies that $(\boldsymbol{X}^h_\ell, \boldsymbol{X}^d_\ell) : I \times \mathbb{R} \to \Delta_1$ is an isotropic map with respect to $K_1$. Therefore $\boldsymbol{X}^h_\ell$ and $\boldsymbol{X}^d_\ell$ are $\Delta_1$-dual each other. Since $\boldsymbol{X}^h_\ell(s,t)$ is a linear Weingarten surface of non-Bryant type, $\boldsymbol{X}^d_\ell(s,t)$ is a linear Weingarten surface of non-Bianchi type by Theorem 5.2. By the same calculation, we can show that the $\Delta_2$-duality between $\boldsymbol{X}^h_\ell(\pm s,t)$ and $\boldsymbol{X}^\ell_\ell(\pm s,t)$, and the $\Delta_3$-duality between $\boldsymbol{X}^d_\ell(\pm s,t)$ and $-\boldsymbol{X}^\ell_\ell(\mp s,t)$ under the assumptions $c_2(t) \equiv 0$, $c_1(t) - c_4(t) \equiv 0$. These assumptions mean that a kind of flatness of $\boldsymbol{X}^h_\ell(s,t), \boldsymbol{X}^d_\ell(s,t)$ and $\boldsymbol{X}^\ell_\ell(s,t)$. For $\boldsymbol{X}^h_\ell(s,t)$, such a flatness is called *horo-flat* in [22].

Furthermore, under the conditions $c_2 \equiv c_3 \equiv 0$ (resp. $c_2 \equiv c_6 \equiv 0$), we have $\left\langle \boldsymbol{X}^d_h, \boldsymbol{X}^\ell_h\right\rangle \equiv 1$ (resp. $\left\langle \boldsymbol{X}^h_d, \boldsymbol{X}^\ell_d\right\rangle \equiv -1$) and $\left\langle \boldsymbol{X}^d_h, d\boldsymbol{X}^\ell_h\right\rangle \equiv 0$ (resp. $\left\langle \boldsymbol{X}^h_d, d\boldsymbol{X}^\ell_d\right\rangle \equiv 0$). Hence $\boldsymbol{X}^d_h$ and $\boldsymbol{X}^\ell_h$ are $\Delta_3$-dual (resp. $\boldsymbol{X}^h_d$ and $\boldsymbol{X}^\ell_d$ are $\Delta_2$-dual) each other. By Theorem 5.2 and the mandala of Legendrian dualities, the surface $\boldsymbol{X}^\ell_\ell(s,t)$ corresponds to the linear Weingarten surfaces of non-Bryant type in $H^3_+(-1)$ and of non-Bianchi type in $S^3_1$.

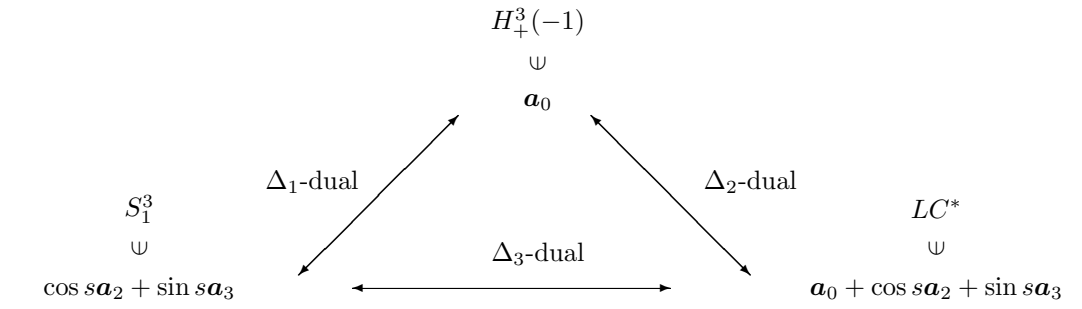
Thus we have the following diagram which expresses the duality for flat surfaces in pseudo-spheres:



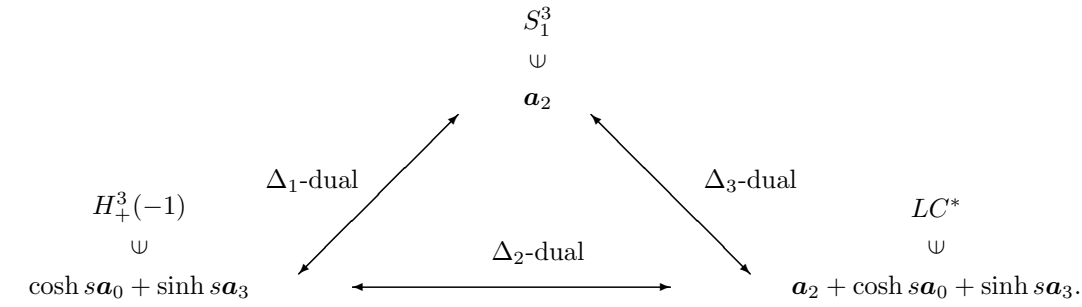

If we start from a curve $\boldsymbol{\ell}$ in the lightcone, we have the following diagram of dualities:




Also we can have the diagram on dualities starting from $\boldsymbol{a}_0$ and $\boldsymbol{a}_2$:




and




We can also have a diagram starting from the curve $\boldsymbol{\ell}_- = \boldsymbol{a}_0 - \boldsymbol{a}_2$. However, the situation is the same as the case for $\boldsymbol{\ell}$, so that we omit it.

# 7 Fundamental properties of parabollatic surfaces

In section 6, we construct the dual surfaces of $\boldsymbol{\ell}$ which are called parabollatic surfaces. The analogous notion in Euclidean space is *ruled surfaces* given by one-parameter families of lines in $\mathbb{R}^3$. For the study of singularities and geometric properties of ruled surfaces, the *striction curve* plays a crucial role ([16]). The striction curve is a curve on the ruled surface which contains the singularities of the surface. Similarly, an analogous notion of the striction curve also plays a crucial role for one-parameter families of circles ([23]). Since surfaces $\boldsymbol{X}^h_\ell$, $\boldsymbol{X}^d_\ell$ and $\boldsymbol{X}^\ell_\ell$ are one-parameter families of parabolas, we try to find the analogous notion of striction curves of ruled surfaces. Here, we only consider the surfaces $\boldsymbol{X}^h_\ell$, $\boldsymbol{X}^d_\ell$ and $\boldsymbol{X}^\ell_\ell$. We remark that surfaces $\boldsymbol{X}^d_h, \boldsymbol{X}^\ell_h, \boldsymbol{X}^h_d$ and $\boldsymbol{X}^\ell_d$ have similar properties as the circular surfaces [23]. We shall investigate these surfaces in the forthcoming paper.

## 7.1 The striction curve of $\boldsymbol{X}^d_\ell$

Let $A = (\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) : I \to SO_0(3,1)$ be a pseudo-orthonormal frame defined in Section 6. The $\Delta_3$-dual surface $\boldsymbol{X}^d_\ell$ of $\boldsymbol{\ell}$ is defined by

$$\boldsymbol{X}^d_{\ell,A}(s,t) := \boldsymbol{a}_2(t) + s\boldsymbol{a}_1(t) - \frac{s^2}{2}\boldsymbol{\ell}(t).$$

For any $t$, the curve $s \mapsto \boldsymbol{X}^d_{\ell,A}(s,t)$ is a parabola. The each parabola called the *generating parabola*.

On the other hand, for any curve

$$\overline{\boldsymbol{a}_2}(t) = \boldsymbol{a}_2(t) + s(t)\boldsymbol{a}_1(t) - \frac{s(t)^2}{2}\boldsymbol{\ell}(t) \tag{7.1}$$

on the $\boldsymbol{X}^d_\ell$, we define

$$\begin{gathered}\overline{\boldsymbol{a}_0}(t) = \left(1 + \frac{s(t)^2}{2}\right)\boldsymbol{a}_0(t) - s(t)\boldsymbol{a}_1(t) + \frac{s(t)^2}{2}\boldsymbol{a}_2(t), \\ \overline{\boldsymbol{a}_1}(t) = -s(t)\boldsymbol{a}_0(t) + \boldsymbol{a}_1(t) - s\boldsymbol{a}_2(t) \text{ and } \overline{\boldsymbol{a}_3}(t) = \boldsymbol{a}_3(t)\end{gathered} \tag{7.2}$$

then $\boldsymbol{X}^d_{\ell,\overline{A}}(s-s(t),t) = \boldsymbol{X}^d_{\ell,A}(s,t)$ holds. Moreover, we define invariants $\overline{C}(t)$ by the formula $\overline{A}'(t) = \overline{C}(t)\overline{A}(t)$, then we have

$$\begin{cases} \overline{c_1} &= \left(\dfrac{1-s(t)^2}{2}\right)c_1 - s'(t) + \dfrac{s(t)^2}{2}c_4 - s(t)c_2 \\ \overline{c_2} &= s(t)c_1 + c_2 - s(t)c_4 \\ \overline{c_3} &= c_3 - s(t)c_5 + \dfrac{s(t)^2}{2}c_{36} \\ \overline{c_4} &= \dfrac{-s(t)^2}{2}c_1 - s(t)c_2 + \left(1 + \dfrac{s(t)^2}{2}\right)c_4 - s'(t) \\ \overline{c_5} &= c_5 - s(t)c_{36} \\ \overline{c_6} &= c_6 + s(t)c_5 - \dfrac{s(t)^2}{2}c_{36}. \end{cases}$$

It follows that

$$\overline{c_1} - \overline{c_4} = c_1 - c_4$$

and

$$\overline{c_1} - \overline{c_4} = \overline{c_2} = 0 \text{ if and only if } c_1 - c_4 = c_2 = 0.$$

This means that the condition $c_1 - c_4 = c_2 = 0$ is invariant under the adopted coordinate changes. Here, a reparameterization $(s,t) \mapsto (S,T)$ of $\boldsymbol{X}^d_{\ell,A}$ is said to be *adopted* if $S = s - s(t)$ and $T = t$. We have the following proposition.

**Proposition 7.1.** *Let $\boldsymbol{X}^d_{\ell,A}$ be a parameterization of a parabollatic surfaces of the form*

$$\boldsymbol{X}^d_{\ell,A}(s,t) = \boldsymbol{a}_2 + s\boldsymbol{a}_1 - \frac{s^2}{2}\boldsymbol{\ell}$$

*such that $c_1 - c_4$ never vanish. Then* Image $\boldsymbol{X}^d_{\ell,A}$ *has an adopted reparameterization of the form*

$$\boldsymbol{X}^d_{\ell,\overline{A}}(s,t) = \overline{\boldsymbol{a}}_2 + s\overline{\boldsymbol{a}}_1 - \frac{s^2}{2}\overline{\boldsymbol{\ell}}$$

*satisfying $\langle \overline{\boldsymbol{a}_0}', \overline{\boldsymbol{a}_2}\rangle = 0$ for any $t$.*

*Proof.* Let us define

$$s(t) = \frac{-c_2(t)}{c_1(t) - c_4(t)}$$

and define curves $\overline{\boldsymbol{a}}_0, \overline{\boldsymbol{a}}_1, \overline{\boldsymbol{a}}_2$ by (7.1) and (7.2). Then $\overline{c_2} \equiv 0$ holds. We do not need to say that $\boldsymbol{X}^d_{\ell,\overline{A}}$ and $\boldsymbol{X}^d_{\ell,A}$ have the same image. Thus the condition of the proposition holds. □

A curve $\boldsymbol{X}^d_{\ell,A}(s(t),t)$ on the surface is called *striction curve* if $\langle \boldsymbol{a}_0'(t), \boldsymbol{a}_2(t)\rangle \equiv 0$ holds. Proposition 7.1 implies that we can take $\boldsymbol{a}_2$ as the striction curve. Singularities of parabollatic surfaces are located on the striction curve. For any parabollatic surfaces satisfying $c_1 - c_4 \neq 0$, there exists a unique striction curve.

**Proposition 7.2.** *Let $\boldsymbol{X}^d_{\ell,A}$ be a parabollatic surface with the striction curve $\boldsymbol{a}_2$ and $c_1 - c_4 \neq 0$. If $(s_0, t_0)$ is a singular point, then $s_0 = 0$ namely, $x_0$ is located on the striction curve. Moreover, if $(0, t_0)$ is a singular point, them the generating parabola at $t_0$ is tangent to the striction curve.*

*Proof.* Direct calculation and $\boldsymbol{a}_2' = -c_4\boldsymbol{a}_1$ yield the conclusion. □

## 7.2 The striction curve of $\boldsymbol{X}^\ell_\ell$

In this section, we study general properties of dual surfaces of $\boldsymbol{\ell}$. Let $A = (\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) : I \to SO_0(3,1)$ be a pseudo-orthonormal frame defined in Section 6. The dual surface $\boldsymbol{X}^\ell_\ell$ of $\boldsymbol{\ell}$ is defined by

$$\boldsymbol{X}^\ell_{\ell,A}(s,t) := \boldsymbol{a}_0(t) - \boldsymbol{a}_2(t) + 2s\boldsymbol{a}_1(t) + s^2\boldsymbol{\ell}(t).$$

For any curve, $\overline{\boldsymbol{a}_0} - \overline{\boldsymbol{a}_2}(t) = \boldsymbol{a}_0 - \boldsymbol{a}_2(t) + 2s(t)\boldsymbol{a}_1(t) + s^2\boldsymbol{\ell}(t)$ on $\boldsymbol{X}^\ell_{\ell,\boldsymbol{a}}$, we define

$$\begin{gathered}\overline{\boldsymbol{a}_0}(t) = \boldsymbol{a}_0 + s(t)\boldsymbol{a}_1 + \frac{s(t)^2}{2}\boldsymbol{\ell}, \quad \overline{\boldsymbol{a}_1}(t) = \boldsymbol{a}_1 + s(t)\boldsymbol{\ell}(t), \\ \overline{\boldsymbol{a}_2}(t) = \boldsymbol{a}_2 - s(t)\boldsymbol{a}_1 - \frac{s(t)^2}{2}\boldsymbol{\ell} \text{ and } \overline{\boldsymbol{a}_3}(t) = \boldsymbol{a}_3(t)\end{gathered} \tag{7.3}$$

then $\boldsymbol{X}^\ell_{\ell,\overline{A}}(s - s(t), t) = \boldsymbol{X}^\ell_{\ell,A}(s,t)$ holds. Moreover, we define invariants $\overline{C}(t)$ by the formula $\overline{A}'(t) = \overline{C}(t)\overline{A}(t)$, then we have

$$\begin{cases} \overline{c_1} &= -\dfrac{s(t)^2}{2}(c_1 - c_4) + c_1 + s'(t) + sc_2 \\ \overline{c_2} &= -s(t)c_1 + c_2 + s(t)c_4 \\ \overline{c_3} &= c_3 + s(t)c_5 + \dfrac{s(t)^2}{2}c_{36} \\ \overline{c_4} &= -\dfrac{s(t)^2}{2}c_1 + s(t)c_2 + \dfrac{s(t)^2}{2}c_4 + s'(t) \\ \overline{c_5} &= c_5 - s(t)c_{36} \\ \overline{c_6} &= c_6 - s(t)c_5 - \dfrac{s(t)^2}{2}c_{36} \end{cases}$$

Thus it follows that

$$\overline{c_1} - \overline{c_4} = c_1 - c_4$$

and

$$\overline{c_1} - \overline{c_4} = \overline{c_2} = 0 \text{ if and only if } c_1 - c_4 = c_2 = 0.$$

A reparameterization $(s,t) \mapsto (S,T)$ of $\boldsymbol{X}^{\ell}_{\ell,A}$ is said to be *adopted* if $S = s - s(t)$ and $T = t$. We have the following proposition.

**Proposition 7.3.** *Let $\boldsymbol{X}^{\ell}_{\ell,A}$ be a parameterization of a parabollatic surfaces of the form*

$$\boldsymbol{X}^{\ell}_{\ell,A}(s,t) = \boldsymbol{a}_0 - \boldsymbol{a}_2 + 2s\boldsymbol{a}_1 + s^2\boldsymbol{\ell}$$

*such that $c_1 - c_4$ never vanish. Then* Image $\boldsymbol{X}^{\ell}_{\ell,A}$ *has an adopted reparameterization of the form*

$$\boldsymbol{X}^{\ell}_{\ell,\overline{A}}(s,t) = \overline{\boldsymbol{a}_0} - \overline{\boldsymbol{a}}_2 + 2s\overline{\boldsymbol{a}}_1 + s^2\overline{\boldsymbol{\ell}}$$

*satisfying $\langle \overline{\boldsymbol{a}_0}', \overline{\boldsymbol{a}_2} \rangle \equiv 0$.*

*Proof.* Let us define

$$s(t) = \frac{c_2(t)}{c_1(t) - c_4(t)}$$

and define curves $\overline{A}$ as (7.1) and (7.2). Then $\overline{c_2} \equiv 0$ holds. We do not need to say that $\boldsymbol{X}^{\ell}_{\ell,\overline{A}}$ and $\boldsymbol{X}^{\ell}_{\ell,A}$ have the same image. Thus the condition of the proposition holds. □

A curve on the surface $\boldsymbol{X}^{\ell}_{\ell,A}(s(t),t)$ is called *striction curve* if $\langle \boldsymbol{a}_0'(t), \boldsymbol{a}_2(t) \rangle \equiv 0$ holds. Proposition 7.3 implies that one can take $\boldsymbol{a}_2$ as the striction curve. Singularities of parabollatic surfaces are located on the striction curve.

**Proposition 7.4.** *Let $\boldsymbol{X}^{\ell}_{\ell,A}$ be a parabollatic surface with the striction curve $\boldsymbol{a}_2$ and $c_1 - c_4 \neq 0$. If $(s_0,t_0)$ is a singular point, then $s_0 = 0$ namely, $x_0$ is located on the striction curve. Moreover, if $(0,t_0)$ is a singular point, then the generating parabola at $t_0$ is tangent to the striction curve.*

*Proof.* For a parabollatic surface $\boldsymbol{X}^{\ell}_{\ell,A}$, point $(s_0,t_0)$ is a singular point if and only if

$$c_2(t_0) - s_0(c_1(t_0) - c_4(t_0)) = 0 \text{ and } c_3(t_0) - c_6(t_0) + 2s_0c_5(t_0) + s_0^2c_{36}(t_0) = 0.$$

Thus if $\boldsymbol{a}_0 - \boldsymbol{a}_2$ is the striction curve, them $s_0 = 0$ and $c_3(t_0) - c_6(t_0) = 0$ holds. Moreover, if $c_3(t_0) - c_6(t_0) = 0$, then the parabola is tangent to the striction curve at $(0,t_0)$. Because of $\boldsymbol{a}_0' - \boldsymbol{a}_2' = -c_2(\boldsymbol{a}_0 - \boldsymbol{a}_2) + (c_1 + c_4)\boldsymbol{a}_1$. □

Singularities of these surface are studied in Section 8. Although we can construct dual surfaces from $\boldsymbol{\ell}_-$, their geometric properties are the same as those of dual surfaces constructed from $\boldsymbol{\ell}$, so that we omit the study of their striction curves.

# 8 Singularities of flat parabollatic surfaces

## 8.1 Criteria for singularities of frontals

All surfaces investigating here have an isotropic lift to some contact manifold. They are called frontals which are originally investigated by Zakalyukin[34, 35]. In order to investigate singularities of concretely parameterized surfaces, the identification problem for singularities are important. Let $f_0$ be a given map germs. The *identification problem for* $f_0$ is to find a condition such that a map germ $f$ satisfies the condition if and only if $f$ is $\mathcal{A}$-equivalent to $f_0$. We call the condition a *criterion* for $f_0$. Such criteria are given by many people now. Simple criteria for the cuspidal edge and the swallowtail were given by Kokubu, Rossman, Saji, Umehara and Yamada [28]. Other criteria for singularities for frontals are investigated in [13, 32, 22]. Here, we briefly review the criteria for frontals. Let $\pi : E \to M$ be a Legendrian fibration from a five-dimensional contact manifold $E$ to a three-dimensional manifold $M$. A $C^\infty$-map $f : U \to M$ is called a frontal (resp. front) if there exists an isotropic lift (resp. Legendrian immersion) $L_f : U \to E$, where $U \subset \mathbb{R}^2$ be an open set. Recall that the image of the Legendrian submanifold is called the wavefront set (see Section 3). By the generalized Darboux theorem (cf., [2], 20.3), any Legendrian fibration $E \to M$ is locally equivalent to the standard fibration $PT\mathbb{R}^3 \to \mathbb{R}^3$. Therefore, we assume that $E \to M$ is $PT\mathbb{R}^3 \to \mathbb{R}^3$ and that $f$ is a $C^\infty$ map germ $(U, p) \to (\mathbb{R}^3, f(p))$. Taking the fiber component, let us denote $L_f = (f, [\nu])$. The *discriminant function* of a frontal $f$ is defined by $\lambda(u, v) = \det(f_u,\ f_v,\ \nu)(u, v)$ using the coordinate system $(u, v)$ on $U$, where $f_u = \partial f/\partial u$, for example. A singular point $p$ of $f$ is *non-degenerate* if $d\lambda(p) \neq \mathbf{0}$ holds. Let $p$ be a non-degenerate singular point of a frontal $f$. In this case, there exists a smooth parameterization $\gamma(t) : (-\varepsilon, \varepsilon) \to U$, $\gamma(0) = p$ of $S(f)$ near $p$. Moreover, there exists a smooth vector field $\eta(t)$ along $\gamma$ satisfying that $\eta(t)$ generates the kernel of $df_{\gamma(t)}$. We call this vector field the *null vector field*. Now we define a function $\phi_f(t)$ on $\gamma$ by

$$\phi_f(t) = \det\big((f \circ \gamma)', \nu \circ \gamma, d\nu(\eta)\big)(t) \tag{8.1}$$

Using these notations, the following criteria have been obtained.

**Theorem 8.1.** [28, 13] *Let $f : U \to \mathbb{R}^3$ be a frontal and $p$ a non-degenerate singular point of $f$ and $\gamma : (\varepsilon, \varepsilon) \to U$, $\gamma(0) = p$ be a smooth parameterization of $S(f)$ near $p$. Then the following assertions hold.*

- *If $\eta\lambda(p) \neq 0$ then $f$ to be a front near $p$ if and only if $\phi_f(0) \neq 0$ holds.*
- *The map germ $f$ at $p$ is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $f$ to be a front near $p$ and $\eta\lambda(p) \neq 0$ hold.*
- *The map germ $f$ at $p$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $f$ to be a front near $p$, $\eta\lambda(p) = 0$ and $\eta\eta\lambda(p) \neq 0$.*
- *The map germ $f$ at $p$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\eta\lambda(p) \neq 0$, $\phi_f(0) = 0$ and $\phi_f'(0) \neq 0$.*

Here, $\eta\lambda : U \to \mathbb{R}$ means the directional derivative of $\lambda$ by the vector field $\tilde{\eta}$, where $\tilde{\eta} \in \mathfrak{X}(U)$ is an extended vector field of $\eta$ to $U$. Moreover, we have the following criterion for the cuspidal butterfly.

**Theorem 8.2.** *Let $f : U \to \mathbb{R}^3$ be a frontal and $p$ a non-degenerate singular point of $f$ and $\gamma : (\varepsilon, \varepsilon) \to U$, $\gamma(0) = p$ be a smooth parameterization of $S(f)$ near $p$. Then the map germ $f$ at $p$ is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if $f$ to be a front near $p$ and $\eta\lambda(p) = \eta\eta\lambda(p) = 0$ and $\eta\eta\eta\lambda(p) \neq 0$.*

A proof of this theorem is given in the appendix. Next we consider a degenerate singularity. Let $p$ be a degenerate singularity of a front $f$. If $\operatorname{rank}(df)_p = 1$, then there exists a non-zero vector field $\eta$ near $p$ such that if $q \in S(f)$ then $\eta(q)$ generates the kernel of $df(q)$. A criterion for the degenerate singularity is given as follows.

**Theorem 8.3.** [*22*] *Let $f$ be a front and $p$ a degenerate singular point of $f$ Then the following assertions hold.*

- *The map germ $f$ at $p$ is $\mathcal{A}$-equivalent to the cuspidal lips if and only if* $\operatorname{rank}(df)_p = 1$ *and* $\det \operatorname{Hess} \lambda(p) > 0$.
- *The map germ $f$ at $p$ is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if* $\operatorname{rank}(df)_p = 1$, $\det \operatorname{Hess} \lambda(p) < 0$ *and* $\eta\eta\lambda(p) \neq 0$.

In order to study singularities of a front in pseudo-Riemannian space, we introduce the following notion.

**Definition 8.4.** ([13]) A lift $L_g : U \to T^*N$ of a $C^\infty$-map $g : U \to N$ to be *admissible* if $g$ never intersect to the zero-section and $g_*(T_pU) \subset \ker(L_g(p))$, where $\ker(L_g(p)) \subset T_{g(p)}N$ is the kernel of a linear map $L_g(p)$.

Using this notion, a criterion for the cuspidal cross cap is stated as follows.

**Theorem 8.5.** ([13, Theorem 1.4]) *Let $g : U \to N$ be a front and $L_g : U \to T^*N$ be an admissible lift of $g$. Let $D$ be an arbitrary linear connection on $N$. Suppose that $\gamma(t)$ $(|t| < \varepsilon)$ is a singular curve on $U$ passing through a non-degenerate singular point $p = \gamma(0)$, and $\xi_g : (-\varepsilon, \varepsilon) \to TN$ is an arbitrarily fixed vector field along $\gamma$ such that*
*(1) $L(\xi_g)$ vanishes on $U$ and*
*(2) $\xi_g$ is transversal to $g_*(T_pU)$ at $p$.*
*We define a function $\psi_g(t)$ by*

$$\psi_g(t) = L\Big(D^g_{\eta(t)}\xi_g\big(\gamma(t)\big)\Big), \tag{8.2}$$

*where $\eta(t)$ is a null vector field on the singular curve parameterized by $t$. Then the germ $g$ at $p$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\psi_g(0) = 0$ and $\psi_g'(0) \neq 0$ hold, and $\eta(0)$ is transversal to $\gamma'(0)$.*

## 8.2 Singularities of dual surfaces of $\boldsymbol{\ell}$

In this subsection, we apply the criteria in Subsection 8.1 for describing the conditions of singularities of dual surfaces of $\boldsymbol{\ell}$. We assume that $c_2 \equiv c_1 - c_4 \equiv 0$ in this section.

**Theorem 8.6.** *The singular set of $\boldsymbol{X}^h_\ell$ is $S(\boldsymbol{X}^h_\ell) = \{(s,t) \mid 2c_3(t) + 2sc_5(t) + s^2c_{36}(t) = 0\}$ and $\boldsymbol{X}^h_\ell$ is a frontal for any $p_0 = (s_0, t_0) \in S(\boldsymbol{X}^h_\ell)$. Then we have the following assertions:*

- *If $c_{36}(t_0)\neq 0$ holds, then $\boldsymbol{X}^h_\ell$ to be a front near $p_0$.*
- *$\boldsymbol{X}^h_\ell$ at $p_0$ is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $c_{36}\neq 0$ and $\alpha^h_\ell := -2c_1(c_5+sc_{36})+2c_3'+2sc_5'+s^2c_{36}'\neq 0$ hold at $p_0$.*
- *$\boldsymbol{X}^h_\ell$ at $p_0$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_{36}\neq 0$, $c_5+sc_{36}\neq 0$, $\alpha^h_\ell=0$ and $c_1(\alpha^h_\ell)_s+(\alpha^h_\ell)'\neq 0$ hold at $p_0$.*
- *$\boldsymbol{X}^h_\ell$ at $p_0$ is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if $c_{36}\neq 0$, $c_5+sc_{36}\neq 0$, $\alpha^h_\ell=0$, $c_1(\alpha^h_\ell)_s+(\alpha^h_\ell)'\neq 0$ and $c_1^2(\alpha^h_\ell)_{ss}+2c_1(\alpha^h_\ell)'_s+c_1'(\alpha^h_\ell)_s+(\alpha^h_\ell)''=0$ hold at $p_0$.*
- *$\boldsymbol{X}^h_\ell$ at $p_0$ is $\mathcal{A}$-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if $c_{36}\neq 0$, $c_5+sc_{36}=0$, $2c_3'+2sc_5'+s^2c_{36}'=0$ and $\det H^h_\ell>0$ (resp. $\det H^h_\ell<0$ and $-2c_1(c_{36}+c_5'+sc_{36}')+(-2c_1(c_5+sc_{36})+c_3'+sc_5'+s^2c_{36}')'\neq 0$) hold at $p_0$, where*

$$H^h_\ell=\begin{pmatrix} 2c_{36} & 2c_5'+2sc_{36}' \\ 2c_5'+2sc_{36}' & 2c_3''+2sc_5''+s^2c_{36}''\end{pmatrix}.$$

- *$\boldsymbol{X}^h_\ell$ at $p_0$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_{36}=0$, $\alpha^h_\ell c_5\neq 0$ and $c_{36}'\neq 0$ hold at $p_0$.*

**Remark 8.7.** Surfaces $\boldsymbol{X}^h_\ell$ satisfying $c_3\equiv 0$ be a horo-flat horo-cyclic surfaces which is investigated in [22]. Substituting $c_3\equiv 0$ in the formulae of Theorem 8.6, we have [22, Theorem 6.2].

**Theorem 8.8.** *The singular set of $\boldsymbol{X}^d_\ell$ is $S(\boldsymbol{X}^d_\ell)=\{(s,t)\mid -2c_6(t)-2sc_5(t)+s^2c_{36}(t)=0\}$ and $\boldsymbol{X}^d_\ell$ is a frontal for any $(s_0,t_0)\in S(\boldsymbol{X}^d_\ell)$. Then we have the following assertions:*

- *If $c_{36}(t_0)\neq 0$ holds, then $\boldsymbol{X}^d_\ell$ to be a front near $p_0$.*
- *$\boldsymbol{X}^d_\ell$ at $p_0$ is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $c_{36}\neq 0$ and $\alpha^d_\ell := 2c_1(c_5-sc_{36})+2c_6'+2sc_5'-s^2c_{36}'\neq 0$ hold at $p_0$.*
- *$\boldsymbol{X}^d_\ell$ at $p_0$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_{36}\neq 0$, $c_5-sc_{36}\neq 0$, $\alpha^d_\ell=0$ and $c_1(\alpha^d_\ell)_s+(\alpha^d_\ell)'\neq 0$ hold at $p_0$.*
- *$\boldsymbol{X}^d_\ell$ at $p_0$ is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if $c_{36}\neq 0$, $c_5-sc_{36}\neq 0$, $\alpha^d_\ell=0$, $c_1(\alpha^d_\ell)_s+(\alpha^d_\ell)'=0$ and $c_1^2(\alpha^d_\ell)_{ss}+2c_1(\alpha^d_\ell)'_s+c_1'(\alpha^d_\ell)_s+(\alpha^d_\ell)''\neq 0$ hold at $p_0$.*
- *$\boldsymbol{X}^d_\ell$ at $p_0$ is $\mathcal{A}$-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if $c_{36}\neq 0$, $2c_6'+2sc_5'-s^2c_{36}'=0$, $c_5-sc_{36}=0$ and $\det H^d_\ell>0$ (resp. $\det H^d_\ell<0$ and $c_1(-2c_1c_{36}+2c_5'-2sc_{36}')+\big(2c_1(c_5-sc_{36})+2c_6'+2sc_5'-s^2c_{36}'\big)'\neq 0$) hold at $p_0$, where*

$$H^d_\ell=\begin{pmatrix} -2c_{36} & 2c_5'-2sc_{36}' \\ 2c_5'-2sc_{36}' & 2c_6''+2sc_5''-s^2c_{36}''\end{pmatrix}.$$

- *$\boldsymbol{X}^d_\ell$ at $p_0$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_{36}=0$, $\alpha^d_\ell c_5\neq 0$ and $c_{36}'\neq 0$ hold at $p_0$.*

**Theorem 8.9.** *The singular set of* $\boldsymbol{X}^\ell_\ell$ *is* $S(\boldsymbol{X}^\ell_\ell) = \{(s,t) \mid c_3(t)(s^2+1)+2sc_5(t)+c_6(t)(s^2-1)=0\}$ *and* $\boldsymbol{X}^\ell_\ell$ *is a frontal for any* $(s_0,t_0) \in S(\boldsymbol{X}^\ell_\ell)$. *Then we have the following assertions:*

- *If* $c_{36}(t_0) \neq 0$ *holds, then* $\boldsymbol{X}^\ell_\ell$ *to be a front near* $p_0$.
- $\boldsymbol{X}^\ell_\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal edge if and only if* $c_{36} \neq 0$ *and* $\alpha^\ell_\ell := -2c_1(c_5+sc_{36})+c_3'(s^2+1)+2sc_5'+c_6'(s^2-1) \neq 0$ *hold at* $p_0$.
- $\boldsymbol{X}^\ell_\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the swallowtail if and only if* $c_{36} \neq 0$, $c_5+sc_{36} \neq 0$, $\alpha^\ell_\ell = 0$ *and* $-c_1(\alpha^\ell_\ell)_s+(\alpha^\ell_\ell)' \neq 0$ *hold at* $p_0$.
- $\boldsymbol{X}^\ell_\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal butterfly if and only if* $c_{36} \neq 0$, $c_5+sc_{36} \neq 0$, $\alpha^\ell_\ell = 0$, $-c_1(\alpha^\ell_\ell)_s(\alpha^\ell_\ell)' = 0$ *and* $c_1^2(\alpha^\ell_\ell)_{ss} - 2c_1(\alpha^\ell_\ell)'_s - c_1'(\alpha^\ell_\ell)_s + (\alpha^\ell_\ell)'' \neq 0$ *hold at* $p_0$.
- $\boldsymbol{X}^\ell_\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if* $c_{36} \neq 0$, $c_3'(s^2+1)+2sc_5'+c_6'(s^2-1)=0$ , $c_5+sc_{36}=0$ *and* $\det H^\ell_\ell > 0$ *(resp.* $\det H^\ell_\ell < 0$ *and* $-c_1\big(-2c_1c_{36}+2(c_5+sc_{36}')\big)+\big(-2c_1(c_5+sc_{36})+c_3'(s^2+1)+2sc_5'+c_6'(s^2-1)\big)' \neq 0$*) hold at* $p_0$, *where*

$$H^\ell_\ell = \begin{pmatrix} 2c_{36} & 2c_5'+2sc_{36}' \\ 2c_5'+2sc_{36}' & c_3''(s^2+1)+2sc_5''+c_6''(s^2-1) \end{pmatrix}.$$

- $\boldsymbol{X}^\ell_\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal cross cap if and only if* $c_{36}=0$, $\alpha^\ell_\ell c_5 \neq 0$ *and* $c_{36}' \neq 0$ *hold at* $p_0$.

## 8.3 Singularities of dual surfaces of $\boldsymbol{a}_0$

In this subsection, we apply the criteria in Subsection 8.1 for describing the conditions of singularities of dual surfaces of $\boldsymbol{a}_0$. We assume that $c_2 \equiv c_3 \equiv 0$.

**Theorem 8.10.** *The singular set of* $\boldsymbol{X}^d_h$ *is* $S(\boldsymbol{X}^d_h) = \{(s,t) \mid c_4(t)\cos s + c_5(t)\sin s = 0\}$ *and* $\boldsymbol{X}^d_h$ *is a frontal for any* $p_0=(s_0,t_0)$. *Then we have the following assertions:*

- *If* $c_1 \neq 0$ *holds, then* $\boldsymbol{X}^d_h$ *to be a front near* $p_0$.
- $\boldsymbol{X}^d_h$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal edge if and only if* $c_1 \neq 0$ *and* $\alpha^d_h := -c_6(c_4\sin s - c_5\cos s) - (c_4'\cos s + c_5'\sin s) \neq 0$ *holds at* $p_0$.
- $\boldsymbol{X}^d_h$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the swallowtail if and only if* $c_1 \neq 0$, $-c_4\sin s + c_5\cos s \neq 0$, $\alpha^d_h = 0$ *and* $-c_6(\alpha^d_h)_s + (\alpha^d_h)' \neq 0$.
- $\boldsymbol{X}^d_h$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal butterfly if and only if* $c_1 \neq 0$, $-c_4\sin s + c_5\cos s \neq 0$, $\alpha^d_h = 0$, $-c_6(\alpha^d_h)_s+(\alpha^d_h)' = 0$ *and* $c_6^2(\alpha^d_h)_{ss} - 2c_6(\alpha^d_h)'_s - c_6'(\alpha^d_h)_s + (\alpha^d_h)'' \neq 0$ *holds at* $p_0$.
- $\boldsymbol{X}^d_h$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if* $c_1 \neq 0$, $c_4'\cos s + c_5'\sin s = 0$, $c_4\sin s - c_5\cos s = 0$, *and* $\det H^d_h > 0$ *(resp.* $\det H^d_h < 0$ *and* $-c_6(\alpha^d_h)_s + (\alpha^d_h)' \neq 0$*) hold at* $p_0$, *where*

$$H^d_h = \begin{pmatrix} -c_4\cos s - c_5\sin s & c_4'\sin s - c_5'\cos s \\ c_4'\sin s - c_5'\cos s & c_4''\cos s + c_5''\sin s \end{pmatrix}.$$

- $\boldsymbol{X}_h^d$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal cross cap if and only if* $c_1 = 0$, $\alpha_h^d(c_4 \sin s - c_5 \cos s) \neq 0$ *and* $c_1' \neq 0$ *hold at* $p_0$.

**Theorem 8.11.** *The singular set of* $\boldsymbol{X}_h^\ell$ *is* $S(\boldsymbol{X}_h^\ell) = \{(s,t) \mid c_1(t) - c_4(t)\cos s - c_5(t)\sin s = 0\}$ *and* $\boldsymbol{X}_h^\ell$ *is a frontal for any* $p_0 = (s_0, t_0)$. *Then we have the following assertions:*

- *If* $c_1(t_0) \neq 0$ *holds, then* $\boldsymbol{X}_h^\ell$ *to be a front near* $p_0$.
- $\boldsymbol{X}_h^\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal edge if and only if* $c_1 \neq 0$ *and* $\alpha_h^\ell := -c_6(c_4 \sin s - c_5 \cos s) + c_1' - c_4' \cos s - c_5' \sin s \neq 0$ *holds at* $p_0$.
- $\boldsymbol{X}_h^\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the swallowtail if and only if* $c_1 \neq 0$, $c_4 \sin s - c_5 \cos s \neq 0$, $\alpha_h^\ell = 0$ *and* $-c_6(\alpha_h^\ell)_s + (\alpha_h^\ell)' \neq 0$ *holds at* $p_0$.
- $\boldsymbol{X}_h^\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal butterfly if and only if* $c_1 \neq 0$, $c_4 \sin s - c_5 \cos s \neq 0$, $\alpha_h^\ell = 0$ *and* $-c_6(\alpha_h^\ell)_s + (\alpha_h^\ell)' = 0$ *and* $c_6^2(\alpha_h^\ell)_{ss} - 2c_6(\alpha_h^\ell)_s' - c_6'(\alpha_h^\ell)_s + (\alpha_h^\ell)'' \neq 0$ *holds at* $p_0$.
- $\boldsymbol{X}_h^\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if* $c_1 \neq 0$, $c_4 \sin s - c_5 \cos s = 0$, $c_1' - c_4' \cos s - c_5' \sin s = 0$ *and* $\det H_h^\ell > 0$ *(resp.* $\det H_h^\ell$ *and* $-c_6(\alpha_h^\ell)_s + (\alpha_h^\ell)' \neq 0$*) holds at* $p_0$, *where*

$$H_h^\ell = \begin{pmatrix} c_4 \cos s + c_5 \sin s & c_4' \sin s - c_5' \cos s \\ c_4' \sin s - c_5' \cos s & c_1'' - c_4'' \cos s - c_5'' \sin s \end{pmatrix}.$$

- $\boldsymbol{X}_h^\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal cross cap if and only if* $c_1 = 0$, $\alpha_h^\ell(c_4 \sin s - c_5 \cos s) \neq 0$ *and* $c_1' \neq 0$ *hold at* $p_0$.

**Remark 8.12.** Surfaces $\boldsymbol{X}_h^\ell$ satisfying $c_1 - c_4 \equiv 0$ is called a *hyperbolic-flat tangent lightcone circular surface* which was investigated in [22]. Substituting $c_1 - c_4 \equiv 0$ in the formulae of Theorem 8.11, we have [22, Theorem 8.2].

## 8.4 Singularities of dual surfaces of $\boldsymbol{a}_2$

In this subsection, we apply criteria in Subsection 8.1 for describing the conditions of singularities of dual surfaces of $\boldsymbol{a}_2$. In this section, we assume that $c_2 \equiv c_6 \equiv 0$.

**Theorem 8.13.** *The singular set of* $\boldsymbol{X}_d^h$ *is* $S(\boldsymbol{X}_d^h) = \{(s,t) \mid c_1(t)\cosh s - c_5(t)\sinh s = 0\}$ *and* $\boldsymbol{X}_d^h$ *is a frontal for any* $p_0 = (s_0, t_0)$. *Then we have the following assertions*:

- *If* $c_4 \neq 0$ *holds, then* $\boldsymbol{X}_d^h$ *is a front near* $p_0$.
- $\boldsymbol{X}_d^h$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal edge if and only if* $c_4 \neq 0$ *and* $\alpha_d^h := -c_3(c_1 \sinh s - c_5 \cosh s) + c_1' \cosh s - c_5' \sinh s \neq 0$ *hold at* $p_0$.
- $\boldsymbol{X}_d^h$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the swallowtail if and only if* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s \neq 0$ *and* $\alpha_d^h = 0$ *and* $-c_3(\alpha_d^h)_s + (\alpha_d^h)' \neq 0$ *hold at* $p_0$.
- $\boldsymbol{X}_d^h$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal butterfly if and only if* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s \neq 0$ *and* $\alpha_d^h = -c_3(\alpha_d^h)_s + (\alpha_d^h)' = 0$, $-c_3(\alpha_d^h)_s + (\alpha_d^h)' = 0$ *and* $c_3^2(\alpha_d^h)_{ss} - 2c_3(\alpha_d^h)_s' - c_3'(\alpha_d^h)_s + (\alpha_d^h)'' \neq 0$ *hold at* $p_0$.

- $\boldsymbol{X}_d^h$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s = 0$, $c_1' \cosh s - c_5' \sinh s = 0$ $\det H_d^h > 0$ *(resp.* $\det H_d^h < 0$ *and* $-c_3(\alpha_d^h)_s + (\alpha_d^h)' \neq 0$*) hold at* $p_0$*, where*

$$H_d^h = \begin{pmatrix} c_1 \cosh s - c_5 \sinh s & c_1' \sinh s - c_5' \cosh s \\ c_1' \sinh s - c_5' \cosh s & c_1'' \cosh s - c_5'' \sinh s \end{pmatrix}.$$

- $\boldsymbol{X}_d^h$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal cross cap if and only if* $c_4 = 0$, $\alpha_d^h(c_1 \sinh s - c_5 \cosh s) \neq 0$ *and* $c_4' \neq 0$ *hold at* $p_0$.

**Theorem 8.14.** *The singular set of* $\boldsymbol{X}_d^\ell$ *is* $S(\boldsymbol{X}_d^\ell) = \{(s,t) \mid -c_4(t) + c_1(t)\cosh s - c_5(t)\sinh s = 0\}$ *and* $\boldsymbol{X}_d^\ell$ *is a frontal for any* $p_0 = (s_0, t_0) \in S(\boldsymbol{X}_d^\ell)$*. Then we have following assertions*:

- *If* $c_4 \neq 0$ *holds, then* $\boldsymbol{X}_d^\ell$ *is a front near* $p_0$.
- $\boldsymbol{X}_d^\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal edge if and only if* $c_4 \neq 0$ *and* $\alpha_d^\ell := -c_3(c_1 \sinh s - c_5 \cosh s) - c_4' + c_1' \cosh s - c_5' \sinh s \neq 0$ *hold at* $p_0$.
- $\boldsymbol{X}_d^\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the swallowtail if and only if* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s \neq 0$, $\alpha_d^\ell = 0$ *and* $-c_3(\alpha_d^\ell)_s + (\alpha_d^\ell)' \neq 0$ *hold at* $p_0$.
- $\boldsymbol{X}_d^\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal butterfly if and only if* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s \neq 0$ $c_1 \sinh s - c_5 \cosh s \neq 0$, $\alpha_d^\ell = 0$ $-c_3(\alpha_d^\ell)_s + (\alpha_d^\ell)' = 0$ *and* $c_3^2(\alpha_d^\ell)_{ss} - 2c_3(\alpha_d^\ell)_s' - c_3'(\alpha_d^\ell)_s + (\alpha_d^\ell)'' \neq 0$ *holds at* $p_0$.
- $\boldsymbol{X}_d^\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal lips (resp. cuspidal beaks)* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s = 0$, $-c_4' + c_1' \cosh s - c_5' \sinh s = 0$ *and* $\det H_d^\ell > 0$ *(resp.* $\det H_d^\ell < 0$ *and* $-c_3(\alpha_d^\ell)_s + (\alpha_d^\ell)' \neq 0$*) hold at* $p_0$*, where*

$$H_d^\ell = \begin{pmatrix} c_1 \cosh s - c_5 \sinh s & c_1' \sinh s - c_5' \cosh s \\ c_1' \sinh s - c_5' \cosh s & -c_4'' + c_1'' \cosh s - c_5'' \sinh s \end{pmatrix}.$$

- $\boldsymbol{X}_d^\ell$ *at* $p_0$ *is* $\mathcal{A}$*-equivalent to the cuspidal cross cap if and only if* $c_4 = 0$, $\alpha_d^\ell(c_1 \sinh s - c_5 \cosh s) \neq 0$ *and* $c_4' \neq 0$ *hold at* $p_0$.

We now give proofs of these theorems.

*Proof of Theorem 8.9.* Let $p_0 = (s_0, t_0)$ be a singular point of $\boldsymbol{X}_\ell^\ell$. Since

$$\begin{aligned} \left(\boldsymbol{X}_\ell^\ell\right)_s &= 2s\boldsymbol{a}_0 + 2\boldsymbol{a}_1 + 2s\boldsymbol{a}_2 \\ \left(\boldsymbol{X}_\ell^\ell\right)' &= 2sc_1\boldsymbol{a}_0 + 2c_1\boldsymbol{a}_1 + 2c_1\boldsymbol{a}_2 + \big(c_3(s^2+1) + 2sc_5 + c_6(s^2-1)\big)\boldsymbol{a}_3, \end{aligned}$$

we have $S(\boldsymbol{X}_\ell^\ell) = \{(s,t) \mid c_3(t)(s^2+1) + 2sc_5(t) + c_6(t)(s^2-1) = 0\}$. Furthermore, an isotropic map $(\boldsymbol{X}_\ell^\ell, \boldsymbol{\ell}) : U \to \Delta_4$ is a Legendrian immersion if and only if $c_{36} \neq 0$ on $S(\boldsymbol{X}_\ell^\ell)$.

In this case $\boldsymbol{X}_\ell^\ell$ is a front near $p_0$. Since $\boldsymbol{a}_0$ and $\boldsymbol{a}_2$ are linearly independent to $TLC^*$, we can choose the discriminant function $\lambda$ as

$$\lambda = \det\left(\left(\boldsymbol{X}_\ell^\ell\right)_s, \left(\boldsymbol{X}_\ell^\ell\right)', \boldsymbol{a}_0, \boldsymbol{a}_2\right) = -2\big(c_3(t)(s^2+1) + 2sc_5(t) + c_6(t)(s^2-1)\big).$$

Since the kernel direction of $d\boldsymbol{X}_\ell^\ell$ on singular set is $\eta = -c_1\partial s + \partial t$ and we can take a transversal vector field $\partial s$, we have

$$\begin{aligned}
\eta\lambda &= -2c_1(c_5 + sc_{36}) + c_3'(s^2+1) + 2sc_5' + c_6'(s^2-1)\\
\eta\eta\lambda &= -c_1\big(-2c_1c_{36} + 2(c_5 + sc_{36}')\big)\\
&\qquad +\big(-2c_1(c_5+sc_{36}) + c_3'(s^2+1) + 2sc_5' + c_6'(s^2-1)\big)'\\
\operatorname{Hess}\lambda &= \begin{pmatrix} 2c_{36} & 2c_5' + 2sc_{36}'\\ 2c_5' + 2sc_{36}' & c_3''(s^2+1) + 2sc_5'' + c_6''(s^2-1)\end{pmatrix}
\end{aligned}$$

Hence we have all assertions of Theorem 8.9 except the case for the condition for the cuspidal cross caps. We give the proof of the condition for the cuspidal cross cap as follows: Let us define a lift $\omega : U \to T^*LC^*$ by

$$\omega_p(v) = \langle v, \boldsymbol{\ell}(p)\rangle\,,\ v \in T_{X_\ell^\ell(p)}LC^*,\ p \in U.$$

Then $\omega$ does not have intersection with the zero section. Since $(\pi\circ\omega)_*(z) = d\boldsymbol{X}_\ell^\ell(z)$ for any vector $z \in T_pU$, we have $\left\langle \boldsymbol{\ell}, d\boldsymbol{X}_\ell^\ell\right\rangle = \theta_4(\boldsymbol{a}_3, \boldsymbol{X}_\ell^\ell) = 0$. Thus we have $(\pi\circ\omega)_*(T_pU) \subset \ker\omega_p$. This means that $\omega$ is the admissible lift of $\boldsymbol{X}_\ell^\ell$. Under the assumption that $c_{36}(t_0) = 0$, $\lambda_s(p_0) \neq 0$ if and only if $c_5(t_0) \neq 0$. Then $S(\boldsymbol{X}_\ell^\ell)$ can be parameterized as $(s(t), t)$ for some function $s(t)$ near $p_0$. Putting $\xi(t) = \boldsymbol{a}_3(t)$, then $\xi$ is a non-zero vector field along $\boldsymbol{X}_\ell^\ell|_{S(X_\ell^\ell)}$. Since $\left\langle \xi, \boldsymbol{X}_\ell^\ell\right\rangle = 0$, the vector field $\xi$ satisfies the conditions of Theorem 8.5. Therefore the function $\psi_{X_\ell^\ell}(t)$ is equal to $\langle\eta\xi, \boldsymbol{\ell}\rangle(t) = -c_{36}(t)$. Since $\alpha_\ell^\ell(p_0) = \eta\lambda(p_0)$ holds, $\alpha_\ell^\ell(p_0) \neq 0$ means that $\eta$ is transversal to $S(X_\ell^\ell)$ at $p_0$. By Theorem 8.5, if $c_{36} = 0$, $\alpha_\ell^\ell c_5 \neq 0$ and $c_{36}' \neq 0$ hold at $p_0$, then $\boldsymbol{X}_\ell^\ell$ at $p_0$ is $\mathcal{A}$-equivalent to the cuspidal cross cap. On the other hand, if $\lambda_s(p_0) = 0$ and $\lambda'(p_0) \neq 0$, then $\boldsymbol{X}_\ell^\ell$ at $p_0$ is not $\mathcal{A}$-equivalent to the cuspidal cross cap. This completes the proof of Theorem 8.9. □

We can give the proofs of Theorems 8.6, 8.8, 8.10, 8.11, 8.13 and 8.14 by the same arguments as those of the above proof. We only state the fundamental data here, and omit the detailed proof. The discriminant function $\lambda$, null vector field $\eta$, the one-form $\omega$ and the vector field $\xi$ for each dual surfaces are shown in the Table 1.

# 9 Dualities of singularities

Comparing Theorems 8.6, 8.8 and 8.9, when singular point is always $(0, t)$, with Theorems 8.10, 8.11, 8.13 and 8.14 we observe a certain duality between the swallowtails and the cuspidal cross caps. It can be summerized as follows.

**Remark 9.1.** The conditions that singular set is equal to the curve $(0, t)$ is

| Surface | $\lambda$ | $\eta$ | $\omega$ | $\xi$ |
|---|---|---|---|---|
| $\boldsymbol{X}_\ell^h$ | $2c_3 + 2sc_5 + s^2c_{36}$ | $(-c_1, 1)$ | $\langle *, \boldsymbol{\ell}\rangle$ | $\boldsymbol{a}_3$ |
| $\boldsymbol{X}_\ell^d$ | $-2c_6 - 2sc_5 + s^2c_{36}$ | $(c_1, 1)$ | $\langle *, \boldsymbol{\ell}\rangle$ | $\boldsymbol{a}_3$ |
| $\boldsymbol{X}_h^d$ | $c_4 \cos s + c_5 \sin s$ | $(-c_6, 1)$ | $\langle *, \boldsymbol{a}_0\rangle$ | $\boldsymbol{a}_1$ |
| $\boldsymbol{X}_h^\ell$ | $c_1 - c_4 \cos s - c_5 \sin s$ | $(-c_6, 1)$ | $\langle *, \boldsymbol{a}_0\rangle$ | $\boldsymbol{a}_1$ |
| $\boldsymbol{X}_d^h$ | $c_1 \cosh s - c_5 \sinh s$ | $(-c_3, 1)$ | $\langle *, \boldsymbol{a}_2\rangle$ | $\boldsymbol{a}_1$ |
| $\boldsymbol{X}_d^\ell$ | $-c_4 + c_1 \cosh s - c_5 \sinh s$ | $(-c_3, 1)$ | $\langle *, \boldsymbol{a}_2\rangle$ | $\boldsymbol{a}_1$ |

Table 1: Fundamental data to recognize the conditions of singularities of dual surfaces

- $c_3 \equiv 0$ for $\boldsymbol{X}_\ell^h$,
- $c_6 \equiv 0$ for $\boldsymbol{X}_\ell^d$,
- $c_3 - c_6 \equiv 0$ for $\boldsymbol{X}_\ell^\ell$
- $c_4 \equiv 0$ for $\boldsymbol{X}_h^d$
- $c_1 - c_4 \equiv 0$ for $\boldsymbol{X}_h^\ell$,
- $c_1 \equiv 0$ for $\boldsymbol{X}_d^h$,
- $c_1 - c_4 \equiv 0$ for $\boldsymbol{X}_d^\ell$.

Moreover, if $c_2 \equiv c_3 \equiv c_1 - c_4 \equiv 0$, then $\boldsymbol{X}_\ell^h$ at $(0, t_0)$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_6 = 0$ and $c_1 c_6' \neq 0$ at $t_0$. This condition is the same as the condition that $\boldsymbol{X}_h^\ell$ at $(0, t_0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap. Furthermore, $\boldsymbol{X}_\ell^h$ at $(0, t_0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_1 = 0$ and $c_6 c_1' \neq 0$ at $t_0$. This condition is the same as the condition that $\boldsymbol{X}_h^\ell$ at $(0, t_0)$ is $\mathcal{A}$-equivalent to the swallowtail. Like as these arguments, we have the same type condition of singular points for dual surfaces when the singular set is equal to $(0, t)$.

We can summerize this situation on the Table 2. In the table, $S$ means the singular set.

We can observe there are some dual relations of conditions for singularities of the swallowtail and the cuspidal cross cap on each dual points of surfaces. Furthermore, the condition of holding the duality and that the singular set is $\{(0, t)\}$ are the same between $\boldsymbol{X}_\ell^h$ and $\boldsymbol{X}_h^\ell$ (resp. between $\boldsymbol{X}_\ell^d$ and $\boldsymbol{X}_d^\ell$):

$$\begin{array}{ccc} \boldsymbol{X}_\ell^h & \xrightarrow{\text{taking singular value}} & \boldsymbol{a}_0 \\ {\scriptstyle \Delta_2\text{-dual}}\uparrow & & \downarrow{\scriptstyle \Delta_2\text{-dual}} \\ \boldsymbol{\ell} & \xleftarrow{\text{taking singular value}} & \boldsymbol{X}_h^\ell \end{array} \quad \text{and} \quad \begin{array}{ccc} \boldsymbol{X}_\ell^d & \xrightarrow{\text{taking singular value}} & \boldsymbol{a}_2 \\ {\scriptstyle \Delta_3\text{-dual}}\uparrow & & \downarrow{\scriptstyle \Delta_3\text{-dual}} \\ \boldsymbol{\ell} & \xleftarrow{\text{taking singular value}} & \boldsymbol{X}_d^\ell. \end{array}$$

Like as the remark, a duality between the swallowtail and the cuspidal cross cap have been pointed out in many researches, for example, [33, 13, 22]. In this section, we give an interpretation for this duality. Firstly, we prove the following lemma.

| | duality | $S=\{(0,t)\}$ | cuspidal edge | swallowtail | cuspidal cross cap |
|---|---|---|---|---|---|
| $\boldsymbol{X}^h_\ell$ | $c_2\equiv 0$<br>$c_1-c_4\equiv 0$ | $c_3\equiv 0$ | $c_6\neq 0$,<br>$c_1c_5\neq 0$ | $c_6c_5\neq 0$,<br>$c_1=0,\ c_1'\neq 0$, | $c_1c_5\neq 0$,<br>$c_6=0,\ c_6'\neq 0$, |
| $\boldsymbol{X}^d_\ell$ | $c_2\equiv 0$<br>$c_1-c_4\equiv 0$ | $c_6\equiv 0$ | $c_3\neq 0$,<br>$c_1c_5\neq 0$ | $c_3c_5\neq 0$,<br>$c_1=0,\ c_1'\neq 0$ | $c_1c_5\neq 0$,<br>$c_3=0,\ c_3'\neq 0$, |
| $\boldsymbol{X}^\ell_\ell$ | $c_2\equiv 0$<br>$c_1-c_4\equiv 0$ | $c_3-c_6\equiv 0$ | $c_{36}\neq 0$,<br>$c_1c_5\neq 0$ | $c_{36}c_5\neq 0$,<br>$c_1=0,\ c_1'\neq 0$ | $c_1c_5\neq 0$,<br>$c_{36}=0,\ c_{36}'\neq 0$, |
| $\boldsymbol{X}^d_h$ | $c_2\equiv 0$<br>$c_3\equiv 0$ | $c_4\equiv 0$ | $c_1\neq 0$<br>$c_5c_6\neq 0$ | $c_1c_5\neq 0$<br>$c_6=0,\ c_6'\neq 0$ | $c_5c_6\neq 0$<br>$c_1=0,\ c_1'\neq 0$ |
| $\boldsymbol{X}^\ell_h$ | $c_2\equiv 0$<br>$c_3\equiv 0$ | $c_1-c_4\equiv 0$ | $c_1\neq 0$<br>$c_5c_6\neq 0$ | $c_1c_5\neq 0$<br>$c_6=0,\ c_6'\neq 0$ | $c_5c_6\neq 0$<br>$c_1=0,\ c_1'\neq 0$ |
| $\boldsymbol{X}^h_d$ | $c_2\equiv 0$<br>$c_6\equiv 0$ | $c_1\equiv 0$ | $c_4\neq 0$<br>$c_3c_5\neq 0$ | $c_4c_5\neq 0$<br>$c_3=0,\ c_3'\neq 0$ | $c_3c_5\neq 0$<br>$c_4=0,\ c_4'\neq 0$ |
| $\boldsymbol{X}^\ell_d$ | $c_2\equiv 0$<br>$c_6\equiv 0$ | $c_1-c_4\equiv 0$ | $c_4\neq 0$<br>$c_3c_5\neq 0$ | $c_4c_5\neq 0$<br>$c_3=0,\ c_3'\neq 0$ | $c_3c_5\neq 0$<br>$c_4=0,\ c_4'\neq 0$ |

Table 2: Dualities of condition for singularity.

**Lemma 9.2.** *Let $M_i$ $(i=1,2)$ be three dimensional manifolds and $\Delta\subset M_1\times M_2$ a five dimensional submanifold with the contact structure. Assume that the canonical projection $\pi_1:\Delta\to M_1$ is a Legendre fibrations. If an isotropic map $L_1=(f_1,\nu_1)$ and a frontal $f_2:U\to M_2$ satisfies that $p$ is a non-degenerate singular point of both $f_i$ $(i=1,2)$ and $\nu_1$ degenerates a curve such that $\nu_1=f_2\circ\sigma$, where $\sigma$ is a submersion $U\to S(f)$. If the null direction of $f_1$ does not parallel to the kernel of $\sigma$, then the following two conditions are equivalent.*

- *$L_1$ is a Legendrian immersion.*
- *The null direction of $f_2$ at $p$ is transversal to $S(f_1)$.*

*Proof.* Since $p$ is a non-degenerate singular point, $L_1$ is a Legendrian immersion if and only if the directional derivative $\eta_1\nu_1$ does not vanish. this is equivalent to the condition that $df_2(\eta_1)(\sigma)$ does not vanish. This is equivalent to that the tangential direction of $S(f_1)$ does not parallel to $\eta_2$. This is equivalent to the condition that $\eta_2$ is transversal to $S(f_1)$. This completes the proof. □

**Theorem 9.3.** *Let $p$ be a non-degenerate singular point of a frontal $f$. Then we have the following criteria of singularities by using the function $\psi_f$ defined in* (8.2).

(1) *If $\psi_f(p)\neq 0$, then $f$ at $p$ is $\mathcal{A}$-equivalent to the cuspidal edge.*

(2) *Assume that $f$ is a front. If $\psi_f(p)=0$ and $(d/dt)\psi_f(p)\neq 0$, then $f$ at $p$ is $\mathcal{A}$-equivalent to the swallowtail.*

(3) *Assume that the null direction at $p$ is transversal to $S(f)$. If $\psi_f(p)=0$ and $(d/dt)\psi_f(p)\neq 0$, then $f$ at $p$ is $\mathcal{A}$-equivalent to the cuspidal cross cap.*

*Proof.* Since the conditions are independent of the choice of coordinates, we take the coordinate system $(u, v)$ satisfying $S(f) = \{v = 0\}$. Under this conditions, $\psi_f$ is proportional to $\phi_f$, where $\phi_f$ is defined in (8.1). Firstly, we prove (1). The condition $\phi_f \neq 0$ implies that $f_u$ and $\eta\nu$ are linearly independent. Since $\nu$ points the kernel direction of $df$, this implies that $f$ to be a front. Moreover, we have $f_u \neq \mathbf{0}$, this implies that $\eta$ does not tangent to $S(f)$. By Theorem 8.1, we have (1).

Next, we assume that $f$ to be a front and $\phi_f = 0$ at $p$. Then this condition implies $f_u(p) = \mathbf{0}$, namely, $\eta$ tangents $S(f)$ at $p$. Thus we can take a function $\beta(u)$ such that $\eta(u) = \partial/\partial u + \beta(u)\partial/\partial v$, $\beta(0) = 0$. By Theorem 8.1, $f$ at $p$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $\beta'(0) \neq 0$. On the other hand, $(d/dt)\phi_f(p) \neq 0$ implies that $\det(f_{uu}, \nu, \nu_u)(p) \neq 0$. Since $f_v$, $\nu$ and $\nu_u$ are linear independent at $p$, this is equivalent to $\langle f_{uu}, f_v \rangle (p) \neq 0$. Since $\eta$ is the null vector field on the $u$-axis, $f_u + \beta(u) + f_v = \mathbf{0}$ holds on the $u$-axis. Thus $\langle f_{uu}, f_v \rangle (p) \neq 0$ implies $\beta'(0) \neq 0$. This completes the proof. The assertion (3) directly holds from Theorem 8.1. □

We can give the alternative proof of Theorem 8.11 in the special case of $c_1 - c_4 \equiv 0$ and $c_5 \neq 0$.

*Proof of Theorem* 8.11. If $c_2 \equiv c_3 \equiv c_1 - c_4 \equiv 0$, and $c_5(t_0) \neq 0$, then by Theorem 8.6 $\boldsymbol{X}^\ell_h$ at $(t_0, 0)$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_6 \neq 0$, $c_1 = 0$ and $c_1' \neq 0$ at $t_0$. Furthermore, $\boldsymbol{X}^\ell_h$ at $(t_0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_1 \neq 0$, $c_6 = 0$ and $c_6' \neq 0$ at $t_0$.

Under the assumptions $c_2 \equiv c_3 \equiv c_1 - c_4 \equiv 0$ and $c_5(t_0) \neq 0$, it holds that $S(\boldsymbol{X}^h_\ell) = S(\boldsymbol{X}^\ell_h) = \{(t, 0)\}$ near $(t_0, 0)$. Hence we can apply Lemma 9.2 and Theorem 9.3. This means that the conditions for the swallowtail and the cuspidal cross cap of the dual surface are obtained by only interchanging the conditions for the cuspidal cross cap and the swallowtail of the original surface. Thus we have that $\boldsymbol{X}^\ell_h$ at $(t_0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_6 \neq 0$, $c_1 = 0$ and $c_1' \neq 0$ at $t_0$. Furthermore, $\boldsymbol{X}^\ell_h$ at $(t_0, 0)$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_1 \neq 0$, $c_6 = 0$ and $c_6' \neq 0$ at $t_0$. This is the same as Theorem 8.11 under the assumption $c_1 - c_4 \equiv 0$. □

# A A criterion for the cuspidal butterfly

In this section, we give a proof of Theorem 8.2. The main tool for the proof is the notion of generating families. Let $G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ which we call an *unfolding of* $g(q) = G(q, \mathbf{0})$. We say that $G$ is a *Morse family of hypersurfaces* if the mapping

$$\Delta^* G = \left( G, \frac{\partial G}{\partial q_1}, \ldots, \frac{\partial G}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(q, x) = (q_1, \ldots, q_k, x_1, \ldots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$. In this case we have a smooth $(n-1)$-dimensional submanifold

$$\Sigma_*(G) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \;\middle|\; G(q, x) = \frac{\partial G}{\partial q_1}(q, x) = \cdots = \frac{\partial G}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\Phi_G : (\Sigma_*(G), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\Phi_G(q,x) = \left( x, \left[ \frac{\partial G}{\partial x_1}(q,x) : \cdots : \frac{\partial G}{\partial x_n}(q,x) \right] \right)$$

is a Legendrian immersion germ. The fundamental result of Arnol'd-Zakalyukin [2, 34] assets that all Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method. We call $G$ a *generating family* of $\Phi_G(\Sigma_*(G))$. Therefore the wave front of $\Phi_G(\Sigma_*(G))$ is

$$W(\Phi_G) = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists q \in \mathbb{R}^k \text{ such that } G(q,x) = \frac{\partial G}{\partial q_1}(q,x) = \cdots = \frac{\partial G}{\partial q_k}(q,x) = 0 \right\}.$$

We also write $\mathcal{D}_G = W(\Phi_G)$ and call it the *discriminant set* of $G$.

We now introduce an equivalence relation among Legendrian submanifold germs. Let $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L',p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian submanifold germs. Then we say that $i$ and $i'$ are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$ such that $H$ preserves fibers of $\pi$ and that $H(L) = L'$.

Since the Legendrian lift $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs[35]:

**Proposition A.1.** *Let $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L',p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that the representative of both the regular sets of the projections $\pi \circ i$ and $\pi \circ i'$ are dense. Then $i$ and $i'$ are Legendrian equivalent if and only if wave front sets $W(i)$ and $W(i')$ are diffeomorphic as set germs.*

The assumption in the above proposition is a generic condition for $i$ and $i'$.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{ h \in \mathcal{E}_n \mid h(\mathbf{0}) = 0 \}$. Let $G_1, G_2 : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that $G_1$ and $G_2$ are *$P$-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(q,x) = (\psi_1(q,x), \psi_2(x))$ for $(q,x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle G_1 \rangle_{\mathcal{E}_{k+n}}) = \langle G_2 \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$ .

Let $\bar{G} : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that $\bar{G}$ is a *$\mathcal{K}$-versal unfolding of $g = \bar{G}|\mathbb{R}^k \times \{\mathbf{0}\}$* if for any unfolding $G : (\mathbb{R}^k \times \mathbb{R}^m, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ of $g$ (i.e., $G(q, \mathbf{0}) = g(q)$), there exists a map germ $\phi : (\mathbb{R}^m, \mathbf{0}) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ such that $\phi^*\bar{G}$ and $G$ are $P$-$\mathcal{K}$-equivalent, where $\phi^*\bar{G}(q,u) = \bar{G}(q, \phi(u))$. For an unfolding $G(t,x)$ of a function $g(t)$ of one-variable, we have the following useful criterion on the $\mathcal{K}$-versal unfoldings in (cf., [4], 6.10): We say that $g$ has an *$A_r$-singularity* at $t_0$ if $g^{(p)}(t_0) = 0$ for all $1 \leq p \leq r$, and $g^{(r+1)}(t_0) \neq 0$. We have the following lemma

**Lemma A.2.** *Let $G$ be an unfolding of $g$ and $g(t)$ has an $A_r$-singularity ($r \geq 1$) at $t_0$. We denote the $(r-1)$-jet of the partial derivative $\partial G/\partial x_i$ at $t_0$ by*

$$j^{(r-1)} \left( \frac{\partial G}{\partial x_i}(t, x_0) \right)(t_0) = \sum_{j=0}^{r-1} \alpha_{ji} (t - t_0)^j$$

*for $i = 1, \ldots, n$. Then $G$ is a $\mathcal{K}$-versal unfolding if and only if the $r \times n$ matrix of coefficients $(\alpha_{ji})$ has rank $r$ ($r \leq n$).*

It follows from the above lemma that the function germ defined by

$$t^{r+1} + x_1 t^{r-1} + x_2 t^{r-2} + \cdots + x_{r-1} t + x_r$$

is a $\mathcal{K}$-versal unfolding of $g(t) = t^{r+1}$. One of the main results in the theory of Legendrian singularities is the following theorem:

**Theorem A.3.** *Let $G_1, G_2 : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then $\Phi_{G_1}$ and $\Phi_{G_2}$ are Legendrian equivalent if and only if $G_1$ and $G_2$ are $P$-$\mathcal{K}$-equivalent.*

As a corollary of Proposition A.1 and Theorem A.3, we have the following proposition.

**Proposition A.4.** *Let $G_1, G_2 : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Suppose that both the regular sets of the representative of projections $\pi \circ \Phi_{G_1}, \pi \circ \Phi_{G_2}$ are dense. Then $(W(\Phi_{G_1}), 0)$ and $(W(\Phi_{G_2}), 0)$ are diffeomorphic as set germs if and only if $G_1$ and $G_2$ are $P$-$\mathcal{K}$-equivalent.*

The following Lemma roles the key of the proof for the criteria. Two function germs $g_i : (\mathbb{R}, \mathbf{0}) \to (\mathbb{R}, 0)$ $(i = 1, 2)$ are $\mathcal{R}$-equivalent if there exists a diffeomorphism germ $\alpha : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\alpha \circ g_1 = g_2$ holds.

**Lemma A.5.** *Let $g : ((\mathbb{R}; t), \mathbf{0}) \to (\mathbb{R}, 0)$ be a function germ such that $\mathcal{R}$-equivalent to $t^5$. If an unfolding $G : ((\mathbb{R} \times \mathbb{R}^3; t, x, y, z), \mathbf{0}) \to (\mathbb{R}, 0)$ of $g$ is a Morse family and a function $\bar{G}(t, x, y, z, w) = G(t, x, y, z) + wt^3$ is a $\mathcal{K}$-versal unfolding of $g$, then $G(t, x, y, z)$ is $P$-$\mathcal{K}$-equivalent to $t^5 + xt^2 + yt + z$.*

*Proof.* Since the condition does not depend on the parameter transformation of $t$, we can assume that $g(t) = t^5$. Moreover, since the map $t^5 + wt^3 + xt^2 + yt + z$ is the versal unfolding of $t^5$, there is a diffeomorphism $(\phi_1, \phi_2, \phi_3, \phi_4) : \mathbb{R}^4 \to \mathbb{R}^4$ such that $G$ is $P$-$\mathcal{K}$-equivalent to

$$t^5 + \phi_1(x, y, z, w)t^3 + \phi_2(x, y, z, w)t^2 + \phi_3(x, y, z, w)t + \phi_4(x, y, z, w).$$

Since $\overline{G} = G + wt^3$ is a $\mathcal{K}$-versal unfolding, and the condition of lemma only depend on the $P$-$\mathcal{K}$-equivalent class, we can rechoose $(x, y, z)$ such that $\overline{G}$ is $P$-$\mathcal{K}$-equivalent to

$$t^5 + \phi_1(x, y, z, w)t^3 + xt^2 + yt + z.$$

Furthermore, since $\overline{G}$ is a versal unfolding and $\partial\phi_1/\partial w(\mathbf{0}) = 0$, we rechoose $w$ such that $\overline{G}$ is $P$-$\mathcal{K}$-equivalent to

$$t^5 + (w - h(x, y, z))t^3 + xt^2 + yt + z$$

for some function $h$. Summerizing up these argument, we can assume that $\overline{G}$ is

$$\overline{G}_h(t, x, y, z) := t^5 + (w - h(x, y, z))t^3 + xt^2 + yt + z.$$

We have the following Zakalyukin's lemma

**Lemma A.6.** [35, Theorem 1.4] *Let $\mathcal{V} : (\mathbb{R} \times \mathbb{R}^4, \mathbf{0}) \to (\mathbb{R}, 0)$ be a $\mathcal{K}$-versal unfolding of the form*

$$\mathcal{V}(t, x, y, z, w) = t^5 + wt^3 + xt^2 + yt + z \tag{A.1}$$

*and $\sigma : (\mathbb{R}^4, \mathbf{0}) \to (\mathbb{R}, 0)$ be a function germ with $(x, y, z, w)$-variables such that $\partial\sigma/\partial w(\mathbf{0}) \neq 0$. Then there exists a diffeomorphism germ $\Theta : (\mathbb{R}^4, \mathbf{0}) \to (\mathbb{R}^4, \mathbf{0})$ such that*

$$\Theta(D_{\mathcal{V}}) = D_{\mathcal{V}} \text{ and } \sigma \circ \Theta(x, y, z, w) = w.$$

Let us continue to prove of Lemma A.5. We apply Lemma A.6 to $\mathcal{V}$ of (A.1) and $w - h(x,y,z)$. Then there exists a diffeomorphism germ $\Theta$ such that

$$\Theta(D_{\mathcal{V}}) = D_{\mathcal{V}} \text{ and } (w - h(x,y,z)) \circ \Theta(x,y,z,w) = w.$$

We define a diffeomorphism germ

$$\Psi(x,y,z,w) = (x,y,z,w - h(x,y,z)),$$

then it holds that $\Psi^*\mathcal{V} = \overline{G}_h$. Define a new diffeomorphism germ $\tilde{\Theta}$ by $\tilde{\Theta} = \Psi \circ \Theta$ then we have

$$\tilde{\Theta}(D_{\mathcal{V}}) = \Psi \circ \Theta(D_{\mathcal{V}}) = \Psi(D_{\mathcal{V}}) = D_{\overline{G}_h}.$$

Hence $D_{\mathcal{V}}$ and $D_{\overline{G}}$ are diffeomorphic. On the other hand, let us define $\pi : (\mathbb{R}^4, \mathbf{0}) \to (\mathbb{R}, 0)$ by $\pi(x,y,z,w) = w$. Since

$$\pi \circ \tilde{\Theta}(x,y,z,w) = \pi \circ \Psi \circ \Theta = (w - h(x,y,z)) \circ \Theta = w,$$

we have $\pi \circ \tilde{\Theta} = \pi$. Since the set of regular points of $D_{\mathcal{V}}$ is dense, by the Zakalyukin theorem ([35], see also [28, Appendix]), there exist a diffeomorphism germ $\Xi : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R} \times \mathbb{R}^4$ of the form

$$\begin{aligned}&\Xi(t,x,y,z,w)\\ &= \big(\xi(t,x,y,z,w), \zeta_1(x,y,z,w), \zeta_2(x,y,z,w), \zeta_3(x,y,z,w), \zeta_4(w)\big)\end{aligned}$$

such that $\Xi^*(\langle \mathcal{V} \rangle_{\mathcal{E}_{1+4}}) = \langle \overline{G}_h \rangle_{\mathcal{E}_{1+4}}$.

If we restrict the above map to $w = 0$, we complete the proof Lemma A.2. □

Using these results, we give the criterion of the $A_4$-singularity of wave fronts.

Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a front and $\nu$ be the normal vector field of $f$. Let $\mathbf{0}$ be a non-degenerate singular point of $f$. Needless to say, the conditions of Theorem 8.2 do not depend on the choice of coordinates and choice of $\nu$. One can prove the following lemma.

**Lemma A.7.** *One can choose the coordinate systems $(u,v)$ of $(\mathbb{R}^2, \mathbf{0})$ and $(X_1, X_2, Z)$ of $(\mathbb{R}^3, \mathbf{0})$ satisfying*

- $\eta \equiv \partial v$.
- $f(u,v) = (f_1(u,v), f_2(u,v), u)$ *and* $(f_1)_u(\mathbf{0}) = (f_2)_u(\mathbf{0}) = 0$.
- $\nu(\mathbf{0}) = (1,0,0)$.

Under this coordinate system, we prove that if $f : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies $\eta\lambda = \eta\eta\lambda = 0$ and $\eta\eta\eta\lambda \neq 0$ at $\mathbf{0}$ then $f$ at $\mathbf{0}$ is $\mathcal{A}$-equivalent to the cuspidal butterfly.

*Proof of Theorem 8.2.* Let us fix a small number $u$ and consider a family of plane curves $\Gamma^u(v) = \Gamma(u,v) = (f_1(u,v), f_2(u,v), u)$ in the plane $\Pi_u = \{(X_1, X_2, Z) | Z = u\}$ and show that these are fronts near $\mathbf{0}$. Denote $\nu = (\nu_1, \nu_2, \nu_3)$ and put

$$[N^u(v)] = [(\nu_1(u,v), \nu_2(u,v), 0)]\,.$$

Then $[N^u(v)]$ is well-defined near $\mathbf{0}$. We put

$$\gamma(u,v) = (f_1(u,v), f_2(u,v)) \text{ and } n(u,v) = (\nu_1(u,v), \nu_2(u,v)).$$

Then, since $\langle \gamma'(u,v), n(u,v) \rangle \equiv 0$, $(\gamma, [n])$ is an isotropic map for all $u$, where $'$ denotes $\partial/\partial v$ and $\langle \cdot, \cdot \rangle$ is the canonical inner product of $\mathbb{R}^3$. Since $\nu_3'(\mathbf{0})$, we have $n'(\mathbf{0}) \neq \mathbf{0}$. This implies that for each $u$ near 0, $(\gamma, [n])$ is a Legendrian immersion germ.

We define two functions $\Psi : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$\Psi(v, X_1, X_2, Z) = n_1(Z,v)(X_1 - f_1(Z,v)) + n_2(Z,v)(X_2 - f_2(Z,v)), \quad \psi(v) = \Psi(v,0,0,0).$$

Then we have $D_\Psi = f(U)$. Hence by Lemma A.6 and the arguments in the above, it is sufficient to prove that $\psi$ has an $A_4$-singularity and $\Psi$ satisfies the conditions of Lemma A.5. In the following context, we put $Z = u$.

**Lemma A.8.** *It holds that* $f'(\mathbf{0}) = f'(\mathbf{0}) = f'''(\mathbf{0}) = \mathbf{0}$, $f''''(\mathbf{0}) \neq \mathbf{0}$ *and* $(\overline{f})'(\mathbf{0}) = (\overline{f})''(\mathbf{0}) = (\overline{f})'''(\mathbf{0}) = \mathbf{0}$, $(\overline{f})''''(\mathbf{0}) \neq \mathbf{0}$.

*Proof.* Since $\partial_v$ is the null vector field, so that we have $f'(\mathbf{0}) = \mathbf{0}$ and $S(f) = \{f_v = \mathbf{0}\}$. By $\eta\lambda = 0$, since $(\partial_v =)\eta_0 \in T_0 S(f)$, it holds that $f''(\mathbf{0}) = \mathbf{0}$. Furthermore, by $\lambda''(\mathbf{0}) = 0$ and $f'(\mathbf{0}) = f''(\mathbf{0}) = \mathbf{0}$, we have $\lambda''(\mathbf{0}) = \det(f_u, f''', \nu)(\mathbf{0})$. Hence it holds that $f'''(\mathbf{0}) \in \mathrm{span}\,\{f_u(\mathbf{0}), \nu(\mathbf{0})\}$.

On the other hand, we have $\langle f_u, f''' \rangle(\mathbf{0}) = \langle (0,0,1), (*,*,0) \rangle = 0$. Differentiating $\langle \nu, f' \rangle = 0$, we have $\langle \nu, f'' \rangle = \langle \nu, f' \rangle' - \langle \nu', f' \rangle$ and $\langle \nu, f''' \rangle = \langle \nu, f'' \rangle' - \langle \nu', f'' \rangle$. Hence $\langle \nu, f'' \rangle \equiv 0$ holds on $S(f)$. Since $\eta_0 \in T_0 S(f)$, it holds that $\langle \nu, f'' \rangle'(\mathbf{0}) = 0$ and $\langle \nu, f''' \rangle(\mathbf{0}) = 0$. Thus we have $f'''(\mathbf{0}) = 0$.

Since $\lambda'''(\mathbf{0}) \neq 0$ and $f'(\mathbf{0}) = f''(\mathbf{0}) = f'''(\mathbf{0}) = \mathbf{0}$, it holds that $0 \neq \lambda'''(\mathbf{0}) = \det(f_u, f'''', \nu)(\mathbf{0})$. In particular, $f''''(\mathbf{0}) \neq \mathbf{0}$ holds. □

To prove Theorem 8.2, firstly we show that $\psi$ has the $A_4$-singularity at $\mathbf{0}$. Differentiating $\langle (\overline{f})', n \rangle \equiv 0$ and by Lemma A.8, we have $\langle (\overline{f})'''', n \rangle(\mathbf{0}) = 0$ and $4\langle (\overline{f})'''', n' \rangle(\mathbf{0}) + \langle (\overline{f})''''', n \rangle(\mathbf{0}) = 0$.

By these formulae and Lemma A.8, we have $\psi'(0) = \psi''(0) = \psi'''(0) = 0$, $\psi''''(0) = -\langle n, (\overline{f})'''' \rangle(\mathbf{0}) = 0$ and $\psi'''''(0) = -\langle n', (\overline{f})'''' \rangle(\mathbf{0})$.

On the other hand, since $n, n'$ is linearly independent at $\mathbf{0}$ and $\langle n, (\overline{f})'''' \rangle(\mathbf{0}) = 0$, we have

$$\langle n', (\overline{f})'''' \rangle(\mathbf{0}) \neq 0 \Longleftrightarrow (\overline{f})''''(\mathbf{0}) \neq \mathbf{0} \Longleftrightarrow f''''(\mathbf{0}) \neq \mathbf{0}.$$

Hence $\psi$ has the $A_4$ singularity at $\mathbf{0}$.

Next, we show that $(\Psi, \Psi', \Psi'')$ is non-singular. If this is satisfied, $\Psi$ satisfies the condition of Lemma A.5 namely, $\Psi$ is a Morse family and $\Psi(v, X_1, X_2, u) + wv^3$ is a $\mathcal{K}$-versal unfolding of $\psi$. Remark that the discriminant set of an unfolding $t^5 + xt^2 + yt + z$ of a function $t^5$ is diffeomorphic to the image of the canonical cuspidal butterfly $(u,v) \mapsto (u, 5v^4 + 2uv, 4v^5 + uv^2 - u^2)$ at $\mathbf{0}$ as set germs. Therefore by Proposition A.4 and Lemma A.5, we can show that $f$ at $\mathbf{0}$ is $\mathcal{A}$-equivalent to the cuspidal butterfly.

Since $\Psi_X(\mathbf{0}) = -n_1(\mathbf{0}), \Psi_Y(\mathbf{0}) = -n_2(\mathbf{0})$ and $\Psi_u = \sum_{i=1,2} \langle (n_i)_u, X_i - f_i \rangle - \langle n_i, (f_i)_u \rangle$, it holds that $\Psi_u(\mathbf{0}) = 0$. By a direct calculation, we have $\Psi'_X(\mathbf{0}) = -n_1'(\mathbf{0}), \Psi'_Y(\mathbf{0}) = -n_2'(\mathbf{0})$

and $\Psi'_u = \sum_{i=1,2} \langle (n_i)'_u, X_i - f_i \rangle - \langle n_i, f' \rangle - \langle n'_i, (f_i)_u \rangle - \langle n_i, (f_i)'_u \rangle$. Since $\langle n_i, (f_i)'_u \rangle = \langle n, (\overline{f})' \rangle_u - \langle n_u, (\overline{f})' \rangle = 0$ holds at $\mathbf{0}$, we have $\Psi'_u(\mathbf{0}) = 0$.

Thus it is sufficient to prove that the matrix

$$\begin{pmatrix} \partial(\Psi,\ \Psi',\ \Psi'')/\partial t \\ \partial(\Psi,\ \Psi',\ \Psi'')/\partial X_1 \\ \partial(\Psi,\ \Psi',\ \Psi'')/\partial X_2 \\ \partial(\Psi,\ \Psi',\ \Psi'')/\partial u \end{pmatrix}(\mathbf{0}) = \begin{pmatrix} 0 & 0 & 0 \\ n_1 & n'_1 & * \\ n_2 & n'_2 & * \\ 0 & 0 & \Psi''_u \end{pmatrix}(\mathbf{0})$$

is of full rank. Hence we show that $\Psi''_u(\mathbf{0}) \neq 0$.

Differentiating $\langle n, (\overline{f})' \rangle \equiv 0$ by $u$ and $v$, and by Lemma A.8, we have

$$\langle n, (\overline{f})''_u \rangle (\mathbf{0}) = \langle n', (\overline{f})'_u \rangle (\mathbf{0}). \tag{A.2}$$

Differentiating $\Psi$ by $u$ and $v$ two times, and by Lemma A.8 and (A.2), we have

$$\Psi''_u(\mathbf{0}) = -\langle n', (\overline{f})'_u \rangle (\mathbf{0}).$$

On the other hand, since $\langle n, (\overline{f})'_u \rangle (\mathbf{0}) = 0$,

$$\langle n', (\overline{f})'_u \rangle (\mathbf{0}) \neq 0 \Longleftrightarrow (\overline{f})'_u(\mathbf{0}) \neq 0 \Longleftrightarrow f'_u(\mathbf{0}) \neq 0.$$

holds. By $\lambda_u(\mathbf{0}) \neq 0$ and $f_v(\mathbf{0}) = \mathbf{0}$, we have

$$0 \neq \lambda_u(\mathbf{0}) = \det(f_u, f'_u, \nu)(\mathbf{0}).$$

In particular, $f'_u(\mathbf{0}) \neq \mathbf{0}$ holds. This implies the desired result.

The converse pert of the theorem is obvious since the conditions and assertions of Theorem 8.2 are independent of the choice of coordinates and the choice of $\nu$, and the canonical $A_4$ singularity satisfies the condition of theorem. □

Remark that since $\mathbf{0}$ is a non-degenerate singular point, we have the parameterization $\gamma(t)$ of $S(f)$. Take the null vector field on $\gamma$ as $\eta(t)$. Define a function of $t$ by

$$\mu(t) = \det(\gamma'(t),\ \eta(t)).$$

One can easily show that $\mu(0) = \mu'(0) = 0$ and $\mu''(0) \neq 0$ and $\eta\lambda(\mathbf{0}) = \eta\eta\lambda(\mathbf{0}) = 0$ and $\eta\eta\eta\lambda(\mathbf{0}) \neq 0$ are equivalent, as a corollary, the following assertion holds.

**Corollary A.9.** *A front germ $f$ at $\mathbf{0}$ is $\mathcal{A}$-equivalent to the $A_4$-singularity if and only if $\mathbf{0}$ is a non-degenerate singular point and $\mu(0) = \mu'(0) = 0$ but $\mu''(0) \neq 0$ holds.*

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