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Combinatorial computation of the motivic Poincaré series

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Abstract

We give an explicit algorithm computing the motivic generalization of the Poincaré series of a plane curve singularity introduced by A. Campillo, F. Delgado and S. Gusein-Zade. It is done in terms of the embedded resolution. The result is a rational function depending of the parameter q, at $q = 1$ it coincides with the Alexander polynomial of the corresponding link. For irreducible curves we relate this invariant to the Heegaard-Floer knot homology constructed by P. Ozsváth and Z. Szabó. Many explicit examples are considered.

Contents

1 Introduction

In the series of articles (e.g. [3],[4]) A. Campillo, F. Delgado and S. Gusein-Zade proved that the Alexander polynomial of the link of the plane curve singularity is related to the generating function arising in the purely algebraic setup.

Let $C = \cup_{i=1}^{r} C_i$ be a germ of a plane curve,

$$\gamma_i : (\mathbb{C}, 0) \to (C_i, 0)$$

are the uniformizations of its components. If $f \in \mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$ is a germ of a function on $(\mathbb{C}^2, 0)$, we define

$$v_i(f) = Ord_0 f(\gamma_i(t)),$$

and the Poincaré series of the curve C is defined ([4]) as the integral with respect to the Euler characteristic

$$P^C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdot \ldots \cdot t_r{}^{v_r} d\chi, \tag{1}$$

where $\mathbb{P}\mathcal{O}$ denotes the projectivization of $\mathcal{O}$ as a vector space. For example, if C is irreducible, we can define the decreasing filtration

$$\mathcal{O} \supset J_1 \supset J_2 \supset \ldots, \qquad J_n = \{f \in \mathcal{O} | v_1(f) \geq n\}, \tag{2}$$

and

$$P^C(t) = \sum_{n=0}^{\infty} t^n \dim J_n / J_{n+1}. \tag{3}$$

Let $\Delta^C(t_1, \ldots, t_n)$ denote the Alexander polynomial of the intersection of C with a small sphere centered at the origin. The theorem of Campillo, Delgado and Gusein-Zade says that if $r = 1$, then

$$(1 - t) P^C(t) = \Delta^C(t), \tag{4}$$

and if $r > 1$, then

$$P^C(t_1, \ldots, t_r) = \Delta^C(t_1, \ldots, t_r).$$

In [5] there was proposed the following natural generalization of the Poincaré series. One can naturally define the motivic measure on the space of functions, and consider the following motivic integral, generalizing (1):

$$P_g^C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdot \ldots \cdot t_r{}^{v_r} d\mu. \tag{5}$$

If $r = 1$, we can rewrite (5) as the generalization of (3):

$$P_g^C(t) = \sum_{n=0}^{\infty} t^n \frac{q^{\operatorname{codim} J_n} - q^{\operatorname{codim} J_{n+1}}}{1-q}, \tag{6}$$

therefore in this case one can deduce $P_g(t)$ from $P(t)$. If r is greater than 1, the situation becomes more complicated. Nevertheless, the explicit algorithm for the computation of the motivic Poincaré series is presented in Theorem 3.

Definition: The reduced motivic Poincaré series is the power series

$$\overline{P}_g(t_1, \ldots, t_r) = (1 - qt_1) \cdot \ldots \cdot (1 - qt_r) \cdot P_g(t_1, \ldots, t_r). \tag{7}$$

We prove that the reduced motivic Poincaré series satisfies the following properties.

1. **Polynomiality.** $\overline{P}_g(t_1, \ldots, t_r; q)$ is a polynomial in variables $t_1, \ldots, t_r$ and q. We give a bound for its degree on $t_1, \ldots, t_r$.

2. **Reduction to the Alexander polynomial.** If $n = 1$, then

$$\overline{P}_g(t; q = 1) = \Delta(t),$$

where Δ denote the Alexander polynomial of the link of the corresponding plane curve singularity. If $n > 1$, then

$$\overline{P}_g(t_1, \ldots, t_r; q = 1) = \Delta(t_1, \ldots, t_r) \cdot \prod_{i=1}^{r} (1 - t_i).$$

3. **Forgetting components.** Let C be a curve with r components, and C_1 be an irreducible curve. Then

$$\overline{P}_g^{C \cup C_1}(t_1, \ldots, t_r, t_{r+1} = 1) = (1 - q)\overline{P}_g^C(t_1, \ldots, t_r). \tag{8}$$

If C has only one component, then

$$\overline{P}_g^C(t = 1) = 1.$$

This property is clear from the equation (5), but seems to be curious and, for example, does not hold for the Alexander polynomial (we cannot reconstruct the Alexander polynomial of a sublink from the Alexander polynomial of a link by setting the corresponding variable to 1).

4. **Symmetry.** Let μ_α be the Milnor number ([2]) of C_α, let $(C_\alpha \circ C_\beta)$ be the intersection index of C_α and C_β, let $\mu(C)$ be the Milnor number of C. Let

$$l_\alpha = \mu_\alpha + \sum_{\beta \neq \alpha} (C_\alpha \circ C_\beta), \quad \delta(C) = (\mu(C) + r - 1)/2.$$

Remark that $\sum_{\alpha=1}^{r} l_\alpha = 2\delta(C)$.

It is known that the Alexander polynomial is symmetric in a sense that

$$\Delta(t_1^{-1}, \ldots, t_r^{-1}) = \prod_{\alpha=1}^{r} t_\alpha^{1-l_\alpha} \cdot \Delta(t_1, \ldots, t_r), \quad r > 1$$

and

$$\Delta(t^{-1}) = t^{-\mu}\Delta(t), \quad r = 1.$$

In Theorem 4 we prove a generalization of this identities that holds for any r, namely,

$$\overline{P}_g(\frac{1}{qt_1}, \ldots, \frac{1}{qt_r}) = q^{-\delta(C)} \prod_{\alpha} t_\alpha^{-l_\alpha} \cdot \overline{P}_g(t_1, \ldots, t_r).$$

5. **Relation to the knot homology.** For irreducible curves we prove that $\overline{P}_g(t)$ can be related by the simple procedure to the Poincaré polynomial of the Heegaard-Floer knot homology constructed by P. Ozsváth and Z. Szabó. This homology theory is a "categorification" of the Alexander polynomial, tightly related with the symplectic topology and Seiberg-Witten theory. Since the origins of our and their construction are quite far, the relation between them seems to be interesting. No conceptual proof for this fact is known, and we just use that both answers are determined by the Alexander polynomial in the same way.

The paper is organized in the following way. In the section 2 we recall the definition of the Poincaré series of a plane curve singularity. Then we recall the definition of the motivic measure on the space of functions and give, following [5], two definitions of the motivic Poincaré series as a motivic integral and in terms of the multi-index filtration associated with the curve. We give the simple method of deduction of the motivic Poincaré series from the ordinary Poincaré series for irreducible curves. In Theorem 2 we recall the formula from [5] expressing the motivic Poincaré series in terms of the embedded resolution of a curve. This formula is proved by Campillo, Delgado and Gusein-Zade using thorough analysis of the geometry of the functional spaces defined by the embedded resolution of a curve.

In the section 3 we apply Theorem 2 to a nonsingular curve and explain step-by-step the calculation of all sums involved. It turns out to be a curious exercise, and this simplest example is a toy model for the consequent combinatorial work.

The section 4 contains several steps of the simplification of Theorem 2. In the result (Lemma 6) the motivic Poincaré series is expressed in terms of some quantities $c_K(n)$. In Lemma 5 the generating function for these quantities is explicitly written in the closed form. This allows to compute the motivic Poincaré series.

Applying Lemma 6 directly, we get a lot of similar summands which cancel after all substitutions, but this cancellation is not clear from lemmas 5 and 6. For example, it is not even clear, that the answer is a polynomial.

Therefore in the rest of section 4 we discuss the analogues of the identity

$$\sum_{n=0}^{\infty} t^n q^{\frac{n^2+3n}{2}} (q^{-n} - tq) = 1$$

arising in the nonsingular case. The result of this investigation is Theorem 3, where we formulate an explicit algorithm of calculation of the motivic Poincaré series. This algorithm does not involve infinite sums, and can be implemented as a short `Mathematica` program.

The algorithm is presented in the same manner as in Lemma 6: the motivic Poincaré series is expressed in terms of some quantities $d_P(n)$, which fit into the explicitly defined generating function $H_P(u)$. This function is generally more complicated than the one from Lemma 5, but in some examples (Lemma 9) it has more or less compact form.

Section 5 contains a bunch of explicit answers for the curves with resolutions containing up to 3 divisors.

In the section 6 we prove the symmetry property for the motivic Poincaré series (Theorem 4). It generalizes the known symmetry property for the Alexander polynomial of a link. From the viewpoint of the algebraic geometry, it is related to the Gorenstein property of the coordinate ring of a curve ([6]), thus it seems to be related to the Kapranov's functional equation ([11],[10]) for the motivic zeta function of a curve.

We prove the symmetry property by proving the analogous statements for all steps of our algorithm: the function $H_P(u)$ is symmetric, what implies some relations for its coefficients $d_P(n)$ and, therefore, for the Poincaré series.

The main result of the section 7 is Theorem 6 describing the surprising relation between the motivic Poincaré series of an irreducible plane curve singularity and another deformation of the Alexander polynomial, namely, the Poincaré polynomial for the Heegaard-Floer knot homology ([18],[19]). The proof is based on the fact that in both cases the Poincaré polynomial (and series) is defined by the Alexander polynomial. We also give some corollaries from this fact which look more geometric. A filtered complex of $\mathbb{Z}[U]$-modules analogous to the Ozsváth-Szabó complex $CFL^{-}(K)$ is constructed. This gives an algebraic model for the minus- and hat-versions of the Heegaard-Floer complexes for algebraic knots.

We also compare the motivic Poincaré series with the Heegaard-Floer homologies of two-component links, corresponding to the singularities of type A_{2n-1}.

The motivic Poincaré series has been independently studied by J. Moyano-Fernandez and W. Zuniga-Galindo in [14]. Their approach is based on the study of the multi-dimensional semigroup of the singularity instead of its resolution. In particular, they gave alternative proofs of the Theorems 3 and 4 of this article.

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2 Poincaré series and its generalization

2.1 Poincaré series

Let $C = \cup_{i=1}^r C_i$ be a reduced plane curve singularity at the origin in $\mathbb{C}^2$, and C_i are its irreducible components. Let $\gamma_i : (\mathbb{C}, 0) \to (C_i, 0)$ be the uniformizations of these components.

We define r integer-valued functions on the space $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$ by the formula

$$v_i(f) = \mathrm{Ord}_0(f(\gamma_i(t)))$$

and $\mathbb{Z}^r$-indexed filtration

$$J_{\underline{v}} = \{f \in \mathcal{O} | v_i(f) \geq v_i\}.$$

Note that $J_{\underline{v}}$ are also defined for negative values of $\underline{v}$. This filtration is decreasing in a sense that if $\underline{v}_1 \prec \underline{v}_2$, then $J_{\underline{v}_1} \supset J_{\underline{v_2}}$. Consider the Laurent series

$$L_C(t_1, \dots, t_r) = \sum_{\underline{v}} t_1^{v_1} \dots t_r^{v_r} \cdot \dim J_{\underline{v}}/J_{\underline{v}+\underline{1}}.$$

Definition:([6], [3]) The Poincaré series of the curve C is defined by the formula

$$P_C(t_1, \dots, t_r) = \frac{L_C(t_1, \dots, t_r) \cdot \prod_{i=1}^r (t_i - 1)}{t_1 \cdot \dots \cdot t_r - 1}.$$

For example, if $r = 1$, we have

$$P_C(t) = \sum_{v=0}^{\infty} t^v \cdot \dim J_v/J_{v+1}.$$

One can prove, that P_C is always a power series. More geometric meaning of this definition is given by the following interpretation of the Poincaré series as an integral with respect to the Euler characteristic.

Proposition.([4]) Let $\mathbb{P}\mathcal{O}$ denote the projectivization of the functional space $\mathcal{O}$ as a vector space. Then the following equation holds:

$$P_C(t_1, \dots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdot \dots \cdot t_r^{v_r} d\chi. \tag{9}$$

On the other hand, consider the link of C – the intersection of C with a small three-dimensional sphere centred at the origin. We denote its multi-variable Alexander polynomial by $\Delta_C(t_1, \dots, t_r)$. Campillo, Delgado and Gusein-Zade proved the following

Theorem 1 *([4]) If $r = 1$ then*

$$P_C(t)(1 - t) = \Delta_C(t), \tag{10}$$

and if $r > 1$ then

$$P_C(t_1, \dots, t_r) = \Delta_C(t_1, \dots, t_r). \tag{11}$$

2.2 Motivic measure

Let $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$ be the space of formal germs of analytic functions at the origin on the plane. It is the set of formal power series $f(x, y)$ (without degree 0 term). Let $\mathcal{O}_n$ be the space of n-jets of such arcs, let $\pi_n : \mathcal{O} \to \mathcal{O}_n$ be the natural projection.

Let $K_0(Var_{\mathbb{C}})$ be the Grothendieck ring of complex quasiprojective varieties. It is generated by the isomorphism classes of complex quasiprojective varieties modulo the relations $[X] = [Y] + [X \setminus Y]$, where Y is a Zariski locally closed subset of X. Multiplication is given by the formula $[X] \cdot [Y] = [X \times Y]$. Let $\mathbb{L} = [\mathbb{C}] \in K_0(Var_{\mathbb{C}})$ be the class of the affine line in this ring.

The Euler characteristic provides a ring homomorphism

$$\chi : K_0(Var_{\mathbb{C}}) \to \mathbb{Z}.$$

Consider the ring $K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}]$ with the following filtration: F_k is generated by the elements of the type $[X] \cdot [\mathbb{L}^{-n}]$ with $n - \dim X \geq k$. Let $\mathcal{M}$ be the completion of the ring $K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}]$ corresponding to this filtration.

On an algebra of subsets of $\mathcal{O}$ Campillo, Delgado and Gusein-Zade ([5]), following the ideas of Kontsevich, Denef and Loeser ([7]) constructed a measure μ with values in the ring $\mathcal{M}$.

Definition:([5]) A subset $A \subset \mathcal{O}$ is said to be cylindric if there exist n and a constructible set $A_n \subset \mathcal{O}_n$ such that $A = \pi_n^{-1}(A_n)$. For the cylindric set A define its *motivic measure* by the formula

$$\mu(A) = [A_n] \cdot \mathbb{L}^{-\frac{(n+1)(n+2)}{2}}.$$

Remark that $\dim \mathcal{O}_n = \frac{(n+1)(n+2)}{2}$, hence the definition of the motivic measure is in fact independent on n. In a full analogy with [7], this measure can be extended to an countable-additive $\mathcal{M}$-valued measure on a suitable algebra of subsets of $\mathcal{O}$.

Definition: A function $f : \mathcal{O} \to G$ with values in an abelian group G is called simple, if its image is countable or finite, and for every $g \in G$ the set $f^{-1}(g)$ is measurable. Using this measure, one can define in the natural way the motivic integral for simple functions on $\mathcal{O}$ as

$$\int_{\mathcal{O}} f d\mu = \sum_{g \in G} g \cdot \mu(f^{-1}(g)),$$

if the right hand side sum converges in $G \otimes \mathcal{M}$.

Remark. Note that for cylindric sets the Euler characteristic can be defined by the formula $\chi(A) = \chi(A_n)$. This gives a $\mathbb{Z}-$valued measure on the algebra of cylindric sets. However, it cannot be extended to the algebra of measurable sets. This measure provides a notion of an integral with respect to the Euler characteristic for functions on $\mathcal{O}$ with cylindric level sets. It is clear that for such functions

$$\chi(\int_{\mathcal{O}} f d\mu) = \int_{\mathcal{O}} f d\chi.$$

Using the same construction, one can define the motivic measure on the projectivization $\mathbb{P}\mathcal{O}$ of the functional space.

As a direct generalisation of the equation (9) Campillo, Delgado and Gusein-Zade proposed the following

Definition: Motivic Poincaré series is the motivic integral

$$P_g^C(t_1,\ldots,t_r)=\int_{\mathbb{P}\mathcal{O}} t_1^{v_1}\cdot\ldots\cdot t_r^{v_r}d\mu \tag{12}$$

As above, this definition can be reformulated in terms of the multi-index filtration on the space of functions. Let $q=\mathbb{L}^{-1}$ be a formal variable. Let $h(\underline{v})=\text{codim}J_{\underline{v}}$, and

$$L_g(t_1,\ldots,t_r,q)=\sum_{\underline{v}\in\mathbb{Z}^r}\frac{q^{h(\underline{v})}-q^{h(\underline{v}+\underline{1})}}{1-q}\cdot t_1^{v_1}\ldots t_r^{v_r}.$$

Then the following equation holds ([5]):

$$P_g^C(t_1,\ldots,t_r;q)=\frac{L_g^C(t_1,\ldots,t_r)\cdot\prod_{i=1}^r(t_i-1)}{t_1\cdot\ldots\cdot t_r-1}. \tag{13}$$

An example of the calculation of the motivic Poincaré series for the singularities of type A_{2n-1} directly from the equation (13) is presented in the section 7.4 below.

2.3 Irreducible case

If $r=1$, the equation (13) has a very clear form, since in this case

$$P_g^C(t)=L_g^C(t).$$

Remark that

$$\text{codim}J_v=\dim\mathcal{O}/J_1+\dim J_1/J_2+\ldots+\dim J_{v-1}/J_v, \tag{14}$$

so the series $P_g^C(t)$ can be reconstructed from the series $P_C(t)$.

The functional $v(f)=\text{Ord}_0 f(\gamma(t))$ is a valuation on the ring $\mathcal{O}$.The set of values of v is an integer semigroup $S=\{\sigma_1,\sigma_2,\sigma_3,\ldots\}$. For example, for the singularity $x^p=y^q$ (its link is the torus (p,q) knot) we have $x(t)=t^q, y(t)=t^p$, so the corresponding semigroup is generated by p and q. The coefficient at t^v in $P_C(t)$ vanishes, if $J_v=J_{v+1}$ (or, equivalently, v does not belong to the semigroup S) , and equals to 1 otherwise. Therefore we have

$$P_C(t)=1+t^{\sigma_1}+t^{\sigma_2}+t^{\sigma_3}+\ldots.$$

Now the equation (14) implies the following formula for the motivic Poincaré series:

$$P_g^C(t;q)=1+qt^{\sigma_1}+q^2t^{\sigma_2}+q^3t^{\sigma_3}+\ldots. \tag{15}$$

Example. Consider the cusp $x^2=y^3$. Its semigroup is generated by 2 and 3, the Poincaré series is equal to

$$P(t)=1+t^2+t^3+t^4+\ldots,$$

the motivic Poincaré series is equal to

$$P_g(t) = 1 + qt^2 + q^2t^3 + q^3t^4 + \dots .$$

Note that

$$P(t)(1-t) = 1 - t + t^2,$$

what equals to the Alexander polynomial of the trefoil knot.

2.4 Formula of Campillo, Delgado and Gusein-Zade

In [5] Campillo, Delgado and Gusein-Zade gave a formula for the generalized Poincaré series in terms of the resolution.

Let $\pi : (X, D) \to (\mathbb{C}^2, 0)$ be an embedded resolution where $D = \cup_{i=1}^{s} E_i$ is the exceptional divisor. Let $E_i^{\bullet}$ be E_i without intersection points of E_i with other components of D, E_i° be $E_i^{\bullet}$ without intersection points of E_i with the components of the strict transform of our curve. Let $A = (E_i \circ E_j)$ be the intersection matrix and $M = -A^{-1}$.

Let $I_0 = \{(i,j) : i < j, E_i \cap E_j = pt\}$, $K_0 = \{1, \dots, r\}$. For $\sigma \in I_0$, $\sigma = (i,j)$ let $i(\sigma) = i$, $j(\sigma) = j$. For $I \subset I_0$, $K \subset K_0$ let

$$\mathcal{N}_{I,K} := \{\underline{\mathbf{n}} = (n_i, n'_\sigma, n''_\sigma, \tilde{n}'_k, \tilde{n}''_k) : n_i \geq 0, i = 1 \dots, s$$

$$n'_\sigma, n''_\sigma, \sigma \in I; \tilde{n}'_k > 0, \tilde{n}''_k > 0, k \in K\}.$$

For $\underline{\mathbf{n}} \in \mathcal{N}_{I,K}, i = 1, \dots, s$, let

$$\hat{n}_i = n_i + \sum_{\sigma \in I : i(\sigma) = i} n'_\sigma + \sum_{\sigma \in I : j(\sigma) = i} n''_\sigma + \sum_{k \in K : i(k) = i} \tilde{n}'_k. \tag{16}$$

Let

$$F(\underline{\mathbf{n}}) = \frac{1}{2}\Big(\sum_{i,j=1}^{s} m_{ij}\hat{n}_i\hat{n}_j + \sum_{i=1}^{s}\hat{n}_i\Big(\sum_{j=1}^{s} m_{ij}\chi(E_j^{\bullet}) + 1\Big)\Big) + \sum_{k \in K}\tilde{n}''_k, \tag{17}$$

$$\overline{F}(\underline{\hat{n}}) = \frac{1}{2}\Big(\sum_{i,j=1}^{s} m_{ij}\hat{n}_i\hat{n}_j + \sum_{i=1}^{s}\hat{n}_i\Big(\sum_{j=1}^{s} m_{ij}\chi(E_j^{\bullet}) + 1\Big)\Big),$$

and

$$\underline{w}(\underline{\mathbf{n}}) = \sum_{i=1}^{s}\hat{n}_i\underline{m}_i, v_k(\underline{\mathbf{n}}) := w_{i(k)}(\underline{\mathbf{n}}) + \tilde{n}''_k.$$

Theorem 2 *([5])*

$$P_g(t_1, \dots, t_r, q) = \sum_{I \subset I_0, K \subset K_0}\ \sum_{\underline{\mathbf{n}} \in \mathcal{N}_{I,K}} q^{F(\underline{\mathbf{n}}) - \sum_{i=1}^{s} n_i - |I| - |K|} \cdot (1-q)^{|I|+|K|} \times$$

$$\times \prod_{i=1}^{s}\left(\sum_{j=0}^{\min\{n_i, 1-\chi(E_i^{\circ})\}} (-1)^j \binom{1-\chi(E_i^{\circ})}{j} q^j\right) \cdot t^{\underline{v}(\underline{\mathbf{n}})}.$$

We briefly recall the sketch of the proof from [5]. Consider a function $f \in \mathcal{O}$ and its pullback $\pi^* f$ on the space of resolution X. Now let $I(f)$ be the set of intersection points in D such that there are components of the strict transform of X passing through them, $K(f)$ is the analogous set of intersection points of strict transform of C with D. Now $n_i(f)$ is the intersection index of the strict transform of f with the smooth part of E_i, n'_σ and n''_σ are intersection indices of the component of the strict transform of f passing through σ with $E_{i(\sigma)}$ and $E_{j(\sigma)}$ respectively, $\tilde{n}'_k$ and $\tilde{n}''_k$ are intersection indices of the component passing through the point k with $E_{i(k)}$ and corresponding component of C respectively.

Given these sets and multiplicities, the value of the function $t_1^{v_1(f)} \cdot \ldots \cdot t_r^{v_r(f)}$ is equal to $t^{\underline{v}(\underline{\mathbf{n}})}$. Every summand in Theorem 2 is equal to this value multiplied by the motivic measure of the set of functions providing such set of data.

3 Example: nonsingular curve

Let us check that for the nonsingular curve the complicated expression from Theorem 2 coincides with the expected one.

We have one divisor and one component of the strict transform of the curve. We have $I_0 = \emptyset$, $K_0 = \{1\}$. Also we have $\chi(E^\circ) = 1, \chi(E^\bullet) = 2$, hence $1 - \chi(E^\circ) = 0$. To sum over $K \subset K_0$, consider two cases:

1) $K = \emptyset$. In this case $F(n) = \frac{1}{2}(n^2 + 3n)$, and we have a sum

$$\sum_{n=0}^{\infty} t^n q^{\frac{n^2+3n}{2}} \cdot q^{-n}$$

2) $K = \{1\}$. In this case $F(n) = \frac{1}{2}(\hat{n}^2 + 3\hat{n}) + n''$, and we have a sum

$$\sum_{\hat{n}=1}^{\infty} q^{\frac{\hat{n}^2+3\hat{n}}{2}} t^{\hat{n}} \sum_{n=0}^{\hat{n}-1} q^{-n-1}(1-q) \sum_{n''=1}^{\infty} q^{n''} t^{n''} = \sum_{\hat{n}=1}^{\infty} q^{\frac{\hat{n}^2+3\hat{n}}{2}} t^{\hat{n}} (q^{-\hat{n}} - 1) \cdot \frac{qt}{1-qt}.$$

Summing these two expressions, we get

$$1 + \sum_{n=1}^{\infty} t^n q^{\frac{n^2+3n}{2}} (q^{-n} + \frac{qt}{1-qt}(q^{-n} - 1)) = 1 + \frac{1}{1-qt} \sum_{n=1}^{\infty} t^n q^{\frac{n^2+3n}{2}} (q^{-n} - qt) =$$

$$1 + \frac{1}{1-qt} (\sum_{n=1}^{\infty} t^n q^{\frac{n(n+1)}{2}} - \sum_{n=1}^{\infty} t^{n+1} q^{\frac{(n+1)(n+2)}{2}}).$$

In the last sum all coefficients at t^n for $n \geq 2$ cancel, therefore

$$P_g(t; q) = 1 + \frac{tq}{1-qt} = \frac{1}{1-qt}.$$

4 Combinatorics

4.1 Preliminary simplification

Let

$$P_{k,n}(q)=\sum_{j=0}^{n}(-1)^j q^j \binom{k}{j}$$

(k can be negative, but n should be non-negative and integer).

Lemma 1 *Let $S^n X$ denote the nth symmetric power of a space X. Then*

$$[S^n(\mathbb{CP}^1 - k\{pt\})] = q^{-n}P_{k-1,n}(q).$$

Proof. If Y denote the union of k points on $\mathbb{C}^1$, then we have

$$S^m(\mathbb{CP}^{\nVdash}) = \sqcup_{i=0}^{m} S^i(Y)\times S^{m-i}(\mathbb{CP}^1\setminus Y),$$

what is equivalent to the following multiplicativity property:

$$\sum_{n=0}^{\infty} t^n[S^n(\mathbb{CP}^1)] = \sum_{n=0}^{\infty} t^n[S^n(Y)]\cdot\sum_{n=0}^{\infty} t^n[S^n(\mathbb{CP}^1\setminus Y)].$$

Since

$$\sum_{n=0}^{\infty} t^n[S^n(\mathbb{CP}^1)] = \sum_{n=0}^{\infty} t^n[\mathbb{CP}^n] = \frac{1}{(1-t)(1-\mathbb{L}t)},$$

we get

$$\sum_{n=0}^{\infty} t^n[S^n(\mathbb{CP}^1 - k\{pt\})] = \frac{(1-t)^{k-1}}{(1-\mathbb{L}t)} =$$

$$\sum_{a,b}(-1)^a\binom{k-1}{a} t^a\mathbb{L}^b t^b = \sum_{n=0}^{\infty} t^n\sum_{a=0}^{n}(-1)^a\binom{k-1}{a}\mathbb{L}^{n-a} =$$

$$\sum_{n=0}^{\infty} t^n q^{-n}P_{k-1,n}(q).$$

□

Let us fix some notations.

Definition: Let

$$f_i(I,K)=\sum_{\sigma\in I:i(\sigma)=i}1+\sum_{\sigma\in I:j(\sigma)=i}1+\sum_{k\in K:i(k)=i}1,$$

$$f_i(I)=\sum_{\sigma\in I:i(\sigma)=i}1+\sum_{\sigma\in I:j(\sigma)=i}1.$$

Note that $\sum_{i=1}^{s} f_i(I,K)=2|I|+|K|, \sum_{i=1}^{s} f_i(I)=2|I|$.

To any divisor E_i we associate the factor

$$\phi_i(I,K,\hat{n}) = P_{1-\chi(E_i^\circ)-f_i(I,K),\hat{n}_i-f_i(I,K)},$$

and let

$$G(I,K,\hat{n}) = q^{|I|}(1-q)^{|I|+|K|}\prod_i \phi_i(I,K,\hat{n}).$$

Now we can start the simplification of the algorithm proposed in Theorem 2. The next two lemmas will allow us to reduce the summation over all quadruples $(n_i, n'_\sigma, n''_\sigma, \widetilde{n}'_k)$ to the summation by a single variable $\hat{n}_i$ defined by (16).

Lemma 2 *Let us fix $\hat{n}_i$. Then*

$$\sum_{n_i,n'_\sigma,n''_\sigma,\widetilde{n}'_k} q^{-n_i-f_i(I,K)}P_{1-\chi(E_i^\circ),n_i}(q) = q^{-\hat{n}_i}\phi_i(I,K,\hat{n}). \tag{18}$$

Proof. By Lemma 1 we have

$$\sum_{n_i,n'_\sigma,n''_\sigma,\widetilde{n}'_k} q^{-n_i-f_i(I,K)}P_{1-\chi(E_i^\circ),n_i}(q) = \sum_{n_i,n'_\sigma,n''_\sigma,\widetilde{n}'_k} q^{-f_i(I,K)}[S^{n_i}(E_i^\circ)].$$

Consider a n_i-tuple of points on E_i°, intersection points $\sigma \in I$ such that $i(\sigma) = i$ with multiplicities $n'_\sigma - 1$, intersection points $\sigma \in I$ such that $j(\sigma) = i$ with multiplicities $n''_\sigma - 1$, intersection points $k \in K$ such that $i(k) = i$ with multiplicities $\widetilde{n}'_k - 1$. We get the unordered $\hat{n}_i - f_i$-tuple of points on $E_i^\circ \cup f_i(I,K)$. Thus the sum (18) equals to

$$q^{-f_i(I,K)}[S^{\hat{n}_i-f_i(I,K)}(E_i^\circ \cup f_i(I,K))] = q^{-\hat{n}_i}P_{1-\chi(E_i^\circ)-f_i(I,K),\hat{n}_i-f_i(I,K)}(q).$$

□

Lemma 3

$$P_g(t_1,\ldots,t_r,q) = \sum_{I\subset I_0,K\subset K_0}\sum_{\hat{n}_i\geq f_i(I,K)} \underline{t}^{M\underline{\hat{n}}}q^{\overline{F}(\underline{\hat{n}})}\prod_{i=1}^{s} q^{-\hat{n}_i}\phi_i(I,K,\hat{n})\times \tag{19}$$

$$q^{|I|}(1-q)^{|I|+|K|}\prod_{k\in K}\frac{qt_k}{1-qt_k}.$$

Proof. First, remark that for every k

$$\sum_{\tilde{n}''_k>0} q^{\tilde{n}''_k}t_k^{\tilde{n}''_k} = \frac{t_k q}{1-t_k q},$$

so from now on we can forget about summation over $\tilde{n}''_k$.

We have

$$q^{-\sum_{i=1}^{s} n_i-|I|-|K|} = q^{|I|}\prod_{i=1}^{s} q^{-n_i-f_i(I,K)},$$

therefore we can reformulate the statement of Theorem 2 in the form

$$P_g(t_1,\ldots,t_r,q)=\sum_{I\subset I_0,K\subset K_0} q^{|I|}(1-q)^{-|I|}\sum_{\hat{n}_i\geq f_i(I,K)} \underline{t}^{M\underline{\hat{n}}}q^{\overline{F}(\underline{\hat{n}})}\times$$

$$\prod_{i=1}^{s}\left[\sum_{n_i,n'_\sigma,n''_\sigma,\widetilde{n}'_k} q^{-n_i-f_i(I,K)}P_{1-\chi(E_i^\circ),n_i}(q)\right].$$

Now the equation (19) follows from the Lemma 2. $\square$

Definition: By the reduced motivic Poincaré series from now on we mean

$$\overline{P}_g(t_1,\ldots,t_r)=P_g(t_1,\ldots,t_r)\cdot\prod_{j=1}^{r}(1-t_jq).$$

Lemma 4

$$\sum u^{\hat{n}}G(K,I,\hat{n})=q^{|I|}(1-q)^{|I|+|K|}\prod_i\frac{u_i^{f_i(K,I)}}{1-u_i}(1-u_iq)^{1-\chi(E_i^\circ)-f_i(I,K)} \tag{20}$$

The proof of this lemma can be found in the Appendix.

Definition: Let

$$c_K(n)=\sum_I\sum_{K_1\subset K}(-1)^{|K|-|K_1|}G(K_1,I,n),$$

$$A_K(u)=\sum_n u^nc_K(n).$$

The next lemma provides a closed formula for the function $A_K(u)$, which can be considered as a generating function for the quantities $c_K(n)$.

Lemma 5

$$A_K(u)=(-1)^{|K|}\prod_i(1-u_iq)^{|\overline{K}\cap E_i|-1}(1-u_i)^{|K\cap E_i|-1}\prod_\sigma(1-qu_{i(\sigma)}-qu_{j(\sigma)}+qu_{i(\sigma)}u_{j(\sigma)}).$$

The proof of this lemma can be found in the Appendix. The next lemma expresses the reduced motivic Poincaré series in terms of the quantities $c_K(n)$.

Lemma 6

$$\overline{P}_g(t_1,\ldots,t_r,q)=\sum_n t^{Mn}q^{F(n)-\sum n_i}\sum_K t_Kq^{|K|}c_K(n). \tag{21}$$

Proof. From the equation (19) we get

$$P_g(t_1,\ldots,t_r,q)=\sum_{I\subset I_0,K\subset K_0}\sum_{\hat{n}_i\geq f_i(I,K)}\underline{t}^{M\underline{\hat{n}}}q^{\overline{F}(\underline{\hat{n}})}\prod_{i=1}^{s}q^{-\hat{n}_i}\phi_i(I,K,\hat{n})\times$$

$$q^{|I|}(1-q)^{|I|+|K|}\prod_{k\in K}\frac{qt_k}{1-qt_k}=$$

$$\sum_{I\subset I_0,K\subset K_0}\sum_{\hat{n}_i\geq f_i(I,K)}\underline{t}^{M\underline{\hat{n}}}q^{\overline{F}(\underline{\hat{n}})}\prod_{i=1}^{s}q^{-\hat{n}_i}\phi_i(I,K,\hat{n})\times q^{|I|}(1-q)^{|I|+|K|}\prod_{k\in K}\frac{qt_k}{1-qt_k}=$$

$$\frac{1}{\prod_{i=1}^{n}(1-qt_i)}\sum_{\hat{n}}t^{Mn}q^{F(n)-\sum n_i}\sum_{K}t_Kq^{|K|}\sum_{I\subset I_0}\sum_{K_1\subset K}(-1)^{|K|-|K_1|}G(K_1,I,\hat{n})=$$

$$\frac{1}{\prod_{i=1}^{n}(1-qt_i)}\sum_{\hat{n}}t^{Mn}q^{F(n)-\sum n_i}\sum_{K}t_Kq^{|K|}c_K(\hat{n}).$$

□

Lemma 6 together with Lemma 5 gives the explicit description of $\overline{P}_g(t)$: it is expressed in terms of some quantities $c_K(n)$, which fit together into the generating function $A_K(u)$. Lemma 5 provides a closed formula for this generating function.

Nevertheless, as the model example with a nonsingular curve shows, lots of summands in the sum (21) have the same power in t, and for n large enough we have a huge number of cancellations.

4.2 Cancellations

We say that a subset $K\subset K_0$ is *proper everywhere*, if for all i $K\cap E_i$ is a proper subset of $K_0\cap E_i$. We denote the set of proper everywhere subsets by $\mathcal{P}$. For any $K\subset K_0$ let $E(K)$ be the set of divisors such that for $i\in E(K)$ the set $K\cap E_i$ is empty. Sometimes we will write $i\in P$, if $i\notin E(P)$.

Using these notations, every subset $K\subset K_0$ can be presented (uniquely) in the following way: we fix a proper everywhere subset $P(K)$ and a set of divisors $E\subset E(P(K))$ where all intersection points with K_0 belong to K.

For a set E of divisors let $\Delta(E)$ be the number of pairs of intersecting divisors from E. Let $\mu_i(E)=1$, if $i\in E$ and $\mu_i(E)=0$ otherwise.

Lemma 7 *For a proper everywhere set P let*

$$\widetilde{H}_P(u_1,\ldots,u_s)=\sum_{E\subset E(P)}(-1)^{|K_0\cap E|}\prod u_i^{-\sum a_{ij}\mu_j}\cdot q^{\Delta(E)}\prod_{i\in E}(q-u_i)^{k_i-1}\prod_{i\notin(P\cup E)}(1-qu_i)^{k_i-1}$$
$$\times\prod_{\sigma}(1-q^{1-\mu_{i(\sigma)}(E)}u_{i(\sigma)}-q^{1-\mu_{j(\sigma)}(E)}u_{j(\sigma)}+q^{1-\mu_{i(\sigma)}(E)-\mu_{j(\sigma)}(E)}u_{i(\sigma)}u_{j(\sigma)}).\tag{22}$$

Then the polynomial $\widetilde{H}_P$ is divisible by $\prod_{i\in E(P)}(1-u_i)$.

The proof of this lemma can be found in the Appendix.

The next lemma explains the relation of the function $\widetilde{H}_P(u_1,\ldots,u_s)$ (which is a modification of the function $A_K(u)$) to the coefficients $c_K(n)$ defined above. It is the main technical instrument in the study of the cancellations.

Lemma 8

$$\sum_n u^n \sum_{E\subset E(P)} q^{-\sum_{i\in E} n_i-\Delta(E)-\sum_{i\in E} a_{ii}-|E|} q^{|K_0\cap E|}\times c_{P\cup E}(n_i+\sum a_{ij}\mu_j(E)) =$$

$$(-1)^{|P|}\prod_{i\in P}[(1-qu_i)^{k_i-p_i-1}(1-u_i)^{p_i-1}]\cdot\frac{1}{\prod_{i\in E(P)}(1-u_i)}\widetilde{H}_P(u_1,\ldots,u_s).$$

The proof of this lemma can be found in the Appendix.

Definition: For a proper everywhere set P define the quantities $d_P(n)$ by the equation

$$H_P(u)=\sum_n d_P(n)u^n d_P(n)=\frac{\prod_{i\in P}[(1-qu_i)^{k_i-p_i-1}(1-u_i)^{p_i-1}]}{\prod_{i\in E(P)}(1-u_i)}\widetilde{H}_P(u_1,\ldots,u_s). \quad (23)$$

Remark that by Lemma 7 the function $H_P(u)$ is polynomial in u, so we have only finite number of non-zero coefficients $d_P(n)$.

Combining the statements of Lemma 6 and Lemma 8, we get the following result.

Theorem 3 *Then*

$$\overline{P}_g(t_1,\ldots,t_r)=\sum_{P\in\mathcal{P}}(-1)^{|P|}q^{|P|}t_P\times\sum_n d_P(n)t^{Mn}q^{F(n)-\sum n_i}.$$

Proof. From Lemma 6 we have

$$\overline{P}(t)=\sum_{n_1} t^{Mn_1}q^{F(n_1)-\sum n_i}\sum_{K\subset K_0} t_K q^{|K|}c_K(n_1)=$$

$$\sum_{P\in\mathcal{P}} q^{|P|}t_P\sum_{n_1} t^{Mn_1}q^{F(n_1)-\sum n_i}\sum_{E\subset E(P)} t_E q^{|K_0\cap E|}c_{P\cup E}(n_1).$$

Let us collect the coefficient at t^{Mn}. We have

$$Mn_1+\sum\mu_j(E)=Mn,\quad n_1=n+\sum a_{ij}\mu_j(E).$$

and

$$(\overline{F}(n)-\sum n_i)-(\overline{F}(n_1)-\sum n_{1i})=\frac{1}{2}[-2\sum m_{ij}n_i a_{js}\mu_j(E)$$

$$-\sum m_{ij}a_{is}\mu_s(E)a_{jl}\mu_l(E)-\sum m_{ij}\chi(E_i^\bullet)a_{js}\mu_s(E)+\sum a_{ij}\mu_j(E)].$$

Remark that

$$\sum_{i\neq j} a_{ij}=2-\chi(E_j^\bullet),$$

hence

$$(\overline{F}(n)-\sum n_i)-(\overline{F}(n_1)-\sum n_{1i})=\sum_{i\in E} n_i+\Delta(E)+\sum_{i\in E} a_{ii}+|E|.$$

Thus

$$\overline{P}(t)=\sum_{P\in\mathcal{P}} q^{|P|}t_P\sum_n t^{Mn}q^{F(n)-\sum n_i}\sum_{E\subset E(P)} q^{-\sum_{i\in E} n_i-\Delta(E)-\sum_{i\in K} a_{ii}-|E|}$$

$$\times q^{|K_0\cap E|}c_{P\cup E}(n+\sum a_{ij}\mu_j(E)).$$

Now we apply Lemma 8.

□

Corollary 1 *The power series $\overline{P}_g(t_1,\ldots,t_r)$ is a polynomial.*

4.3 The algorithm

If every line E_i is intersected by the one component of the strict transform, any proper everywhere set should be empty. Therefore we get the following statement as a corollary of Theorem 3.

Lemma 9 *Suppose that each divisor E_i is intersected by exactly one component of the strict transform of the curve. Then the reduced motivic Poincaré series can be computed using the following algorithm.*

1. Consider the polynomial

$$A(u_1,\ldots,u_r)=\prod_\sigma(1-qu_{i(\sigma)}-qu_{j(\sigma)}+qu_{i(\sigma)}u_{j(\sigma)}).$$

2. Consider the Laurent polynomial

$$\widetilde{H}(u_1,\ldots,u_t)=\sum_{K\subset K_0}(-1)^{|K|}q^{\Delta(K)}\prod u_i^{-\sum a_{ij}\mu_j}\cdot A(u_1q^{-\mu_1(K)},\ldots,u_rq^{-\mu_r(K)}).$$

3. This polynomial is divisible by $\prod(1-u_i)$. Let

$$H(u_1,\ldots,u_r)=\frac{\widetilde{H}(u_1,\ldots,u_r)}{\prod_{i=1}^r(1-u_i)}.$$

4. Expand this polynomial:

$$H(u_1,\ldots,u_r)=\sum d_{\underline{n}}u^{\underline{n}},$$

and now

$$\overline{P}_g(t_1,\ldots,t_r)=\sum d_{\underline{n}}t^{M\underline{n}}q^{F(\underline{n})-\sum n_i}.$$

5 Examples

5.1 One divisor

We consider the singularity

$$x^{k_0} - y^{k_0} = 0,$$

which is geometrically a union of k_0 pairwise transversal lines. Its minimal resolution has one divisor and k_0 components of the strict transform intersecting it. In particular, for $k_0 = 1$ we get a non-singular case considered above. For $0 < k < k_0$ let the numbers $c_k(n)$ be defined by the equation

$$A_k(u) = \sum_{n=0}^{\infty} u^n c_k(n) = (1-uq)^{k_0-k-1}(1-u)^{k-1},$$

and for $k = 0$ let the numbers $c_0(n)$ be defined by the equation

$$A_0(u) = \sum_{n=0}^{\infty} u^n c_0(n) = \frac{(1-uq)^{k_0-1} - u(u-q)^{k_0-1}}{1-u}.$$

The polynomials $A_k(u)$ have degree $k_0 - 2$ for $k > 0$, $A_0(u)$ has degree $k_0 - 1$, so we have a finite number of non-zero $c_k(n)$.

From the Theorem 3 we conclude that

$$\overline{P}_g(t_1, \ldots, t_{k_0}) = \sum_{K \subset_{\neq} K_0} (-1)^{|K|} q^{|K|} t_K \sum_{n=0}^{\infty} c_{|K|}(n)(t_1 \ldots t_{k_0})^n q^{\frac{n(n+1)}{2}}.$$

For example, if $k_0 = 2$,

$$A_1(u) = 1, A_0(u) = \frac{1 - uq - u(u-q)}{1-u} = 1 + u,$$

so

$$\overline{P}_g(t_1, t_2) = 1 - qt_1 - qt_2 + qt_1t_2.$$

If $k_0 = 3$,

$$A_1(u) = 1 - qu, A_2(u) = 1 - u, A_0(u) = 1 + (1 - 2q - q^2)u + u^2,$$

so

$$\overline{P}_g(t_1, t_2, t_3) = 1 - q(t_1 + t_2 + t_3) + q^2(t_1t_2 + t_1t_3 + t_2t_3) + q(1 - 2q - q^2)t_1t_2t_3 +$$
$$q^3t_1t_2t_3(t_1 + t_2 + t_3) - q^3t_1t_2t_3(t_1t_2 + t_1t_3 + t_2t_3) + q^3t_1^2t_2^2t_3^2.$$

This answer can be rewritten as

$$\overline{P}_g(t_1, t_2, t_2) = (1 - qt_1)(1 - qt_2)(1 - qt_3) - q^3t_1t_2t_3(1 - t_1)(1 - t_2)(1 - t_3) + q(1 - q)^2t_1t_2t_3.$$

5.2 Two divisors

Suppose that the second divisor is intersected by two components of the strict transform, and the first one by one component. This corresponds to the singularity

$$x \cdot (y - x^2) \cdot (y + x^2) = 0.$$

The matrix M is equal to

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

$$\chi(E_1^\bullet) = \chi(E_2^\bullet) = 1,$$

so

$$F(n_1, n_2) = \frac{1}{2}(n_1^2 + 2n_1n_2 + 2n_2^2 + 2n_1 + 3n_2).$$

If $P = \emptyset$, we get

$$\widetilde{H}_\emptyset(u_1, u_2) = (1 - qu_1 - qu_2 + qu_1u_2)(1 - qu_2) - (1 - u_1 - qu_2 + u_1u_2)(1 - qu_2)u_1^2u_2^{-1}$$

$$+(1 - qu_1 - u_2 + u_1u_2)(q - u_2)u_1^{-1}u_2 - q(1 - u_1 - u_2 + q^{-1}u_1u_2)(1 - qu_2)u_1 =$$

$$\frac{1}{u_1u_2}(1 - u_1)(1 - u_2)(-u_1^3 + u_1u_2 + u_1^2u_2 - qu_1^2u_2 - q^2u_1^2u_2 + qu_1^3u_2$$

$$+qu_2^2 + u_1u_2^2 - qu_1u_2^2 - q^2u_1u_2^2 + u_1^2u_2^2 - u_2^3),$$

if P is one point on the second divisor, we get

$$\widetilde{H}_{pt}(u_1, u_2) = (1 - qu_1 - qu_2 + qu_1u_2) - (1 - u_1 - qu_2 + u_2)u_1^2u_2^{-1} =$$

$$-\frac{1}{u_2}(1 - u_1)(u_1^2 - u_2 - u_1u_2 + qu_1u_2 - u_1^2u_2 + qu_2^2).$$

Finally we get the following answer (t_0 corresponds to the first divisor):

$$\overline{P}_g(t_0, t_1, t_2) = 1 - qt_0 - qt_1 + q^2t_0t_1 - qt_2 + q^2t_0t_2 + q^2t_1t_2 + qt_0t_1t_2 - q^2t_0t_1t_2 - q^3t_0t_1t_2$$

$$-q^2t_0t_1^2t_2 + q^3t_0t_1^2t_2 - q^2t_0t_1t_2^2 + q^3t_0t_1t_2^2 + q^2t_0t_1^2t_2^2 - q^3t_0t_1^2t_2^2 - q^4t_0t_1^2t_2^2 + q^4t_0^2t_1^2t_2^2$$

$$+q^4t_0t_1^3t_2^2 - q^4t_0^2t_1^3t_2^2 + q^4t_0t_1^2t_2^3 - q^4t_0^2t_1^2t_2^3 - q^4t_0t_1^3t_2^3 + q^4t_0^2t_1^3t_2^3.$$

This answer can be rewritten as

$$\overline{P}_g(t_0, t_1, t_2) = (1 - qt_0)(1 - qt_1)(1 - qt_2) - q^4t_0t_1^2t_2^2(1 - t_0)(1 - t_1)(1 - t_2)$$

$$+(1 - q)qt_0t_1t_2(1 - qt_1 - qt_2 + qt_1t_2).$$

If $q = 1$, we get the known Alexander polynomial:

$$\overline{P}_g(t_0, t_1, t_2; q = 1) = (1 - t_0)(1 - t_1)(1 - t_2)(1 - t_0t_1^2t_2^2).$$

If $t_2 = 1$, we get the known answer for A_1 singularity:

$$\overline{P}_g(t_0, t_1, 1) = (1 - q)(1 - qt_0 - qt_1 + qt_0t_1).$$

If $t_0 = 1$, we get the answer for A_3 singularity:

$$\overline{P}_g(1, t_1, t_2) = (1-q)(1 - qt_1 - qt_2 + qt_1t_2 + q^2t_1t_2 - q^2t_1^2t_2 - q^2t_1t_2^2 + q^2t_1^2t_2^2),$$

so

$$\overline{P}_g^{A_3}(t_1, t_2) = (1-qt_1)(1-qt_2) + qt_1t_2(1 - qt_1 - qt_2 + qt_1t_2) =$$
$$(1-qt_1)(1-qt_2) + q^2t_1t_2(1-t_1)(1-t_2) + (1-q)qt_1t_2.$$

This answer agrees with the general answer for the singularities of type A_{2n-1} in the section 7.5.

5.3 Three divisors

For simplicity we assume that each divisor is intersected by one component of the strict transform. This corresponds to the singularity

$$x \cdot y \cdot (x^2 - y^3) = 0.$$

Matrix M is equal to

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 6 \end{pmatrix},$$

$$\chi(E_1^\bullet) = \chi(E_2^\bullet) = 1, \chi(E_3^\bullet) = 0,$$

so

$$F(n_1, n_2, n_2) = \frac{1}{2}(n_1^2 + 2n_2^2 + 6n_3^2 + 2n_1n_2 + 4n_1n_3 + 6n_2n_3 + n_1 + 2n_2 + 4n_3).$$

Now

$$A(u_1, u_2, u_3) = (1 - qu_1 - qu_3 + qu_1u_3)(1 - qu_2 - qu_3 + qu_2u_3),$$

so

$$E(u_1, u_2, u_3) = \frac{1}{u_1u_2u_3^2}({u_2}^3u_3u_1 - {u_1}^3{u_3}^2q + {u_1}^4u_3u_2 - {u_1}^2{u_2}^2{u_3}^2 - {u_2}^2{u_3}^2u_1 +$$
$${u_1}^4{u_2}^3u_3 - {u_3}^3{u_1}^2q - {u_1}^3u_2{u_3}^2 + {u_1}^3{u_2}^3u_3 + {u_1}^2{u_2}^3u_3 - {u_3}^3qu_2 -$$
$${u_1}^3{u_2}^2{u_3}^2 - {u_3}^3u_1q - {u_2}^2{u_3}^2q - {u_1}^2u_2{u_3}^2 - {u_3}^2u_1u_2 + {u_2}^2{u_1}^4u_3 - {u_1}^3{u_2}^3qu_3 +$$
$${u_2}^2{u_3}^2{u_1}^2q - {u_1}^4u_3{u_2}^2q - {u_1}^4{u_3}^2u_2q - {u_2}^3{u_3}^2u_1q - {u_2}^3u_3{u_1}^2q + {u_3}^3u_1q^2u_2 +$$
$${u_2}^2{u_3}^2u_1q^2 + {u_1}^3{u_2}^2u_3q^2 + {u_1}^3{u_3}^2u_2q^2 - {u_1}^4{u_2}^3 + {u_1}^2{u_3}^3 + {u_3}^3u_1 + {u_3}^2{u_1}^2u_2q +$$
$${u_1}^3{u_3}^3 + {u_3}^3{u_2}^2 + {u_3}^3u_2 + {u_3}^3 - {u_3}^4),$$

and

$$\overline{P}_g(t_1, t_2, t_3) = 1 - t_3q + {t_1}^2{t_2}^3{t_3}^7q^7 + {t_1}^2{t_2}^2{t_3}^5q^5 + t_1t_2{t_3}^3q^3 + t_1{t_2}^2{t_3}^4q^4 - {t_1}^2{t_2}^4{t_3}^7q^7 +$$
$$t_2t_3q^2 - t_1t_2{t_3}^3q^2 + t_1t_2q^2 - t_1{t_2}^2{t_3}^4q^3 - {t_1}^2{t_2}^2{t_3}^5q^4 - t_1{t_2}^2{t_3}^2q^2 - {t_1}^2{t_2}^3{t_3}^5q^5 -$$

$$t_1{}^3t_2{}^3t_3{}^7q^7 - t_1{}^3t_2{}^4t_3{}^6q^7 + t_1{}^2t_2{}^3t_3{}^5q^4 + t_1{}^2t_2{}^2t_3{}^4q^3 + t_1{}^2t_2{}^2t_3{}^3q^4 - t_1{}^2t_2{}^2t_3{}^3q^3 +$$
$$t_1{}^2t_2{}^3t_3{}^4q^5 + t_1{}^2t_2{}^4t_3{}^6q^7 + t_1t_2{}^2t_3{}^2q^3 - t_1{}^2t_2{}^3t_3{}^6q^7 - t_1{}^2t_2{}^2t_3{}^4q^5 - t_1t_2{}^2t_3{}^3q^4 -$$
$$t_1t_2t_3q^3 + t_1t_2{}^2t_3{}^3q^2 - t_2q + t_1t_3q^2 - t_1t_2t_3{}^2q^2 + t_1{}^3t_2{}^4t_3{}^7q^7 +$$
$$t_1t_2t_3{}^2q - t_1q - t_1{}^2t_2{}^3t_3{}^4q^4 + t_1{}^3t_2{}^3t_3{}^6q^7.$$

It can be rewritten as

$$\overline{P}_g(t_1,t_2,t_3) = (1-t_1q)(1-t_2q)(1-t_3q) - t_1{}^2t_2{}^3t_3{}^6q^7(1-t_1)(1-t_2)(1-t_3) -$$
$$t_1t_2t_3^2q(q-1)(1-t_2q)(1-t_3q) - t_1^2t_2^2t_3^4q^4(q-1)(1-t_2)(1-t_3) -$$
$$t_1t_2^2t_3^3q^2(q-1)(1-t_1q) + t_1t_2^2t_3^4q^3(q-1)(1-t_1).$$

In this presentation the symmetry of $\overline{P}_g$ is clear, since every line in the right hand side is invariant under the change $t_i \leftrightarrow q^{-1}t_i^{-1}$.

If we set $q = 1$, we get

$$\overline{P}_g(t_1,t_2,t_3,q=1) = (1-t_1^2t_2^3t_3^6)(1-t_1)(1-t_2)(1-t_3).$$

If we consider only singularity of type A_2, we set $t_1 = t_2 = 1, t_3 = t$, and

$$\overline{P}_g(1,1,t) = (1-q)^2(1-tq+t^2q),$$

so

$$P_g(1,1,t) = \frac{1-tq+t^2q}{1-tq} = 1 + \sum_{k=2}^{\infty} t^k q^{k-1}.$$

This answer coincides with the one obtained in the section 2.3.

6 Symmetry

In this section we prove the symmetry property for the reduced motivic Poincaré series (Theorem 4). The strategy of the proof passes along the lines of the computation described in Lemma 6: namely, we prove the symmetry property for the generating function $A_K(u)$ in Lemma 10, deduce from it a certain relations on its coefficients $c_K(n)$ in Lemma 11. Since we can express the motivic Poincaré series in terms of $c_K(n)$, we can finish the proof by fitting this relations to the statement of Theorem 4.

Lemma 10

$$A_K\left(\frac{1}{qu_1},\ldots,\frac{1}{qu_s}\right) = q^{1-|K|}\prod_{i=1}^{s} u_i^{\chi(E_i^\circ)} \cdot A_{\overline{K}}(u_1,\ldots,u_s).$$

Proof.

$$A_K(\frac{1}{qu}) = (-1)^{|K|}\prod_i(1-\frac{1}{u_i})^{|\overline{K}\cap E_i|-1}(1-\frac{1}{u_iq})^{|K\cap E_i|-1}\prod_\sigma(1-\frac{1}{u_{i(\sigma)}}-\frac{1}{u_{j(\sigma)}}+\frac{1}{qu_{i(\sigma)}u_{j(\sigma)}}) =$$

$$A_{\overline{K}}(u)\prod_i u_i^{1-|\overline{K}\cap E_i|}u_i^{1-|K\cap E_i|}q^{1-|K\cap E_i|}\prod_\sigma(qu_{i(\sigma)}u_{j(\sigma)})^{-1} =$$

$$A_{\overline{K}}(u)q^{s-|K|-|I_0|}\prod u_i^{2-|K_0\cap E_i|+\chi(E_i^\bullet)-2}.$$

It rests to note that $|I_0| = s-1$ and $\chi(E_i^\circ) = \chi(E_i^\bullet) - |K_0\cap E_i|$. □

Lemma 11

$$c_K(n_1,\ldots,n_s) = q^{1-|K|+n}c_{\overline{K}}(-\chi(E_1^\circ)-n_1,\ldots,-\chi(E_s^\circ)-n_s),$$

where $n = \sum_{i=1}^s n_i$.

Proof.

$$A_K(\frac{1}{qu_1},\ldots,\frac{1}{qu_s}) = \sum_{\underline{n}} c_K(n_1,\ldots,n_s)\underline{u}^{-\underline{n}}q^{-n} = q^{1-|K|}\prod u_i^{\chi(E_i^\circ)}\sum_{\underline{z}} c_{\overline{K}}(z_1,\ldots,z_s)\underline{u}^{\underline{z}}.$$

We have

$$z_i+\chi(E_i^\circ) = -n_i,\ \ z_i = -\chi(E_i^\circ)-n_i.$$

□

Theorem 4 *Let μ_α be the Milnor number of C_α, and $(C_\alpha\circ C_\beta)$ is the intersection index of $C_\alpha\circ C_\beta$, $\mu(C)$ is the Milnor number of C. Let $l_\alpha = \mu_\alpha+\sum_{\beta\neq\alpha}(C_\alpha\circ C_\beta)$ and $\delta(C) = (\mu(C)+r-1)/2$. Then*

$$\overline{P}_g(\frac{1}{qt_1},\ldots,\frac{1}{qt_r}) = q^{-\delta(C)}\prod_\alpha t_\alpha^{-l_\alpha}\cdot\overline{P}_g(t_1,\ldots,t_r).$$

The theorem follows from Lemma 11 describing the symmetry of the coefficients $c_K(n)$ and Lemma 6 describing $\overline{P}_g(t_1,\ldots,t_r)$ in terms of $c_K(n)$. The detailed proof is rather technical and can be found in the Appendix.

Corollary 2 *The degree of the polynomial $\overline{P}_g(t_1,\ldots,t_r)$ with respect to the variable t_i is equal to l_i. The greatest monomial in it equals to $q^{\delta(C)}\prod_{i=1}^r t_i^{l_i}$.*

Alternative proof of the symmetry property for the motivic Poincaré series can be found in [14], where it is deduced from the theorem of Campillo, Delgado and Kiyek on the symmetry of the multi-variable Poincaré series of a plane curve singularity.

7 Relation to the Heegaard-Floer knot homology

7.1 Heegaard-Floer homology

In the series of articles (e.g. [18],[19],[20],[22], see also [23]) P. Ozsváth and Z. Szabó constructed new powerful knot invariants, Heegaard-Floer knot (and link) homology. To each link $L = \cup_{i=1}^{r} K_i$ they assign the collection of homology groups $\widehat{HFL}_d(L, \underline{h})$, where d is an integer and $\underline{h}$ belongs to some r-dimensional lattice. Their original description was based on the constructions from the symplectic topology, later ([12],[13]) there were elaborated combinatorial models for them. All of these homologies are invariants of the link L, and they have the following properties ([19], [13]).

First, they give a "categorification" of the Alexander polynomial of L: if $r = 1$, then

$$\sum_h \chi(\widehat{HFL}_*(L, h))t^h = \Delta^s(t),$$

where $\Delta^s(t) = t^{-\deg \Delta/2}\Delta(t)$ is a symmetrized Alexander polynomial of L. If $r > 1$, then

$$\sum_{\underline{h}} \chi(\widehat{HFL}_*(L, h))\underline{t}^h = \prod_{i=1}^{r}(t_i^{1/2} - t_i^{-1/2}) \cdot \Delta^s(t_1, \ldots, t_r).$$

Second, they have the symmetry extending the symmetry of the Alexander polynomial:

$$\widehat{HFL}_d(L, h) \cong \widehat{HFL}_{d-2H}(L, -h),$$

where $H = \sum_{i=1}^{r} h_i$.

These properties are similar to the ones of the polynomials $\overline{P}_g(t)$, and one could be interested in comparison of these objects. It turns out, that for knots (of course, $\overline{P}_g(t)$ is defined only for the algebraic ones) this comparison can be done.

In [22] for the relatively large class of knots, containing all algebraic knots, the following statement was proved.

Theorem 5 *([22]) Let the symmetrized Alexander polynomial have the form*

$$\Delta^s(t) = (-1)^k + \sum_{i=1}^{k}(-1)^{k-i}(t^{n_i} + t^{-n_i})$$

for some integers $0 < n_1 < n_2 < \ldots < n_k$. Let $n_{-j} = -n_j, n_0 = 0$. For $-k \leq i \leq k$ let us introduce the numbers δ_i by the formula

$$\delta_i = \begin{cases} 0, & \text{if } i{=}k \\ \delta_{i+1} - 2(n_{i+1} - n_i) + 1, & \text{if } k\text{-}i \text{ is odd} \\ \delta_{i+1} - 1, & \text{if } k\text{-}i > 0 \text{ is even.} \end{cases}$$

Then $\widehat{HFL}(K, j) = 0$, if j does not coincide with any n_i, and $\widehat{HFL}(K, n_i) = \mathbb{Z}$ belongs to the homological grading δ_i.

In what follows we will need more detailed algebraic structure of the Heegaard-Floer homology which can be described in the following way ([19]).

Consider the ring
$$R = \mathbb{Z}[U_1, \dots, U_r].$$
For every r-component link L there exists a $\mathbb{Z}^r$-filtered chain complex $CFL^-(S^3, L)$ of R-modules, whose filtered homotopy type is an invariant of the link L. Filtrations naturally correspond to the components of the link L. The operators U_i lowers the homological grading by 2 and the filtration level by $\underline{1}$. The homologies of the associated graded object are denoted as $HFL^-(S^3, L)$. If one sets $U_1 = U_2 = \ldots = U_r = 0$, he gets a new $\mathbb{Z}^r$-filtered chain complex of $\mathbb{Z}$-modules, which will be denoted as $\widehat{CFL}(L)$. The homology of the associated graded object are denoted as $\widehat{HFL}(L)$, and they are the homology discussed above.

The filtration on the second complex is compatible with the forgetting of components (proposition 7.1 in [19]). Namely, let M be the two-dimensional graded vector space with one generator in grading 0 and one in grading -1.

Proposition. Let L be an oriented, r-component link in S^3 and distinguish the first component K_1. Consider the complex $\widehat{CFL}(L)$ viewed as a $\mathbb{Z}^{n-1}$-filtered chain complex where the filtration corresponding to the first component is omitted. The filtered homotopy type of this complex is identified with $\widehat{CFL}(L - K_1) \otimes M$.

If we forget all components of L, we get either the complex
$$\hat{CF}(S^3) \otimes M^{r-1},$$
where $\hat{CF}(S^3)$ has one-dimensional homology in grading 0 or
$$CF^-(S^3) = \mathbb{Z}[U],$$
where all U_i acts by the multiplication by U.

This proposition is a direct analogue to the equation (8).

The three-manifolds with simplest Heegaard-Floer homology are the rational homology spheres Y, for which the rank of the Heegaard-Floer homology is equal to the order of the first (singular) homology, i.e.
$$\text{rk} \ \ \widehat{HF(Y)} = |H_1(Y; Z)|.$$
These manifolds are called L-spaces, for example, lens spaces are L-spaces. In the case that some positive surgery on K gives an L-space, we call K an L-space knot. It was proved by M. Hedden in [9] that all algebraic knots (i.e. links of irreducible plane curve singularities) belong to the class of L-space knots.

It was proved in [22], that for the L-space knot K and any filtration level n
$$\text{rk} \ \ H^*(CFL^-(K, n)/U_1(CFL^-(K, n))) = 1. \tag{24}$$
This is a key geometric ingredient in the proof of Theorem 5.

7.2 Matching the answers

Consider the Poincaré polynomial for the Heegaard-Floer homologies:

$$HFL(t,u)=\sum u^d t^s \dim \widehat{HFL}_{d,s}(K).$$

It categorifies the Alexander polynomial in the sense that

$$HFL(t,-1)=t^{-\deg\Delta/2}\Delta(t).$$

Remark that the coefficients in $\overline{P}_g(t,q)$ are always equal to 0 or to ± 1. It can be proved from the equation (15).

Theorem 6 *Take $\overline{P}_g(t,q)$ and let us make a following change in it: $t^\alpha q^\beta$ is transformed to $t^\alpha u^{-2\beta}$, and $-t^\alpha q^\beta$ is transformed to $t^\alpha u^{1-2\beta}$. We get a polynomial $\widetilde{\Delta}_g(t,u)$. Then*

$$\widetilde{\Delta}_g(t^{-1},u)=t^{-\deg\Delta/2}HFL(t,u). \tag{25}$$

Example. For $(3,5)$ torus knot we have

$$P_g(t,q)=1+qt^3+q^2t^5+q^3t^6+\frac{q^4t^8}{1-qt},$$

$$\overline{P}_g(t,q)=1-qt+qt^3-q^2t^4+q^2t^5-q^4t^7+q^4t^8,$$

$$\widetilde{\Delta_g}(t,q)=1+u^{-1}t+u^{-2}t^3+u^{-3}t^4+u^{-4}t^5+u^{-7}t^7+u^{-8}t^8,$$

and

$$HFL(t,u)=t^4+u^{-1}t^3+u^{-2}t+u^{-3}t^0+u^{-4}t^{-1}+u^{-7}t^{-3}+u^{-8}t^{-4}.$$

Proof. To prove (25) we match Theorem 5 with the equation (15).

In the notation of Theorem 5 the non-symmetrized Alexander polynomial equals to

$$\Delta=\sum_{i=k}^{-k}(-1)^{k-i}t^{n_k-n_i}=\sum_{i=0}^{2k}(-1)^i t^{n_k-n_{k-i}},$$

$$P(t)=\frac{\Delta}{1-t}=\sum_{i=0}^{k-1}\sum_{j=n_k-n_{k-2i}}^{n_k-n_{k-2i-1}-1}t^j+\frac{t^{2n_k}}{1-t}.$$

Note that for $i>0$

$$\delta_{k-2i}=\delta_{k-2i+1}-1=\delta_{k-2(i-1)}-2(n_{k-2i+2}-n_{k-2i+1}),$$

so

$$P_g(t,q)=\sum_{i=0}^{k-1}\sum_{j=n_k-n_{k-2i}}^{n_k-n_{k-2i-1}-1}q^{(j-n_k+n_{k-2i})-\delta_{k-2i}/2}t^j+\frac{t^{2n_k}q^{n_k}}{1-qt},$$

$$\overline{P}_g(t,q)=\sum_{i=0}^{k-1}(q^{-\delta_{k-2i}/2}t^{n_k-n_{k-2i}}-q^{-\delta_{k-2i-1}/2}t^{n_k-n_{k-2i-1}})+t^{2n_k}q^{n_k}.$$

Now

$$\widetilde{\Delta_g}(t,u)=\sum_{i=0}^{k-1}(u^{\delta_{k-2i}}t^{n_k-n_{k-2i}}+u^{\delta_{k-2i-1}}t^{n_k-n_{k-2i-1}})+t^{2n_k}u^{-2n_k},$$

$$t^{n_k}\widetilde{\Delta_g}(t^{-1},u)=\sum_{i=0}^{k-1}(u^{\delta_{k-2i}}t^{n_{k-2i}}+q^{\delta_{k-2i-1}}t^{n_{k-2i-1}})+t^{2n_k}u^{-2n_k}=\sum_{i=-k}^{k}u^{\delta_i}t^{n_i}=HFL(t,u).$$

□

7.3 Comparing filtered complexes

In this section we try to describe the relation between the knot filtration on the Heegaard-Floer complexes and the filtration on the space of functions defined by a curve.

To be more close to the algebraic setup, we reverse all signs for filtrations and for the homological (Maslov) grading as well (so we get cohomology groups). The Alexander grading is also changed to get the non-symmetrized Alexander polynomial. In another words, the Poincaré polynomial of the resulting cohomology coincides with $\widetilde{\Delta_g}(t,u^{-1})$. The operator U will now increase the homological grading by 2.

Consider a $\mathbb{Z}_{\geq 0}$-indexed filtration J_n by vector subspaces (with finite codimensions) on a infinite-dimensional complex vector space J_0. It induces a filtration by projective subspaces $\mathbb{P}J_n$ on $\mathbb{P}J_0=\mathbb{CP}^\infty$:

$$\mathbb{P}J_0\overset{j_1}{\hookleftarrow}\mathbb{P}J_1\overset{j_2}{\hookleftarrow}\mathbb{P}J_2\overset{j_3}{\hookleftarrow}\ldots,$$

so we have a sequence of corresponding Gysin maps in cohomology:

$$H^*(\mathbb{P}J_0)\overset{(j_1)_*}{\hookleftarrow}H^{*-2\cdot\mathrm{codim}_{J_1}}\mathbb{P}J_1\overset{(j_2)_*}{\hookleftarrow}H^{*-2\cdot\mathrm{codim}_{J_2}}\mathbb{P}J_2\overset{(j_3)_*}{\hookleftarrow}\ldots.$$

We get a $\mathbb{Z}_{\geq 0}$-indexed filtration

$$F_k=(j_k)_*(H^*(\mathbb{P}J_k))$$

in $H^*(\mathbb{CP}^\infty)=\mathbb{Z}[U]$, which is compatible with the multiplication by U. If we also know (as for the filtration defined by the orders on the curve), that $\dim J_k/J_{k+1}\leq 1$, we conclude that U increase the filtration level at least by 1.

The motivic Poincaré series in this setup can be written as

$$P_g(t,q)=\sum_{k,n}t^kq^{n/2}\dim H^n(F_k/F_{k+1}).$$

The situation is similar to the Heegaard-Floer complexes, but U may increase the filtration level more that by 1. To avoid this problem, we should modify the complex.

Example. Consider the following filtered complex T: it has generators U^ka_0, U^ka_1 and U^ka_2. The homological degree of U^la_j equals to $2l+j$ and its filtration level equals to $l+j$. The differential is defined as

$$d(a_1)=a_2+Ua_0.$$

One can check that

$$\sum_{k,n} t^k u^n \dim H^n(T_k/T_{k+1}) = 1 + u^2t^2 + u^4t^3 + u^6t^4 + \dots$$

(so this complex corresponds to minus-version of the Heegaard-Floer homology of the trefoil knot) and $\mathrm{rk} H^*(T_k/UT_k) = 1$ for all k. Remark that if $\widehat{T}^k = T_k/UT_{k-1}$, then

$$\sum_{k,n} t^k u^n \dim H^n(\widehat{T}_k/\widehat{T}_{k+1}) = 1 + ut + u^2t^2,$$

what is the Poincaré polynomial for the hat-version of the Heegaard-Floer homology of the trefoil.

Let us turn to the general case. Consider the complex

$$\mathcal{C}_0 = F_0[U_1] + (F_0[1])[U_1] \tag{26}$$

with the filtration

$$\mathcal{C}_n = \bigoplus_{k+l=n} U_1^l F_k \oplus \bigoplus_{k+l=n-1} U_1^l F_k[1]$$

and the natural action of the operator U_1 of homological degree 2. The differential is given by the equation

$$d(x) = U_1 \cdot x + Ux.$$

One can check that this differential preserves the filtration $\mathcal{C}_n$ and commutes with U_1.

Lemma 12

$$H^*(\mathcal{C}_n/\mathcal{C}_{n+1}) = F_n/F_{n+1}, \mathrm{rk} \quad H^*(\mathcal{C}_n/U_1(\mathcal{C}_n)) = 1.$$

Proof. We have

$$\mathcal{C}_n/\mathcal{C}_{n+1} = \bigoplus_{k+l=n} U_1^l(F_k/F_{k+1}) \oplus \bigoplus_{k+l=n-1} U_1^l(F_k/F_{k+1})[1].$$

Since the U_1-increasing component of the differential

$$d_1(U_1^l x[1]) = U_1^{l+1} x$$

gives the isomorphism

$$d_1 : U_1^l(F_k/F_{k+1}) \to U_1^{l+1}(F_k/F_{k+1}),$$

we have

$$H^*(\mathcal{C}_n/\mathcal{C}_{n+1}) = F_n/F_{n+1}.$$

Also we have

$$\mathcal{C}_n/U_1(\mathcal{C}_n) = F_0 \oplus F_0[1] \bigoplus_{k+l=n, l>0} U_1^l(F_k/F_{k+1}) \oplus \bigoplus_{k+l=n-1, l>0} U_1^l(F_k/F_{k+1})[1],$$

and up to the isomorphisms d_1 we have the complex $F_0 \oplus F_0[1]$ with the differential

$$d_2(x[1]) = Ux,$$

so

$$rk\ \ H^*(\mathcal{C}_n/U_1(\mathcal{C}_n)) = 1.$$

□

The properties of the complex $\mathcal{C}_0$ are similar to the ones of the complex $CFL^-(K)$. More precisely, the calculations of [22] (lemma 3.1 and lemma 3.2) imply the following

Proposition. Suppose that a cochain complex $\mathcal{C}$ has a filtration $\mathcal{C}_k,\ \ k \geq 0$ and an injective operator U of homological degree 2 acting on it such that

1)$U(\mathcal{C}_k) \subset C_{k+1}$ and $U^{-1}(\mathcal{C}_k) \subset \mathcal{C}_{k-1}$ (this means that U increase the level of filtration exactly by 1)

2)$H^*(\mathcal{C}_k/U(\mathcal{C}_k))$ has rank 1 for all k.

Then

3) For all k the rank of $H^*(\mathcal{C}_k/\mathcal{C}_{k+1})$ is at most 1.

Let $\{0, \sigma_1, \sigma_2, \ldots\}$ is the set of k such that this rank is 1. Then

4) $H^*(\mathcal{C}_{\sigma_k}/\mathcal{C}_{\sigma_k+1})$ belongs to degree $2k$.

Let

$$Q(t, q) = \sum_{k=0}^{\infty} q^k t^{\sigma_k},\ \ \overline{Q}(t, q) = Q(t, q)(1 - qt).$$

Let us make a following change in $\overline{Q}$: $t^\alpha q^\beta$ is transformed to $t^\alpha u^{2\beta}$, and $-t^\alpha q^\beta$ is transformed to $t^\alpha u^{2\beta-1}$.

5) The result is equal to

$$\sum_{k,n} t^k u^n \dim H^n(\mathcal{C}_k/(\mathcal{C}_{k+1} + U\mathcal{C}_{k-1})).$$

The second condition is analogous to the equation (24) for the Heegaard-Floer homology of the L-space knots.

The last result can be reformulated as follows. Consider the complex $\widehat{\mathcal{C}_k} = \mathcal{C}_k/U\mathcal{C}_{k-1}$, then the last homology is the homology of the associated graded object $\widehat{\mathcal{C}}_k/\widehat{\mathcal{C}}_{k-1}$. The multiplication by $1 - qt$ corresponds to the exact sequence

$$0 \to \mathcal{C}_{k-1}/\mathcal{C}_k \xrightarrow{U} \mathcal{C}_k/\mathcal{C}_{k+1} \to \widehat{\mathcal{C}}_k/\widehat{\mathcal{C}}_{k+1} \to 0.$$

As a corollary we get that the series $Q(t, 1)$ determines completely all discussed cohomology. Since for the filtered complexes $\mathcal{C}$ and CFL^- we have $Q(t, 1) = \Delta(t)/(1 - t)$ for both, we have the equality of the cohomology of the associated graded objects and the more clear proof of the Theorem 6. As an another corollary, we get the equation

$$H^*(CFL^-(S^3)/CFL_s^-(S^3,K)) \cong H^*(\mathbb{P}(\mathcal{O}/J_s)), \tag{27}$$

which looks more geometric than the Theorem 6.

Remarks.

1. It would be interesting to construct the analogous $\mathbb{Z}^n$-filtered complex of $\mathbb{Z}[U_1,\ldots,U_n]$ for multi-component links which would carry the information about the Poincaré series of the corresponding multi-index filtration.

2. It would be also interesting to compare these results with the ones of [15], [16] and [17] computing the Seiberg-Witten and Heegaard-Floer invariants of links of surface singularities.

7.4 Example: A_{2n-1} singularities

Since the algorithm of computation of the (reduced) motivic Poincaré series is quite complicated, it is useful to have a series of answers where the motivic Poincaré series and the Heegaard-Floer link homology can be computed.

Proposition. Consider the singularity of type A_{2n-1} given by the equation

$$y^2 = x^{2n}.$$

From the topological viewpoint this corresponds to the 2-component link, whose components are unknotted, all intersections are positive and the linking number of the components equals to n. Then

$$P_g(t_1,t_2) = 1 + qt_1t_2 + \ldots + q^{n-1}t_1^{n-1}t_2^{n-1} + \frac{q^n(1-q)t_1^nt_2^n}{(1-t_1q)(1-t_2q)}.$$

Proof. For the proof we use the equation (13). Parametrisations of the components are

$$(x(t_1),y(t_1)) = (t_1,t_1^n), \quad \text{and} \quad (x(t_2),y(t_2)) = (t_2,-t_2^n),$$

so

$$x^ay^b|_{C_1} = t_1^{a+bn}, \quad x^ay^b|_{C_2} = (-1)^bt_2^{a+bn}.$$

If $a < n$, then every function with order a on C_1 has a form $x^a + \ldots$, so its order on C_2 is also equal to a.

For every $a, b \geq n$ consider the function $x^{a-n}(x^n+y) + x^{b-n}(x^n-y)$. Its restrictions on C_1 and C_2 are respectively equal to $2t_1^a$ and $2t_2^b$, therefore

$$\dim J_{a,b}/J_{a+1,b} = \dim J_{a,b}/J_{a,b+1} = 1.$$

The codimensions $h(v_1,v_2)$ are equal to v_1+v_2-n, if $v_1,v_2 \geq n$, to v_2, if $v_1 < n, v_2 \geq n$, to v_1, if $v_2 < n, v_1 \geq n$, and to $\max(v_1,v_2)$, if $0 \leq v_1, v_2 < n$. We have

$$L_g^{A_{2n-1}}(t_1,t_2,q) = \sum_{0\leq \max(v_1,v_2);\min(v_1,v_2)<n} t_1^{v_1}t_2^{v_2}q^{\max(v_1,v_2)} + (1+q)\sum_{v_1,v_2=n}^{\infty} t_1^{v_1}t_2^{v_2}q^{v_1+v_2-n},$$

hence

$$L_g^{A_{2n-1}}(t_1-1)(t_2-1) = -1+(1-q)t_1t_2+\ldots+(q^{n-2}-q^{n-1})t_1^{n-1}t_2^{n-1}+q^{n-1}(1-q+q^2)t_1^nt_2^n$$
$$+\frac{q^{n+1}t_1^{n+1}t_2^n(q-1)}{1-qt_1}+\frac{q^{n+1}t_1^nt_2^{n+1}(q-1)}{1-qt_2}+\frac{q^nt_1^{n+1}t_2^{n+1}(1+q)(1-q)^2}{(1-qt_1)(1-qt_2)},$$

and

$$P_g^{A_{2n-1}} = \frac{L_g^{A_{2n-1}}(t_1-1)(t_2-1)}{t_1t_2-1} = 1+qt_1t_2+\ldots+q^{n-1}t_1^{n-1}t_2^{n-1}+\frac{q^n(1-q)t_1^nt_2^n}{(1-qt_1)(1-qt_2)}.$$

□

Corollary 3

$$\overline{P}_g^{A_{2n-1}}(t_1,t_2) = [1+(q+q^2)t_1t_2+\ldots+(q^{n-1}+q^n)t_1^{n-1}t_2^{n-1}+q^nt_1^nt_2^n] \tag{28}$$
$$-(t_1+t_2)[q+q^2t_1t_2+\ldots+q^nt_1^{n-1}t_2^{n-1}].$$

In [19] Ozsváth and Szabó computed the Heegaard-Floer homology of the corresponding links. In their notation the answer has the following form (everywhere we write the Poincaré polynomials of the corresponding complexes). Let

$$Y_{(d)}^l(t_1,t_2,u) = u^d(t_1^l+t_1^{l-1}t_2+\ldots+t_2^l)+u^{d-1}(t_1^{l-1}+\ldots+t_2^{l-1}),$$
$$B_{(d)}(t_1,t_2,u) = u^d+(t_1+t_2)u^{d+1}+u^{d+2}t_1t_2.$$

Then

$$HFL_{A_{2n-1}}(t_1,t_2,u) = Y_{(0)}^0t_1^{n/2}t_2^{n/2}+Y_{(-1)}^1t_1^{n/2-1}t_2^{n/2-1}+\sum_{i=2}^{n}B_{(-2i)}t_1^{n/2-i}t_2^{n/2-i}.$$

Since $Y_{(0)}^0=1$ and $Y_{(-1)}^1=u^{-1}(t_1+t_2)+u^{-2}$ one can simplify this as

$$HFL_{A_{2n-1}}(t_1,t_2,u) = t_1^{n/2}t_2^{n/2}+(u^{-1}(t_1+t_2)+u^{-2})t_1^{n/2-1}t_2^{n/2-1}$$
$$+\sum_{i=2}^{n}(u^{-2i}+(t_1+t_2)u^{-2i+1}+u^{-2i+2}t_1t_2)t_1^{n/2-i}t_2^{n/2-i},$$

so

$$t_1^{n/2}t_2^{n/2}HFL_{A_{2n-1}}(t_1^{-1},t_2^{-1},u) = 1+(u^{-1}(t_1+t_2)+u^{-2}t_1t_2)$$
$$+\sum_{i=2}^{n}(u^{-2i}t_1^it_2^i+(t_1+t_2)u^{-2i+1}t_1^{i-1}t_2^{i-1}+u^{-2i+2}t_1^{i-1}t_2^{i-1}) =$$
$$[1+2u^{-2}t_1t_2+\ldots+2u^{-2n+2}t_1^{n-1}t_2^{n-1}+u^{-2n}t_1^nt_2^n]$$
$$-(t_1+t_2)[u^{-1}+u^{-3}t_1t_2+\ldots+u^{-2n+1}t_1^{n-1}t_2^{n-1}].$$

The last expression is similar to (28) in analogy with the Theorem 6.

8 Appendix

Proof of Lemma 4.

We have

$$\sum u^{\hat{n}_i}\phi_i(I,K,\hat{n}) = \sum_j \sum_{\hat{n}=j+f_i(K,I)}^{\infty} u^{\hat{n}_i}(-1)^j \binom{1-\chi(E_i^\circ)-f_i(I,K)}{j} q^j =$$

$$\frac{u^{f_i(K,I)}}{1-u}\sum_j(-1)^j\binom{1-\chi(E_i^\circ)-f_i(I,K)}{j}(uq)^j = \frac{u^{f_i(K,I)}}{1-u}(1-uq)^{1-\chi(E_i^\circ)-f_i(I,K)},$$

and

$$\sum u^{\hat{n}} G(K,I,\hat{n}) = q^{|I|}(1-q)^{|I|+|K|}\prod_i \frac{u_i^{f_i(K,I)}}{1-u_i}(1-u_iq)^{1-\chi(E_i^\circ)-f_i(I,K)}.$$

□

Proof of Lemma 5

$$A_K(u) = \sum_I q^{|I|}(1-q)^{|I|}\sum_{K_1}(-1)^{|K|-|K_1|}(1-q)^{|K_1|}\sum_n u^n\prod_i \phi_i(I,K_1,n).$$

We have

$$\sum_n u^n \prod_i \phi_i(I,K_1,n) = \prod_i \frac{u_i^{f_i(K,I)}(1-u_iq)^{1-\chi(E_i^\circ)-f_i(I,K)}}{1-u_i}.$$

Now

$$\sum_{K_{1i}\subset(K\cap E_i)}(-1)^{|K\cap E_i|-|K_{i1}|}(1-q)^{|K_{1i}|}\frac{1}{1-u_i}u_i^{f_i(K_1,I)}(1-u_iq)^{1-\chi(E_i^\circ)-f_i(I,K_1)} =$$

$$\frac{1}{1-u_i}u_i^{f_i(K,I)}(1-u_iq)^{1-\chi(E_i^\circ)-f_i(K,I)}\times$$
$$\sum_{K_{1i}}(-1)^{|K\cap E_i|-|K_{1i}|}(1-q)^{|K_{1i}|}u_i^{|K_{1i}|-|K\cap E_i|}(1-u_iq)^{|K\cap E_i|-|K_{1i}|} =$$
$$\frac{1}{1-u_i}u_i^{f_i(K,I)}(1-u_iq)^{1-\chi(E_i^\circ)-f_i(K,I)}(1-q-\frac{1-u_iq}{u_i})^{|K\cap E_i|} =$$
$$\frac{1}{1-u_i}(-1)^{|K\cap E_i|}u_i^{f_i(K,I)-|K\cap E_i|}(1-u_iq)^{1-\chi(E_i^\circ)-f_i(K,I)}(1-u_i)^{|K\cap E_i|}.$$

Remark that $f_i(K,I)-|K\cap E_i| = f_i(I)$ and

$$\chi(E_i^\circ)+f_i(K,I) = \chi(E_i^\bullet)-|K_0\cap E_i|+|K\cap E_i|+f_i(I),$$

hence the last expression can be rewritten in a form

$$(-1)^{|K\cap E_i|}u_i^{f_i(I)}(1-u_iq)^{1-\chi(E_i^\bullet)+|\overline{K}\cap E_i|-f_i(I)}(1-u_i)^{|K\cap E_i|-1}.$$

Also

$$\sum_I q^{|I|}(1-q)^{|I|}\prod_i u_i^{f_i(I)}(1-u_iq)^{-f_i(I)}=\prod_\sigma(1+q(1-q)u_{i(\sigma)}u_{j(\sigma)}(1-u_{i(\sigma)}q)^{-1}(1-u_{j(\sigma)}q)^{-1})=$$

$$\prod_i(1-u_iq)^{\chi(E_i^\bullet)-2}\prod_\sigma(1-qu_{i(\sigma)}-qu_{j(\sigma)}+qu_{i(\sigma)}u_{j(\sigma)}).$$

Therefore

$$A_K(u)=(-1)^{|K|}\prod_i(1-u_iq)^{1-\chi(E_i^\bullet)+|\overline{K}\cap E_i|}(1-u_i)^{|K\cap E_i|-1}\times$$

$$\times\prod_i(1-u_iq)^{\chi(E_i^\bullet)-2}\prod_\sigma(1-qu_{i(\sigma)}-qu_{j(\sigma)}+qu_{i(\sigma)}u_{j(\sigma)})=$$

$$(-1)^{|K|}\prod_i(1-u_iq)^{|\overline{K}\cap E_i|-1}(1-u_i)^{|K\cap E_i|-1}\prod_\sigma(1-qu_{i(\sigma)}-qu_{j(\sigma)}+qu_{i(\sigma)}u_{j(\sigma)}).$$

□

Proof of Lemma 7

We have to prove that $\widetilde{H}_P=0$ at $u_\beta=1$ for $\beta\in E(P)$. Suppose that E_β is intersected by $E_{\alpha_1},\ldots,E_{\alpha_k}$. For every set E of divisors not containing E_β let us compare the summands corresponding to E and to $E\cup E_\beta$.

For E at $u_\beta=1$ we have

$$\prod_{i\neq\beta}u_i^{-\sum a_{ij}\mu_j}(-1)^{|K_0\cap E|}q^{\Delta(E)}\prod_{i\in E}(q-u_i)^{k_i-1}(1-q)^{k_\beta-1}\prod_{i\notin(P\cup E)}(1-qu_i)^{k_i-1}$$

$$\times\prod_{\sigma\notin E_\beta}(1-q^{1-\mu_{i(\sigma)}(E)}u_{i(\sigma)}-q^{1-\mu_{j(\sigma)}(E)}u_{j(\sigma)}+q^{1-\mu_{i(\sigma)}(E)-\mu_{j(\sigma)}(E)}u_{i(\sigma)}u_{j(\sigma)})\cdot(1-q)^k.$$

For $E\cup E_1$ at $u_\beta=1$ we have

$$\prod_{j=1}^k u_{\alpha_j}\prod_{i\neq\beta}u_i^{-\sum a_{ij}\mu_j}(-1)^{k_\beta+|K_0\cap E|}q^{\Delta(E\cup E_1)}(q-1)^{k_\beta-1}\prod_{i\in E}(q-u_i)^{k_i-1}\prod_{i\notin(E\cup P)}(1-qu_i)^{k_i-1}$$

$$\times\prod_{\sigma\notin E_\beta}(1-q^{1-\mu_{i(\sigma)}(E)}u_{i(\sigma)}-q^{1-\mu_{j(\sigma)}(E)}u_{j(\sigma)}+q^{1-\mu_{i(\sigma)}(E)-\mu_{j(\sigma)}(E)}u_{i(\sigma)}u_{j(\sigma)})\cdot\prod_{j=1}^k(1-q)q^{-\mu_{\alpha_j}(E)}u_{\alpha_j}.$$

It rests to note that $\Delta(E\cup E_\beta)-\Delta(E)=\sum_{j=1}^k\mu_{\alpha_j}(E)$.

□

Proof of Lemma 8.

$$\sum_n u^n\sum_{E\subset E(P)}q^{-\sum_{i\in E}n_i-\Delta(E)-\sum_{i\in E}a_{ii}-|E|}q^{|K_0\cap E|}\times c_{P\cup E}(n_i+\sum a_{ij}\mu_j(E))=$$

$$\sum_{E\subset E(P)}\prod u_i^{-\sum a_{ij}\mu_j(E)}\cdot q^{\sum a_{ij}\mu_i(E)\mu_j(E)}\cdot q^{-\Delta(E)-\sum_{i\in I}a_{ii}+|K_0\cap E|-|E|}$$

$$\times \sum_{n_1} \prod_i (u_i q^{-\mu_i(E)})^{n_{1i}} \cdot c_{P\cup E}(n_1) =$$

$$\sum_{E\subset E(P)} \prod u_i^{-\sum a_{ij}\mu_j(E)} \cdot A_{P\cup E}(u_i q^{-\mu_i(E)}) q^{\Delta(E)+|K_0\cap E|-|E|} =$$

$$(-1)^{|P|} \sum_{E\subset E(P)} \prod u_i^{-\sum a_{ij}\mu_j(E)} \cdot (-1)^{|K_0\cap E|} q^{\Delta(E)+|K_0\cap E|-|E|} \prod_{i\in E} [(1-u_i)^{-1}(1-u_i q^{-1})^{k_i-1}]$$

$$\times \prod_{i\in P} [(1-qu_i)^{k_i-p_i-1}(1-u_i)^{p_i-1}] \prod_{i\notin(P\cup E)} [(1-qu_i)^{k_i-1}(1-u_i)^{-1}]$$

$$\times \prod_\sigma (1-q^{1-\mu_{i(\sigma)}(E)}u_{i(\sigma)} - q^{1-\mu_{j(\sigma)}(E)}u_{j(\sigma)} + q^{1-\mu_{i(\sigma)}(E)-\mu_{j(\sigma)}(E)}u_{i(\sigma)}u_{j(\sigma)}) =$$

$$(-1)^{|P|} \prod_{i\in P} [(1-qu_i)^{k_i-p_i-1}(1-u_i)^{p_i-1}] \cdot \frac{1}{\prod_{i\in E(P)}(1-u_i)}$$

$$\times \sum_{E\subset E(P)} (-1)^{|K_0\cap E|} \cdot \prod u_i^{-\sum a_{ij}\mu_j(E)} \cdot q^{\Delta(E)} \prod_{i\in E} (q-u_i)^{k_i-1} \prod_{i\notin E} (1-qu_i)^{k_i-1}$$

$$\times \prod_\sigma (1-q^{1-\mu_{i(\sigma)}(E)}u_{i(\sigma)} - q^{1-\mu_{j(\sigma)}(E)}u_{j(\sigma)} + q^{1-\mu_{i(\sigma)}(E)-\mu_{j(\sigma)}(E)}u_{i(\sigma)}u_{j(\sigma)}).$$

□

Proof of Theorem 4.

Let $k_i = |K_0 \cap E_i|$. From Lemma 6 we get

$$\overline{P}_g(\frac{1}{qt_1}, \ldots, \frac{1}{qt_r}) = (t_1 \cdot \ldots \cdot t_r)^{-1} \sum_{\underline{n}} \underline{t}^{-M\underline{n}} q^{-\sum m_{ij}k_i n_j} q^{F(n)-\sum n_i} \sum_K t_{\overline{K}} c_K(n) =$$

$$\underline{t}^{-\underline{1}-M\chi(\underline{E}^\circ)} \sum_{\underline{n}} \underline{t}^{M(\chi(\underline{E}^\circ)-\underline{n})} q^{-\sum m_{ij}k_i n_j} q^{F(n)-\sum n_i}$$

$$\times \sum_K q^{1-|K|+n} \cdot t_{\overline{K}} \cdot c_{\overline{K}}(-\chi(E_i^\circ)-n_i). \tag{29}$$

Let

$$\xi_i = -\chi(E_i^\circ), \quad \underline{n}_1 = \underline{\xi} - \underline{n}.$$

Then

$$F(n) - \sum n_i = \frac{1}{2}[\sum m_{ij}n_i n_j + \sum m_{ij} n_i \chi(E_j^\bullet) - \sum n_i],$$

so

$$2[F(n_1) - \sum n_{1i} - F(n) + \sum n_i] =$$

$$\sum m_{ij}(\xi_i - n_i)(\xi_j - n_j) + \sum m_{ij}(\xi_i - n_i)\chi(E_j^\bullet) - \sum(\xi_i - n_i)$$

$$-\sum m_{ij}n_i n_j - \sum m_{ij} n_i \chi(E_j^\bullet) + \sum n_i =$$

$$-2\sum m_{ij}(\xi_i + \chi(E_i^\bullet))n_j + 2\sum n_j + 2(F(\xi) - \sum \xi_i) =$$

$$-2\sum m_{ij}k_in_j + 2\sum n_j + 2(F(\xi) - \sum \xi_i).$$

Thus (29) is equal to

$$t^{-1-M\xi}q^{-F(\xi)+\sum\xi_i}q^{1-|K_0|}\sum t^{Mn_1}q^{F(n_1)-\sum n_{1i}}\sum_K t_{\overline{K}}q^{|\overline{K}|}c_{\overline{K}}(n_1).$$

It rests to compute the powers of t_α and of q.

Remark that $\sum \xi_i = |K_0| - 2$, so $\sum \xi_i + 1 - |K_0| = -1$.

Also

$$2F(\xi) = \sum m_{ij}k_ik_j - 2\sum m_{ij}k_i\chi(E_j^\bullet) + \sum m_{ij}\chi(E_i^\bullet)\chi(E_j^\bullet) +$$
$$\sum m_{ij}k_i\chi(E_j^\bullet) - \sum m_{ij}\chi(E_i^\bullet)\chi(E_j^\bullet) + \sum \xi_i =$$
$$\sum m_{ij}k_ik_j - \sum m_{ij}k_i\chi(E_j^\bullet) + |K_0| - 2.$$

The formula of A'Campo ([1]) says that

$$1 - \mu = \sum m\chi(S_m) = \sum \chi(E_i^\circ)m_{ij}k_j = \sum m_{ij}(\chi(E_i^\bullet) - k_i)k_j,$$

so

$$2F(\xi) = \mu - 1 + |K_0| - 2 = 2\delta - 2.$$

Thus $-F(\xi) - 1 = -\delta$.

Also for every α one has

$$1 - \mu_\alpha = \sum_{j\neq i(\alpha)} m_{i(\alpha)j}\chi(E_j^\bullet) + m_{i(\alpha),i(\alpha)}(\chi(E_{i(\alpha)}^\bullet) - 1),$$

and for $\beta \neq \alpha$

$$C_\alpha \circ C_\beta = m_{i(\alpha),i(\beta)},$$

so

$$\sum_{\beta\neq\alpha} C_\alpha \circ C_\beta = \sum_{j\neq i(\alpha)} m_{i(\alpha),j}k_j + m_{i(\alpha),i(\alpha)}(k_{i(\alpha)} - 1)$$

and

$$1 - \mu_\alpha - C_\alpha \circ C_\beta = \sum_j m_{i(\alpha),j}\chi(E_j^\circ).$$

□

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