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KAHN'S CORRESPONDENCE AND COHEN–MACAULAY MODULES OVER ABSTRACT SURFACE AND CURVE SINGULARITIES

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Abstract. We generalize Kahn's correspondence between Cohen–Macaulay modules over normal surface singularities over an algebraically closed field and vector bundles over some projective curves to abstract surface singularities, which need not be algebras over a field. As a consequence, we also generalize to the abstract case the Drozd–Greuel criterion for tameness of curve singularities [4].

1. Introduction

In [7] Kahn established a connection between the categories of (maximal) Cohen–Macaulay modules over normal surface singularities and vector bundles over a special divisor on the exceptional locus of the resolution of this singularity. In the case of arbitrary normal surface singularity Kahn's conditions from [7, Theorem 1.4] are quite cumbersome and difficult to deal with. However, in the case of minimally elliptic singularities there is a simple and explicit description of this connection. Kahn himself [7] and Drozd, Greuel and Kashuba [6] used this result to obtain a classification of Cohen–Macaulay modules over simple elliptic and cusp singularities, to establish a tame-wild criterion for minimally elliptic singularities, as well to study Cohen–Macaulay modules over some curve singularities. In particular, in [6] it is obtained a classification of Cohen–Macaulay modules over the curve singularities of type T_{pq}, which gave a new proof that they are Cohen–Macaulay tame. Previously this result was obtained quite implicitly using deformations and semicontinuity theorems [4].

In this paper we extend these results to a more general situation. Namely, we consider *abstract surface and curve singularities*, that is complete noetherian rings of Krull dimension 2 or 1, which need not be algebras over an algebraically closed field or over a field at all. We establish that Kahn's results hold for this situation too, though some details of the proof change. Using them, we also generalize the tame-wild criteria from [4, 6] to the abstract situation. Unfortunately, the case when the residue field is of characteristic 2 remains unconsidered, since we have to use the suspension trick of Knörrer [10] which does not work in this case. We also have to change the definition of T_{pq} singularities using a parametrization description instead of equations. It is necessary since the classification of such singularities in positive characteristic, moreover, in the abstract situation is still inaccessible, though some important results have been obtained in [2]. Note that in the abstract case one cannot use the deformation arguments from [4], since there are no appropriate results about semicontinuity (in the paper [3] only the case of algebras over an algebraically closed field is considered and this restriction is unavoidable there).

2. The result of Kahn

We are going to generalize the results of Kahn [7] in the following situation. Let (X, x) be a spectrum of a local complete and normal noetherian ring $(R, \mathfrak{m})$ of Krull dimension 2 with maximal ideal $\mathfrak{m}$ and residue field k. We call such schema an (*abstract*) *normal surface singularity*, since in general the ring R is not supposed to be an algebra over the field. Such singularity is *isolated*, that means its closed point x, which corresponds to $\mathfrak{m}$, is a unique singular point of X. It is known from [11] that there exists a *resolution* of (X, x), that is a birational projective morphism $\pi : (\widetilde{X}, E) \to (X, x)$, where $\widetilde{X}$ is smooth and π induces an isomorphism $\widetilde{X} \setminus E \simeq X \setminus \{x\}$, where $E = \pi^{-1}(x)_{\text{red}}$ is an exceptional locus. The resolution is obtained by finite sequence $\overline{X} = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow ... \leftarrow X_n$ where $\overline{X}$ is normalization of X and each X_{i+1} is obtained from X_i by blowing up all the singular points of X_i and normalizing the resulting surface $\widetilde{X} = \overline{X}_n$. Put $E = \bigcup_{i=1}^{l} E_i$, where E_i are the irreducible components of E (all of them are projective curves over k). Note that $\widetilde{X}$ is minimal resolution if and only if there is no component E_i which is a smooth rational curve with self-intersection index $E_i \cdot E_i = -1$. Recall some necessary definitions we need (for details cf. [7]).

Definition 1. A module M over R is called *reflexive* if $M^{\vee\vee} \simeq M$, where $M^{\vee} = \operatorname{Hom}_R(M, R)$. In our case such modules coincide with maximal Cohen-Macaulay modules over R and we denote by $\operatorname{MCM}(X)$ the category of reflexive R-modules.

A locally free sheaf $\mathcal{F}$ on $\widetilde{X}$ is called *full* if $\mathcal{F} \simeq (\pi^* M)^{\vee\vee}$ for some $M \in \operatorname{MCM}(X)$.

An effective divisor $Z > 0$ on $\widetilde{X}$ is called a *reduction cycle* if

(i) $\mathcal{O}_Z(-Z)$ is generically generated by global sections (i.e. generated outside a finite set);

(ii) $\mathrm{H}^1(E, \mathcal{O}_Z(-Z)) = 0$;

(iii) $\omega_Z^{\vee}$ is generically generated by global sections, where $\omega_Z = \omega_{\widetilde{X}}(Z) \otimes \mathcal{O}_Z$.

For a reduction cycle Z the functor $R_Z : \operatorname{MCM}(X) \to \operatorname{VB}(Z)$ from category of reflexive modules to category of locally free sheaves on Z is defined by $R_Z(M) = (\pi^* M)^{\vee\vee} \otimes \mathcal{O}_Z$.

Theorem 2. [7, Theorem 1.4] *Let $\pi : (\widetilde{X}, E) \to (X, x)$ be a resolution of a normal surface singularity, and let Z be a reduction cycle on $\widetilde{X}$. Then the functor R_Z maps non-isomorphic objects from $\operatorname{MCM}(X)$ to non-isomorphic ones from $\operatorname{VB}(Z)$ and a vector bundle $F \in \operatorname{VB}(Z)$ is isomorphic to $R_Z M$ for some reflexive module M if and only if it is generically generated by global sections and there is an extension of F to a vector bundle F_2 on $2Z$ such that the exact sequence*

$$0 \longrightarrow F(-Z) \longrightarrow F_2 \longrightarrow F \longrightarrow 0$$

induces a monomorphism $\mathrm{H}^0(E, F(Z)) \to \mathrm{H}^1(E, F)$.

The proof of this theorem is divided into the following two propositions. Note that in our case the proof of Proposition 3 is absolutely analogous to the original one, so we can omit it here.

Proposition 3. [7, Proposition 1.6] *Let $Z > 0$ be a cycle on $\widetilde{X}$ such that $\mathcal{O}_Z(-Z)$ is generically generated by global sections and $\mathrm{H}^1(E, \mathcal{O}_Z(-Z)) = 0$. Assume that $\mathcal{F}$ is a locally free sheaf on $\widetilde{X}$ and denote by $F = \mathcal{F} \otimes \mathcal{O}_Z$ its restriction to Z. Then, $\mathcal{F}$ is full if and only if*

(i) F is generically generated by global sections;

(ii) The coboundary map $\mathrm{H}^0(E, F(Z)) \to \mathrm{H}^1(E, F)$ is injective.

Proposition 4. [7, Proposition 1.9] *Let Z be a reduction cycle on $\widetilde{X}$ and assume that F is a locally free sheaf on Z such that*

(i) F is generically generated by global sections;

(ii) There is an extension of F to a locally free sheaf over $\mathcal{O}_{2Z}$ such that $\mathrm{H}^0(E, F(Z)) \to \mathrm{H}^1(E, F)$ is injective.

Then there is a full sheaf $\mathcal{F}$ on $\widetilde{X}$ with $\mathcal{F} \otimes \mathcal{O}_Z \cong F$. The extension of F to a full sheaf on $\widetilde{X}$ is unique up to isomorphism.

Proof. We follow the proof of Kahn with some obvious changes, which are necessary due to the fact that an underlying field may not exist, or if existing it may not be algebraically closed. We omit most of details that shall not be changed in our case. Fix an extension F_2 of sheaf F. The sheaf $\mathcal{F}$ from Proposition 3 is constructed as the projective limit $\varprojlim F_n$ of sheaves F_n, where sheaf F_{n+1} is such locally free extension of the sheaf F_n over $\mathcal{O}_{(n+1)Z}$ that the following sequence is exact

$$0 \to F(-nZ) \to F_{n+1} \to F_n \to 0.$$

Note that these extensions exist because the obstructions for the existence belong to $\mathrm{H}^2(E, F^\vee \otimes F(-nZ))$, which is zero by dimensional reasons.

To prove the uniqueness it is enough to show that at every step of the construction given above the next sheaf is defined uniquely up to isomorphism.

We can consider an extension of the sheaf F_n to a locally-free sheaf F_{n+1} over $\mathcal{O}_{(n+1)Z}$ as an element of the group $\mathrm{Ext}^1_{\mathcal{O}_{(n+1)Z}}(F_n, F(-nZ))$. Choosing an element e from this space such that e is locally-free over $\mathcal{O}_{(n+1)Z}$, we obtain that all locally-free extensions of F_n correspond to the points of the coset $e + \mathrm{Ext}^1_{\mathcal{O}_{nZ}}(F_n, F(-nZ))$. There is an obvious action of the group of automorphisms of the sheaf F_n on the group $\mathrm{Ext}^1_{\mathcal{O}_{(n+1)Z}}(F_n, F(-nZ))$. Namely, if e is an extension of F_n by $F(-nZ)$ and φ is an automorphism of F_n then by the product φ^*e we mean the pull-back of e with respect to φ. Then it is enough to show that an arbitrary locally-free extension is of the form φ^*e for some $\varphi \in \mathrm{Aut}(F_n)$. Put $\varphi = \mathrm{id} + h$, where $h \in \mathrm{End}(F_n)$. Then $\varphi^*e = e + h^*e$ and we can consider the product h^*e as the element of $\mathrm{H}^1(E, F^\vee \otimes F(-nZ)) \cong \mathrm{Ext}^1_{\mathcal{O}_{nZ}}(F_n, F(-nZ))$. We have that $h^*e = \delta(h)$, where the map $\delta : \mathrm{H}^0(E, F_n^\vee \otimes F_n) \to \mathrm{H}^1(F^\vee \otimes F(-nZ))$ is induced by the exact sequence $0 \to F^\vee \otimes F(-nZ)) \to F_{n+1}^\vee \otimes F_{n+1} \to F_n^\vee \otimes F_n \to 0$. It is possible to consider only those h that belong to the set $\mathrm{H}^0(E, F^\vee \otimes F(-(n-1)Z))$, which is contained in the radical of $\mathrm{End}(F_n)$ in the case $n > 1$. Note that the natural restriction of the coboundary map to $\delta' : \mathrm{H}^0(E, F^\vee \otimes F(-(n-1)Z)) \to \mathrm{H}^1(E, F^\vee \otimes F(-nZ))$ is surjective by [7, Claim, p.148]. This proves the statement for $n > 1$.

Let us consider the case $n = 1$. The reduction cycle Z is a scheme over the ring $\overline{R} = R/\mathfrak{m}^l$ for some $l > 0$. Put $B = \mathrm{End}_{\overline{R}}(F)$. It is necessary to note that not every element $h \in \mathrm{H}^0(E, F^\vee \otimes F)$ defines an invertible morphism $\varphi = \mathrm{id} + h$, because $\mathrm{H}^0(E, F^\vee \otimes F) \cong B$ which is not a vector space and there exists locally-free extensions of F which are not full. Consider at first the situation when the underlying field exists (this means $l = 1$) and it is infinite. The set of those $h \in B$ for which $\mathrm{id} + h$ is invertible form an open and dense subset in B. Since δ' is linear and surjective, the $\mathrm{Aut}(F)$-orbit of e is open and dense in $e + \mathrm{Ext}^1_{O_Z}(F, F(-Z))$ and contains only full extensions of F. Therefore, any two such orbits have common points, hence, coincide.

Consider now the case when the field k is finite. Consider the sheaf F over $\mathcal{O}_Z$ and its locally free extension F_2 over $\mathcal{O}_{2Z}$. Consider the schema $Y = X \otimes_k K$ and its resolution $\widetilde{Y} = \widetilde{X} \otimes_k K$, where K is an algebraic closure of k. Reduction cycle on $\widetilde{Y}$ is $\overline{Z} = Z \otimes_k K$. The sheaf F corresponds then to the sheaf $\overline{F}$ over $\mathcal{O}_{\overline{Z}}$ such that for every open U on Z $\overline{F}(\overline{U}) \cong F(U) \otimes_k K$. In the same way the sheaf $\overline{F}_2$ corresponds to F_2 and it is a locally free extension of $\overline{F}$ over $\mathcal{O}_{2\overline{Z}}$. For the scheme $\widetilde{Y}$ and the sheaf $\overline{F}$ Proposition 4 holds. In particular if F_2' is a locally free extension of F over $\mathcal{O}_{2Z}$, not equal to F_2, then $\overline{F}_2 \cong \overline{F}'_2$. Denote by $E(F)$ the set of isomorphism classes of locally free extension of F over $\mathcal{O}_{2Z}$. Fix some sheaf $F_2 \in E(F)$. The group $G = \mathrm{Gal}(K/k)$ acts on $\mathrm{Aut}_K(\overline{F}_2)$ in an obvious way and $E(F)$ is the set of all K/k-forms of sheaf F_2 in the sense of [14]. It is easy to see (analogous to [14]) that there exists an injective map of sets $E(F) \longrightarrow \mathrm{H}^1(G, \mathrm{Aut}_K(\overline{F}_2))$. At last, we are going to show that $\mathrm{H}^1(G, \mathrm{Aut}_K(\overline{F}_2)) = \{1\}$. Put $\overline{B} = \mathrm{End}_K(\overline{F}_2)$. We have that $\overline{B} = B \otimes_k K$, $\overline{B}^\times = \mathrm{Aut}_K(\overline{F}_2)$ and $\overline{B}^\times/(1 + \mathrm{rad}\,\overline{B}) \cong \prod \mathrm{GL}_{n_i}(K)$.

On the other hand there exists a chain of subgroups $1+\operatorname{rad}\overline{B}=G_0\supset G_1\supset\ldots\supset G_m=0$ such that $G_i/G_{i+1}\cong K^+$. From the fact that $\mathrm{H}^1(G,\mathrm{GL}_n(K))=\mathrm{H}^1(G,K^+)=\{1\}$ and the exact sequence for Galois cohomology we are done.

Suppose now that there is no underlying field. Let e_1 be some full extension. Then, by the result obtained above $e_1\equiv e_0+h^*e_0(\operatorname{mod}\mathfrak{m})$ for some $h\in B$. This means that $e_1=e_0+h^*e_0+u$ for some $u\in\mathfrak{m}\operatorname{Ext}^1_{O_Z}(F,F(-Z))$. Since δ' is surjection, $u=\delta'(v)=v^*e_0$ for some $v\in\mathfrak{m}B$, so$e_1=e_0+h^*e_0+v^*e_0=e_0+(h+v)^*e_0=\varphi^*e_0$ for the invertible $\varphi=\mathrm{id}+(h+v)$. □

For minimally elliptic singularities we fix some notations. We define a *minimally elliptic singularity* such that R be a Gorenstein ring, i.e. $\omega_X\simeq\mathcal{O}_X$ and $\mathrm{H}^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})\simeq K$, where K is the residue field of $\mathrm{H}^0(\widetilde{X},\mathcal{O}_{\widetilde{X}})$. As in the original work of Kahn we can choose in a role of a reduction cycle Z on $\widetilde{X}$ the fundamental cycle, for which we have $\mathcal{O}_Z\simeq\omega_Z$. The restriction char $K=0$ (which allows to use the Grauert-Riemenschneider vanishing theorem) can be omitted since $\mathrm{H}^1(E,\mathcal{O}_Z(-Z))$ by Serre's duality is dual to $\mathrm{H}^0(E,\omega_Z(Z))=\mathrm{H}^0(E,\mathcal{O}_Z(Z))=0$, because $\deg\mathcal{O}_Z(Z)=Z\cdot Z<0$.

Note that in the minimally elliptic case Z coincides with the *cohomological cycle* [12, p. 99], that is the unique cycle Z_1 on $\widetilde{X}$ such that Z_1 is smallest among all divisors supported on E and the length of the module $\mathrm{H}^1(E,\mathcal{O}_{Z_1})$ takes the maximal value. From isomorphisms $R^1\pi_*\mathcal{O}_{\widetilde{X}}\simeq\mathrm{H}^1(E,\mathcal{O}_Z)$, $R^1\pi_*\mathcal{O}_{\widetilde{X}}\simeq\mathrm{H}^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})$ we have that $\mathrm{H}^1(E,\mathcal{O}_Z)\simeq\mathrm{H}^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})$ and by Serre's duality $\mathrm{H}^1(E,\mathcal{O}_Z)$ is dual to $\mathrm{H}^0(E,\mathcal{O}_Z)$, so $\mathrm{H}^0(E,\mathcal{O}_Z)\simeq K$. For a full sheaf $\mathcal{F}$ on $\widetilde{X}$ and the locally free sheaf F of $\mathcal{O}_Z$-modules define h^i as the dimension over K of the i-th cohomology.

Theorem 5. [7, Theorem 2.1] *Let $\pi:(\widetilde{X},E)\to(X,x)$ be a resolution of a minimally elliptic singularity and let Z be the fundamental cycle on $\widetilde{X}$. Then there is a bijective correspondence, induced by R_Z, between isomorphism classes of non-projective indecomposable reflexive R-modules and isomorphism classes of locally free sheaves of $\mathcal{O}_Z$-modules of the form $n\mathcal{O}_Z\oplus G$ with G indecomposable and*

(i) G is generically generated by global sections;

(ii) $\mathrm{H}^1(E,G)=0$;

(iii) $n=\mathrm{h}^0(E,G(Z))$.

The proof is analogous to the Kahn's situation.

3. Curve singularities

During this section we suppose that A is a reduced, complete, local and noetherian ring of Krull dimensional 1 with maximal ideal $\mathfrak{m}$ and residue field k. Such a ring A is said to be a *curve singularity*. By Cohen's structure theorem A is a finite algebra over a discrete valuation ring $\mathcal{O}$ with residue field equal to k. We assume $\mathcal{O}$ and k are of characteristic not equal to 2 and k is perfect (another possible assumption for k is that char $k\neq 2,3$). For uniformizing parameter $x\in\mathcal{O}$, $p,q\in\mathbb{N}$ such that $\frac{1}{p}+\frac{1}{q}\leqslant\frac{1}{2}$ and $\alpha\in\mathcal{O}^\times$ define the singularities T_{pq} by the parametrization in the following way:

if p and q are both odd, T_{pq} is isomorphic to the subalgebra of $\mathcal{O}^2$ generated by (x^2,x^{q-2}) and $(\alpha x^{p-2},x^2)$;

if p is odd and q is even, T_{pq} is isomorphic to the subalgebra of $\mathcal{O}^3$ generated by $(x,x,\alpha x^{p-2})$ and $(0,x^{q/2-1},x^2)$;

if p and q are both even, T_{pq} is isomorphic to the subalgebra of $\mathcal{O}^4$ generated by $(x,x,\alpha x^{p/2-1},0)$ and $(x^{q/2-1},0,x,x)$ if $(p,q)\neq(4,4)$ and by $(x,0,x,x)$ and $(0,x,x,\alpha x)$ with $\alpha\neq 0,1$ if $(p,q)=(4,4)$.

From the parametrization given above we can represent singularity T_{pq} as the quotient $\mathcal{O}[[y]]/(f)$, where $f = (x^2 - y^{q-2})(x^{p-2} - \alpha^2 y^2)$ and note that it is isolated. Also note that in the case when $\mathcal{O}$ is the ring of power series over a field of characteristic zero or char k doesn't divide p or q (by [2]) this definition of T_{pq} coincides with the standard one from the Arnold's classification list [1]. We are going to prove that the following theorem still holds:

Theorem 6. [4, Theorem 1] *Let A be a curve singularity of infinite Cohen-Macaulay type. Then it is of tame type if and only if it dominates one of the singularities T_{pq}.*

For proving sufficiency we must prove the tameness of T_{pq}. We can do it in the following way.

Define the singularity T_{pq2} as $\mathcal{O}[[y,z]]/(z^2+f)$. The minimal resolution of T_{pq2} is given for instance in [8] and [9], where it is obtained by blowing-up and normalizations and proved that the exceptional locus of a minimal resolution of T_{pq2} is either elliptic curve or a so-called *cyclic configuration*. The description of the vector bundles on cyclic configurations is given in [5] and according to our case this description implies that indecomposable vector bundles on E are in one-to-one correspondence to triples of the form $(\mathbf{d}, m, p(t))$, where $\mathbf{d}$ is some finite sequence of integers called *aperiodic* [6], m is positive integer and $p(t) \in k[t] \setminus \{0\}$. Applying the result of Kahn we immediately obtain that T_{pq2} is of tame type.

The tameness of T_{pq} then follows from the tameness of T_{pq2} by the results of Knörrer. Namely, there exist functors from the categories of matrix factorizations (we keep the notations of paper [10]) $G : \mathcal{MF}(f) \to \mathcal{MF}(f+z^2)$ and $\mathrm{Rest} : \mathcal{MF}(f+z^2) \to \mathcal{MF}(f)$ such that every $X \in \mathcal{MF}(f)$ is a direct summand in $\mathrm{Rest} \circ G(X)$. This implies that one-parameter families of Cohen–Macaulay modules over T_{pq2} exhaust the category of Cohen–Macaulay modules over T_{pq}, so T_{pq} is of tame type.

For proving necessity we produce similar observations as in [4]. Let $\bar{A}$ be the normalization of A in its full ring of fractions and $\bar{A} = \prod_{i=1}^{s} A_i$ be its decomposition into a product of discrete valuation rings. By t_i we denote the uniformizing element of the ring A_i. For a ring B such that $A \subset B \subset \bar{A}$ let $B/\mathfrak{m}B \simeq B_1 \times ... \times B_m$, where each B_i is a local algebras of dimension d_i. Set $d(B) = d_1 + ... + d_m$. Let e_i be the idempotent of $\bar{A}$ such that $A_i = e_i A$, $t = (t_1, ..., t_s)$ and $\theta \in \mathfrak{m}$ such that $\bar{A}\mathfrak{m} = \theta \bar{A}$. Put $A' = t\bar{A} + A$, $A'' = \theta t \bar{A} + A$ and $A'_i = A' + \mathcal{O} e_i$. In this situation the overring conditions of the [4, Theorem 3] rewrites in the following form

Theorem 7. [4, Theorem 3] *Let A be a curve singularity of infinite CM type. The following condition are necessary and sufficient for A to be of tame CM type:*

(O1) $d(\bar{A}) \leqslant 4$ *and* $t^2\bar{A} \subset \mathfrak{m}$,

(O2) $d(A') \leqslant 3$ *and* A'_i *has no local 3-dimensional factor,*

(O3) *if* $d(\bar{A}) = 3$*, then* $d(A'') \leqslant 2$.

The conditions (O1-O3) are invariant under separable field extensions of k and under our assumptions for k we can choose separable extension K of k such that the residue fields of all algebras above are equal to K. Then we obtain the original conditions from [4, Theorem 3]. The proof of the theorem is analogous to the original one. Namely, necessity follows from geometric observation just as in [4]. Then using the parametrization of T_{pq} we prove that if A satisfies the conditions (O1-O3) of the previous theorem then it dominates some of T_{pq}.

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