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A VOLUME-PRESERVING NORMAL FORM FOR A REDUCED NORMAL CROSSING FUNCTION GERM

ADRIAN SZAWLOWSKI

Abstract. We will derive a volume-preserving normal form for holomorphic function germs that are right-equivalent to the product of all coordinates.

1. Introduction and Statement of Result

The complex version of the Morse lemma asserts that a holomorphic critical germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, whose Hessian determinant is nonzero at the origin, is right equivalent to $x_1^2 + \ldots + x_n^2$. If one tightens the notion of right equivalence by stipulating that the coordinate change has to be volume-preserving, then one gets the classical theorem by J. Vey ([Vey77]), asserting that there is a volume-preserving coordinate transformation mapping f to $\Psi(x_1^2 + \ldots + x_n^2)$ where $\Psi \in \mathbb{C}\{t\}$. There is another proof of this result by M. Garay ([Gar04]) even of a much more general statement, see the third section for further explanation. And there is a third proof by J.-P. Françoise in [Fra78]. His idea was the following. Assume that you already have the desired relation $f \circ \Phi(\mathbf{x}) = \Psi(x_1^2 + \ldots + x_n^2)$ with $\Psi(t) = t + o(t)$, say. Putting $\Psi(t) = tu(t)^2$ for some u with $u(0) \neq 0$ one rewrites the relation as $f \circ \Psi(\mathbf{x}) = [x_1 u(x_1^2 + \ldots + x_n^2)]^2 + \ldots + [x_n u(x_1^2 + \ldots + x_n^2)]^2$. Then the map $(x_1, \ldots, x_n) \mapsto (x_1 u(x_1^2 + \ldots + x_n^2), \ldots, x_n u(x_1^2 + \ldots + x_n^2))$ is a coordinate transformation and once it is applied, we can reduce the problem to a problem on the Brieskorn module. It is interesting to note that both Françoise and Garay use this module.
In this article we generalize the approach by Françoise to quasihomogeneous polynomials P instead of the $x_1^2 + \ldots + x_n^2$ in the lemmas 2.1, 2.2 and 3.1. They deal with the above-mentioned coordinate change which was only roughly sketched in Françoise's paper. Having established this, we can use a nonisolated version of the Brieskorn module which was already considered in [Fra82] to deduce a normal form for $P(x_1, \ldots, x_n) = x_1 \cdots x_n$:

Theorem 1.1. *Consider a holomorphic germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ that is right equivalent to the product of all coordinates: $f \sim x_1 \cdots x_n$. Then there exists a volume-preserving automorphism $\Phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and an automorphism $\Psi\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that*

$$f(\Phi(\mathbf{x})) = \Psi(x_1 \cdots x_n).$$

Ψ is uniquely determined by f up to a sign.

The uniqueness of Ψ is established in the fourth section by the technique of integrating over the fibre of f. In the final section we make several comments regarding the search for volume-preserving normal forms in general.

As usual, $\mathcal{O}_{\mathbb{C}^n,0}$ denotes the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$ and $\mathfrak{m}_{\mathbb{C}^n,0}$ its maximal ideal. Writing $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is equivalent to $f \in \mathfrak{m}_{\mathbb{C}^n,0}$. The group of biholomorphisms between sufficiently small neighbourhoods of the origin in $\mathbb{C}^n$ is denoted by

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$\mathrm{Aut}(\mathbb{C}^n, 0)$. An element of this group provides a right-equivalence. Such an element is volume-preserving if the Jacobian determinant of the automorphism is equal to the constant function one in a neighbourhood of the origin.

2. Main Lemma

For the proof of the main lemma we need the following fact from linear algebra which is easily proved by looking at the eigenvalues of the matrix vw^t.

Lemma 2.1. *For $v, w \in \mathbb{C}^n$ (written as column vectors) and $a, b \in \mathbb{C}$ we have*

$$\det(aI + bvw^t) = a^{n-1}(a + bv^t w).$$

Let $w_1, \dots, w_n$ and N be positive integers. A polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ is called quasihomogeneous of type $(w_1, \dots, w_n; N)$ if for all $\mathbf{x} \in \mathbb{C}^n$ and all $\lambda \in \mathbb{C}$ the relation

$$P(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^N P(\mathbf{x})$$

holds.

Lemma 2.2 (Main Lemma)**.**
Let P be quasihomogeneous of type $(w_1, \dots, w_n; N)$. Let $u \in \mathcal{O}_{\mathbb{C},0}$ with $u(0) \neq 0$. Then the map

$$A\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0), \quad \mathbf{x} \mapsto (u(P(\mathbf{x}))^{w_1} x_1, \dots, u(P(\mathbf{x}))^{w_n} x_n)$$

defines an automorphism of $(\mathbb{C}^n, 0)$ with the following properties:

a) There exists a unique $v \in \mathcal{O}_{\mathbb{C},0}$ such that the inverse map A^{-1} is given by

$$\mathbf{z} \mapsto (v(P(\mathbf{z}))^{w_1} z_1, \dots, v(P(\mathbf{z}))^{w_n} z_n).$$

Furthermore $v(0) \neq 0$.

b) With this v, the Jacobian determinant of A^{-1} is given by

$$\det(DA^{-1}(\mathbf{z})) = \left(v(P)^w + \frac{N}{w} P \frac{d}{dP} v(P)^w \right)\Bigg|_{P=P(\mathbf{z})}.$$

Here we have put $w := w_1 + \ldots + w_n$.

c) If we denote the assignment $u \mapsto v$ by $E\colon \mathrm{Units}(\mathcal{O}_{\mathbb{C},0}) \to \mathrm{Units}(\mathcal{O}_{\mathbb{C},0})$, then $E \circ E = \mathrm{id}$.

Proof. The assignment

$$A\colon \mathbf{x} \mapsto \mathbf{z} := (u(P(\mathbf{x}))^{w_1} x_1, \dots, u(P(\mathbf{x}))^{w_n} x_n)$$

is an automorphism of $(\mathbb{C}^n, 0)$ since its Jacobian at the origin is regular:

$$DA(0) = \begin{pmatrix} u(0)^{w_1} & 0 & \dots \\ & \dots & \\ \dots & 0 & u(0)^{w_n} \end{pmatrix}.$$

It is clear that the inverse of A is of the form $A^{-1}\colon \mathbf{z} \to (\tilde{v}_1(\mathbf{z}) z_1, \dots, \tilde{v}_n(\mathbf{z}) z_n)$ for some $\tilde{v} \in \mathcal{O}_{\mathbb{C}^n,0}$. (In fact, if we write $x_i = x_i(\mathbf{z})$ for the components of $A^{-1}(\mathbf{z})$, then $z_i = u(P(\mathbf{x}(\mathbf{z})))^{w_i} x_i(\mathbf{z})$, so that z_i must divide $x_i(\mathbf{z})$.)

In the sequel we are going to show that it is even of the form

$$\mathbf{z} \mapsto (v(P(\mathbf{z}))^{w_1} z_1, \dots, v(P(\mathbf{z}))^{w_n} z_n)$$

for some $v \in \mathcal{O}_{\mathbb{C},0}$! We also show that v is uniquely determined by u and that $v(0) \neq 0$.

Let $\mathbf{z} = A(\mathbf{x})$. From

$$\begin{aligned}\mathbf{z} &= (u(P(\mathbf{x}))^{w_1} x_1, \dots, u(P(\mathbf{x}))^{w_n} x_n) \\ &= (u(P(\mathbf{x}))^{w_1} \tilde{v}_1(\mathbf{z}) z_1, \dots, u(P(\mathbf{x}))^{w_n} \tilde{v}_n(\mathbf{z}) z_n)\end{aligned}$$

we conclude that

$$1 = u(P(\mathbf{x}))^{w_i} \tilde{v}_i(\mathbf{z}) \text{ for } i = 1, \dots, n.$$

Hence for the function $\hat{v} \in \mathcal{O}_{\mathbb{C}^n,0}$ defined by

$$\hat{v}(\mathbf{z}) := \frac{1}{u(P(A^{-1}(\mathbf{z})))},$$

we have

$$\tilde{v}_i(\mathbf{z}) = \hat{v}(\mathbf{z})^{w_i} \text{ for all } i.$$

Now let us show that the function $\hat{v}$ factors through $P(\mathbf{z})$. First we rewrite its defining equation

$$\begin{aligned}1 &= u(P(\mathbf{x}))\, \hat{v}(\mathbf{z}) \\ &= u(P(\tilde{v}_1(\mathbf{z}) z_1, \dots, \tilde{v}_n(\mathbf{z}) z_n))\, \hat{v}(\mathbf{z}) \\ &= u(P(\hat{v}(\mathbf{z})^{w_1} z_1, \dots, \hat{v}(\mathbf{z})^{w_n} z_n))\, \hat{v}(\mathbf{z}) \\ &= u(P(z_1, \dots, z_n) \hat{v}(\mathbf{z})^N)\, \hat{v}(\mathbf{z}). \end{aligned} \tag{2.1}$$

To see factorization through P, we apply twice the implicit function theorem as follows.

(1) The implicit equation $u(v^N t)v = 1$ for v has a unique local solution $v = v(t)\colon (\mathbb{C}, 0) \to (\mathbb{C}, 1/u(0))$. Indeed, the point $(t = 0, v = 1/u(0))$ is a solution and the derivative after v is nonzero at this point:

$$\partial_v\ (u(0)v)|_{v=1/u(0)} = u(0) \neq 0.$$

(2) The implicit equation $u(V^N P(\mathbf{z}))V = 1$ for V has a unique local solution $V = V(\mathbf{z})\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 1/u(0))$. Indeed, the point $(\mathbf{z} = 0, V = 1/u(0))$ is a solution and the derivative after V at this point is nonzero:

$$\partial_V\ (u(0)V)|_{V=1/u(0)} = u(0) \neq 0.$$

Now by the first item (only existence is used), $v(P(\mathbf{z}))$ fulfils $v(P(0)) = 1/u(0)$ and solves the equation $u(v(P(\mathbf{z}))^N P(\mathbf{z}))v(P(\mathbf{z})) = 1$. Comparing this result and equation (2.1) we can deduce from the second item (only uniqueness is used) that

$$v(P(\mathbf{z})) = \hat{v}(\mathbf{z}). \tag{2.2}$$

Hence $\tilde{v}_i(\mathbf{z}) = \hat{v}(\mathbf{z})^{w_i} = v(P(\mathbf{z}))^{w_i}$, hence A^{-1} is of the desired form as stated in part a) of the assertion. Note that $\hat{v}(0) = 1/u(0)$ by its definition and therefore also $v(0) = 1/u(0)$.

The proof of part a) is not yet quite complete. What about the uniqueness of v when we have just given u? By its very definition, $\hat{v}$ is uniquely determined by u (and P). Since $v(P(\mathbf{z})) = \hat{v}(\mathbf{z})$ and since P is surjective onto a neighbourhood of zero, also v is uniquely determined by u.

For part c) of the assertion we note that the operator E which asigns to u the function v is given by solving the implicit equation $u(v^N t)v = 1$ with $v(0) = 1/u(0)$. So let $E(u) = v$

and $E(v) = w$. Then we also have $v(w(s)^N s)w(s) = 1$ for all $s \in (\mathbb{C}, 0)$. If in the equation $u(v(t)^N t)v(t) = 1$ we substitute $t = w(s)^N s$, we get

$$\begin{aligned} u\left[v(w(s)^N s)^N w(s)^N s\right] v(w^N s) &= 1 \\ u[1^N s] \cdot 1/w(s) &= 1 \\ u(s) &= w(s). \end{aligned}$$

This shows part c).

It remains to prove part b). The (i,j)th entry in the Jacobian matrix of the transformation

$$A^{-1}\colon \quad \mathbf{z} \mapsto (v(P(\mathbf{z}))^{w_1} z_1, \ldots, v(P(\mathbf{z}))^{w_n} z_n)$$

is given by

$$\begin{aligned} \partial_i (v(P(\mathbf{z}))^{w_j} z_j) &= w_j (v(P(\mathbf{z})))^{w_j - 1} v'(P(\mathbf{z})) \partial_i P(\mathbf{z}) z_j + v(P(\mathbf{z})^{w_j} \delta_{ij} \\ &= (v(P(\mathbf{z})))^{w_j - 1} \left[w_j v'(P(\mathbf{z})) \partial_i P(\mathbf{z}) z_j + v(P(\mathbf{z})) \delta_{ij} \right] \end{aligned}$$

In order to compute its determinant we use lemma 2.1 from above. This together with the Euler relation for weighted homogeneous polynomials yields

$$\begin{aligned} &\det(DA^{-1}(\mathbf{z})) \\ &= \prod_{j=1}^{n} (v(P(\mathbf{z})))^{w_j - 1} \cdot \det(v'(P(\mathbf{z})) \partial_i P(\mathbf{z}) w_j z_j + v(P(\mathbf{z})) \delta_{ij}) \\ &= \prod_{j=1}^{n} (v(P(\mathbf{z})))^{w_j - 1} \cdot (v(P(\mathbf{z})))^{n-1} \left[v(P(\mathbf{z})) + v'(P(\mathbf{z})) \sum_{j=1}^{n} w_j z_j \partial_j P(\mathbf{z}) \right] \\ &= v(P(\mathbf{z}))^{w_1 + \ldots + w_n - n + n - 1} \cdot [v(P(\mathbf{z})) + v'(P(\mathbf{z})) N P(\mathbf{z})] \\ &= \left(v(P)^w + \frac{N}{w} P \frac{d}{dP} v(P)^w \right) \Big|_{P = P(\mathbf{z})}, \end{aligned}$$

where we used the abbreviation $w = w_1 + \ldots + w_n$. $\square$

Given u, we get the map A of the lemma which we also denote by A_u. Then we have

$$A_{E(u)} \circ A_u = \mathrm{id}\,.$$

We make a remark which however will not be used elsewhere in the paper. Assume that instead of u we have just given the map A (of the form A_u with an unspecified u). Of course the $\tilde{v}_i z_i$ which are the component functions of A^{-1} are uniquely determined by A. Then from $\tilde{v}_i(\mathbf{z}) = \hat{v}(\mathbf{z})^{w_i}$ we infer that the function $\hat{v}$ is uniquely determined up to the multiplication with some number $\xi \in \mathbb{C}$ which fulfills $\xi^{w_i} = 1$ for all i. If we demand that the greatest common divisor of the $w_1, \ldots, w_n$ is equal to one, then $\xi = 1$ and so $\hat{v}$ and also v are uniquely determined by A. Applying this argument to A^{-1} we see that given a map A (of the form A_u with some unknown $u \in \mathrm{Units}(\mathcal{O}_{\mathbb{C},0})$) the u is uniquely determined if $\gcd(w_1, \ldots, w_n) = 1$.

3. Existence of the Normal Form

By a germ of a volume form at the origin in $\mathbb{C}^n$ we understand a germ of a holomorphic n-form which does not vanish at the origin. Let (f_0, Ω_0) be a pair consisting of a germ of a function $f_0 \in \mathcal{O}_{\mathbb{C}^n,0}$ which vanishes at the origin and a germ of a volume form $\Omega_0 \in \Omega^n_{\mathbb{C}^n,0}$. Then the group $\mathrm{Aut}(\mathbb{C}^n, 0)$ acts on the set of such pairs by the usual pulling back of functions resp. forms. A normal form for a pair (f_0, Ω_0) should then be a nicely chosen pair in the same orbit. One way to achieve this is to look only at pairs in the orbit of (f_0, Ω_0) with the same $f = f_0$. Another way would be to consider only those pairs in the orbit of (f_0, Ω_0) with the same $\Omega = \Omega_0$. The latter would give us an Ω_0-preserving normal form for functions which are right equivalent to f_0. That these two approaches are interchangeable when the right normal form is chosen is the content of the following lemma which we will later only use in the direction $(ii) \Rightarrow (i)$.

Lemma 3.1 (Exchange Lemma)**.**
Let P be quasihomogeneous of type $(w_1, \dots, w_n; N)$. For a holomorphic function germ $f = f(\mathbf{y})\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ the following statements are equivalent:

i) There exist an automorphism $\Phi \in \mathrm{Aut}(\mathbb{C}^n, 0), \mathbf{y} \mapsto \mathbf{x}$ and an automorphism $\Psi \in \mathrm{Aut}(\mathbb{C}, 0)$ such that

$$f(\Phi^{-1}(\mathbf{x})) = \Psi(P(\mathbf{x})) \text{ and } (\Phi^{-1})^* d^n\mathbf{y} = d^n\mathbf{x}.$$

ii) There exist an automorphism $\phi \in \mathrm{Aut}(\mathbb{C}^n, 0), \mathbf{z} \mapsto \mathbf{y}$ and a function $\psi \in \mathcal{O}_{\mathbb{C},0}$ with $\psi(0) \neq 0$ such that

$$f(\phi(\mathbf{z})) = P(\mathbf{z}) \text{ and } \phi^* d^n\mathbf{y} = \psi(P(\mathbf{z})) d^n\mathbf{z}.$$

Proof. We start with the implication $(i) \Rightarrow (ii)$. Since $\Psi'(0) \neq 0$ there is a germ $u \in \mathcal{O}_{\mathbb{C},0}, u(0) \neq 0$ with $\Psi(t) = tu(t)^N$. From the quasihomogeneity of P we get

$$\begin{aligned}\Psi(P(\mathbf{x})) &= P(\mathbf{x})u(P(\mathbf{x}))^N \\ &= P(u(P(\mathbf{x}))^{w_1}x_1, \dots, u(P(\mathbf{x}))^{w_n}x_n).\end{aligned}$$

If we define the map

$$A\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0), \mathbf{x} \mapsto \mathbf{z} := (u(P(\mathbf{x}))^{w_1}x_1, \dots, u(P(\mathbf{x}))^{w_n}x_n)$$

then $\Psi(P(\mathbf{x})) = P(A(\mathbf{x}))$. The first part of item (i), $f(\Phi^{-1}(\mathbf{x})) = \Psi(P(\mathbf{x}))$ can therefore be rewritten as $f(\Phi^{-1}(\mathbf{x})) = P(A(\mathbf{x}))$. Since by lemma 2.2 the map A is an automorphism of $(\mathbb{C}^n, 0)$, we can rewrite this again: we let $\phi \in \mathrm{Aut}(\mathbb{C}^n, 0), \mathbf{z} \mapsto \mathbf{y}$ with $\phi := \Phi^{-1} \circ A^{-1}$, then it follows $f(\phi(\mathbf{z})) = P(\mathbf{z})$. This is the first assertion of item (ii).

Again by lemma 2.2 there is $v \in \mathcal{O}_{\mathbb{C},0}, v(0) \neq 0$ with

$$\det(DA^{-1}(\mathbf{z})) = \left(v(P)^w + \frac{N}{w} P \frac{d}{dP} v(P)^w \right)\bigg|_{P=P(\mathbf{z})}.$$

If we define $\psi\colon (\mathbb{C}, 0) \to \mathbb{C}$ by this bracket, i.e.

$$\psi(t) := v(t)^w + \frac{N}{w} t \frac{d}{dt} \left(v(t)^w \right),$$

then $\psi(0) \neq 0$ and we can write the pullback of the volume form as

$$\begin{aligned}\phi^* d^n\mathbf{y} &= (A^{-1})^*(\Phi^{-1})^* d^n\mathbf{y} \\ &= (A^{-1})^* d^n\mathbf{x} \\ &= \psi(P(\mathbf{z})) d^n\mathbf{z}.\end{aligned}$$

This is the second assertion of item (ii).

Now we prove the converse direction. So let us assume (ii) is valid. First we seek a solution $v\colon (\mathbb{C},0) \to \mathbb{C}, v(0) \neq 0$ of the equation

$$\left(v(t)^w + \frac{N}{w} t \frac{d}{dt} v(t)^w \right) = \psi(t) \tag{3.1}$$

where ψ is the function as given in statement (ii), i.e. $\psi\colon (\mathbb{C},0) \to \mathbb{C}, \psi(0) \neq 0$. A solution can be obtained from a power series ansatz, namely if $\psi(t) = \sum a_i t^i$ and $v^w = \sum b_i t^i$, then comparison of the coefficients shows that the stipulation

$$b_i := \frac{a_i}{1 + (Ni/w)}$$

will provide a solution v^w of the differential equation. Since $\psi(0)$ is nonzero so is $v^w(0)$. Hence, taking some wth root v of v^w will give us v.

Now we define $u \in \mathcal{O}_{\mathbb{C},0}$ as $u = E^{-1}(v)$, cf. lemma 2.2. Then $\det(DA_u^{-1}(\mathbf{z})) = \det(DA_v(\mathbf{z})) = \psi(P(\mathbf{z}))$ by that lemma and the definition of v. Now define $\Phi := A_u^{-1}\phi^{-1}$. Then

$$\begin{aligned}
(\Phi^{-1})^* d^n\mathbf{y} &= (\phi \circ A_u)^* d^n\mathbf{y} \\
&= A_u^* \phi^* d^n\mathbf{y} \\
&= A_u^*(\psi(P(\mathbf{z})) d^n\mathbf{z}) \\
&= \psi(P(A_u(\mathbf{x}))) \det DA_u(\mathbf{x}) d^n\mathbf{x} \\
&= \psi(P(\mathbf{z})) \det DA_u(\mathbf{x}) d^n\mathbf{x} \\
&= d^n\mathbf{x}
\end{aligned}$$

Finally when we insert into the given relation $f(\phi(\mathbf{z})) = P(\mathbf{z})$ the expression $\mathbf{z} = A_u(\mathbf{x})$ we can rewrite it as

$$\begin{aligned}
f(\Phi^{-1}(\mathbf{x})) &= P(A_u(\mathbf{x})) \\
&= P(u(P(\mathbf{x}))^{w_1} x_1, \ldots, u(P(\mathbf{x}))^{w_n} x_n) \\
&= P(\mathbf{x}) u(P(\mathbf{x}))^N.
\end{aligned}$$

So letting $\Psi(t) := tu(t)^N$ we have the statement $f \circ \Phi^{-1}(\mathbf{x}) = \Psi(P(\mathbf{x}))$ of our assertion. We note $\Psi'(0) = u(0)^N = 1/v(0)^N \neq 0$, so Ψ is an automorphism of $(\mathbb{C},0)$. This completes the proof. □

We now show that part (ii) in lemma 3.1 is true for $P = x_1 \cdots x_n$. Prior to this a digression on the Brieskorn modules is neccessary.

In the seminal paper [Bri70] Brieskorn has introduced different $\mathbb{C}\{t\}$-modules for the investigation of the monodromy of an isolated singularity. One of these modules is given for an isolated singularity $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ by

$$H_f'' = \frac{\Omega^n_{\mathbb{C}^n,0}}{df \wedge d\Omega^{n-2}_{\mathbb{C}^n,0}}.$$

Here $\Omega^k_{\mathbb{C}^n,0}$ denotes the vector space of germs of holomorphic k-forms at the origin in $\mathbb{C}^n$. The $\mathbb{C}\{t\}$-module structure of this module comes from multiplication with f. It is shown in the cited paper together with Sebastiani's paper [Seb70], see also Malgrange [Mal74], that this is a free module with rank equal to the Milnor number $\mu(f,0)$ of f at the origin. This classical Brieskorn module was extended to apply for isolated complete intersection singularities by Greuel in [Gre75]. It is this "parametrized version" of the Brieskorn module which allowed Garay in [Gar04] to proof his volume-preserving versal unfolding theorem from which one can deduce the theorem of Vey. The former theorem roughly states that there are $\mu(f,0)$ holomorphic moduli for volume-preserving right equivalence. One can ask if it possible to gain similar results for nonisolated singularities. Following analogy we face the problem of choosing the right nonisolated version of the Brieskorn module. Such nonisolated versions were e.g. looked at in the paper by van Straten [vSt87]. But also Françoise in his study of normal forms was already considering

$$F_f := \frac{\Omega^n_{\mathbb{C}^n,0}}{\{d\eta |\ df \wedge \eta = 0\}},$$

which is again a $\mathbb{C}\{t\}$-module. For isolated singularities F_f equals H''_f by the de Rham lemma. But for arbitrary singularities not much is known. At least for $n = 2$ Barlet has shown (cf. [BS07]) that this module is free of finite rank. However, in more than two dimensions freeness of F_f is in general not given ([BS07]). If $P(\mathbf{x}) = x_1^{m_1} \dots x_n^{m_n}$ then F_P has $\gcd(m_1, \dots, m_n)$ generators which are given explicitely in [Fra82]. For $P = x_1 \cdots x_n$, F_P is generated by the single form $d^n\mathbf{x} = dx_1 \wedge \dots \wedge dx_n$.

Now let f be right equivalent to this P. Choosing $\Phi_1 \in \mathrm{Aut}(\mathbb{C}^n, 0)$ with $\Phi_1^* f = P$ there exists $\psi \in \mathbb{C}\{t\}$ and η with $dP \wedge \eta = 0$ such that $\Phi_1^* d^n\mathbf{x} = \psi \circ P\, d^n\mathbf{x} + d\eta$. Now it is important to note - as shown in the proof by Françoise - that among the power series terms on the left-hand side only the constant term, i.e. $\det(D\Phi_1)(0)$, will contribute to the constant term of ψ and they are *equal*. In particular $\psi(0) \neq 0$. Finally we note that $\eta(0) = 0$.

We now make use of the

Lemma 3.2. *Let $g \in \mathfrak{m}_{\mathbb{C}^n,0}$ and Ω_1, Ω_2 two n-forms on $(\mathbb{C}^n, 0)$ with the* same nonzero value *at the origin. If there is an $(n-1)$-form η with $\Omega_1 - \Omega_2 = d\eta$ and $\eta(0) = 0$ such that $dg \wedge \eta = 0$, then there exists $\Phi_2 \in \mathrm{Aut}(\mathbb{C}^n, 0)$ with $\Phi_2^* g = g$ and $\Phi_2^* \Omega_1 = \Omega_2$.*

The proof is based on the path method and can be found in [Fra82].

Applying it to $\Omega_1 := \Phi_1^* d^n\mathbf{x}, \Omega_2 := \psi \circ P d^n\mathbf{x}$ and $g := P$ we get an automorphism Φ_2 with $\Phi_2^* \Phi_1^* d^n\mathbf{x} = \psi \circ P\, d^n\mathbf{x}$ and $\Phi_2^* P = P$. So if we put $\phi := \Phi_1 \circ \Phi_2$ we have

$$\phi^* d^n\mathbf{x} = \psi \circ P\, d^n\mathbf{x} \quad \text{and} \quad \phi^* f = P.$$

This is item (ii) of lemma 3.1. The implication $(ii) \Rightarrow (i)$ thus yields the existence of the normal form.

4. Uniqueness of the Normal Form

We now address the question of unicity of Ψ. The equation $f \circ \Phi(y) = \Psi(P(y))$ can be written as a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^n, 0) & \xleftarrow{\ \Phi\ } & (\mathbb{C}^n, 0) \\ \Big\downarrow f & & \Big\downarrow P \\ (\mathbb{C}, 0) & \xleftarrow[\ \Psi\ ]{} & (\mathbb{C}, 0) \end{array}$$

For sufficiently small $\epsilon > 0$ and for all sufficiently small $0 < \delta \ll \epsilon$ we have the Milnor-Lê fibration ([Lê77]) $f\colon B_\epsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$ where B_ϵ is the open ϵ-ball around $0 \in \mathbb{C}^n$ and D^*_δ is the open δ-ball around the origin in $\mathbb{C}$ minus this point. The general fibre is called the Milnor fibre $\mathrm{Mil}_{f,0}$ of f. For a quasihomogeneous P we can compute the Milnor fibre as the general fibre of the global affine fibration $P\colon \mathbb{C}^n \setminus P^{-1}(0) \to \mathbb{C}^*$, see ([Dim92], p. 68 - 72). Hence the Milnor fibre of P over $s \in D_\delta^*$

$$\mathrm{Mil}_{P,0}(s) = \{\mathbf{x} \in B_\epsilon |\, x_1 \cdots x_n = s\}$$

is diffeomorphic to

$$\{(x_1, \ldots, x_n) \in \mathbb{C}^n |\, x_1 \cdots x_n = 1\} \cong \{(x_2, \ldots, x_n) \in (\mathbb{C}^*)^{n-1}\}.$$

Similar statements hold if replace the standard ball B_ϵ by a ball defined by a rug function $(B_\epsilon(\rho) = \{\mathbf{x} \in \mathbb{C}^n | \rho(\mathbf{x}) < \epsilon\}$ where $\rho\colon (\mathbb{C}^n, 0) \to \mathbb{R}_{\geq 0}$ is real analytic such that $\rho^{-1}(0) = \{0\}$.) So $H_{n-1}(\mathrm{Mil}_{P,0}(s); \mathbb{Z}) \cong \mathbb{Z}$ with generator $\gamma(P, s)$ given by the product of $(n-1)$ circles. In fact for s real and $s < (\epsilon/\sqrt{n})^n$ we have a map

$$\underbrace{S^1 \times \ldots \times S^1}_{n-1} \hookrightarrow \mathrm{Mil}_{P,0}(s),\ (z_2, \ldots, z_n) \mapsto (s^{1/n}/(z_2 \ldots z_n), z_2 s^{1/n}, \ldots, z_n s^{1/n}),$$

which is easily checked to be an embedding. Along this cycle we can integrate any holomorphic $(n-1)$-form λ and if we choose λ as a holomorphic primitive of $d^n\mathbf{x}$, e.g. $\lambda = x_1 dx_2 \wedge \ldots \wedge dx_n$, then we evaluate the integral of λ over one of the generators of $H_{n-1}(\mathrm{Mil}_{P,0}(s); \mathbb{Z})$ as

$$\int_{\gamma(P,s)} \lambda = \int_{S^1 \times \ldots \times S^1} s \frac{dz_2}{z_2} \ldots \frac{dz_n}{z_n} = \pm (2\pi i)^{n-1} s.$$

(Of course, if we had chosen the canonical orientation of $\mathrm{Mil}_{P,0}(s)$ as a complex manifold we would get a plus sign, but it is not important here.)

Finally let $\gamma(f, \cdot)$ be a locally constant section of the $(n-1)$st homological fibration of f, obtained by parallel translating one of the two homology generators of a single reference fibre. Then we get an a priori multivalued holomorphic function germ $t \mapsto \int_{\gamma(f,t)} \lambda$. From the commutativity of the above diagram it follows that an integral homology generator of $\mathrm{Mil}_{P,0}(\Psi^{-1}(t))$ is sent via Φ_* to one of the two generators of $H_{n-1}(\mathrm{Mil}_{f,0}(t); \mathbb{Z})$ and so we obtain

$$\int_{\gamma(f,t)} \lambda = \pm \int_{\Phi_* \gamma(P, \Psi^{-1}(t))} \lambda.$$

Now Φ being volume-preserving, it preserves λ up to a differential, so the right-hand side becomes

$$\begin{aligned} &\int_{\gamma(P,\Psi^{-1}(t))} \Phi^* \lambda \\ &= \int_{\gamma(P,\Psi^{-1}(t))} \lambda \\ &= \pm (2\pi i)^{n-1} \Psi^{-1}(t). \end{aligned}$$

Hence $\Psi^{-1}(t) = \pm \left(\frac{1}{2\pi i}\right)^{n-1} \int_{\gamma(f,t)} \lambda$ so that Ψ is uniquely determined by f, possibly up to a sign.

And indeed we show that the alleged ambiguity in the choice of Ψ's sign cannot be eliminated: Take any permutation matrix $S \in \mathbb{C}^{n \times n}$ with determinant -1 and let c be any number with $c^n = -1$. Then the linear map $\mathbf{x} \mapsto \Phi(\mathbf{x}) := cS\mathbf{x}$ is volume-preserving and transforms $x_1 \cdots x_n$ to $(cx_1) \cdots (cx_n) = -x_1 \cdots x_n$.

5. Comments

Stokes theorem in the real two-dimensional plane asserts that $\oint_C xdy$ computes the area of the interior that is surrounded by the simple closed curve C. When we think of x, y as complex variables and the curve to be a cycle lying in some smooth fibre of a function $f \in \mathfrak{m}_{\mathbb{C}^2,0}$, then one is led to believe that such an integral should be significant for the study of volume-preserving equivalence. And indeed it is, as we have seen for example in section four. Recalling that Garay's unfolding theorem can roughly be interpreted that an isolated singularity f has $\mu(f,0)$ continuous moduli for volume-preserving equivalence, it seems natural to expect that when $\mu(f,0) = 1$ there is only one continuous obstruction. This obstruction should then be the aforementioned integral. And in fact it is the function Ψ from Vey's statement. It is natural to conjecture that we can find volume-preserving normal forms for nonisolated singularities f as well, as long as $H_{n-1}(\mathrm{Mil}_{f,0})$ has rank one. This has been done in this article when f is right equivalent to $x_1 \cdots x_n$. What about other cases? For a singularity f in two variables $H_{n-1}(\mathrm{Mil}_{f,0})$ has rank one if and only if f is right equivalent to $x^a y^b$ with $\gcd(a,b) = 1$. (This should be well-known; it follows e.g. if we compare the homotopy exact sequences of the Milnor fibrations associated to f itself and its reduced version f_{red}. For details the reader is sent to [Sza12].) For $a = b = 1$ we have Vey's lemma, but for other values of a and b, we cannot use the above methods anymore: Instead of $[1 \cdot dx \wedge dy]$, according to Françoise, $[x^{a-1}y^{b-1}\, dx \wedge dy]$ if a generator of F_f but then lemma 3.2 has to be applied e.g. to $\Omega_1 = \Phi_1^* dx \wedge dy$ and $\Omega_2 = dx \wedge dy + \psi \circ P \cdot x^{a-1}y^{b-1}\, dx \wedge dy$ which will however not yield the statement of lemma 3.1(ii).

Finally we can check that the integral of $\lambda = x\, dy$ over a generator of $H_1(\mathrm{Mil}_{f,0})$ is zero: Choose real numbers $0 < s \ll \epsilon \ll 1$ such that $\mathrm{Mil}_f(s) = \{(x,y) \in B_\epsilon(0) |\ x^a y^b = s\}$ is the Milnor fibre of $f(x,y) = x^a y^b$ where $a, b \in \mathbb{N}$ are coprime integers. Then we can embed S^1 into the Milnor fibre over s using the map

$$S^1 \ni z \mapsto (x(t), y(t)) = (z^b s^{1/a+b}, z^{-a} s^{1/a+b}).$$

In fact this map is an injective immersion of a compact space, hence an embedding. (The injectivity follows from $\gcd(a,b) = 1$.) We now integrate the form $x^m y^n dy$ along this cycle:

$$\begin{aligned}\int_{S^1} x^m y^n\, dy &= \int_{S^1} z^{mb} s^{m/(a+b)} z^{-an} s^{n/(a+b)} (-a) z^{-a-1} s^{1/(a+b)} dz \\ &= -a s^{(m+n+1)/(a+b)} \int_{S^1} z^{mb-an-a-1} dz.\end{aligned}$$

This integral is nonzero if and only if $mb - an - a = 0$. Of course there are choices of m, n where this is achieved. Thus, the embedded circle is homologically nontrivial, i.e. represents a generator of $H_1(\mathrm{Mil}_{f,0}; \mathbb{C}) \cong \mathbb{C}$. Now we let $m = 1$ and $n = 0$, so that $\lambda = x\, dy$ is a primitive of the volume form. Its integral is nonzero if and only if $b - a = 0$. But since $\gcd(a,b) = 1$ this holds only if $a = b = 1$.

So a normal form for functions which are right equivalent to $x^a y^b$ with coprime a, b is unlikely to exist in the simple form $f \circ \Phi = \Psi(x^a y^b)$ with Φ volume-preserving. But at least we believe that it might exist in more complicated form.

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Georg-August Universität Göttingen, Mathematisches Institut, D-37073 Göttingen, Germany