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MULTIDIMENSIONAL RESIDUE THEORY AND THE LOGARITHMIC DE RHAM COMPLEX

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Abstract. We study logarithmic differential forms with poles along a reducible hypersurface and the multiple residue map with respect to the complete intersection given by its components. Some applications concerning computation of the kernel and image of the residue map and the description of the weight filtration on the logarithmic de Rham complex for hypersurfaces whose irreducible components are defined by a regular sequence of functions are considered. Among other things we give an easy proof of the de Rham theorem (1954) on residues of closed meromorphic differential forms whose polar divisor has rational quadratic singularities, and correct some inaccuracies in its original formulation and later citations.

Keywords: logarithmic de Rham complex; regular meromorphic forms; multiple residues; complete intersections; weight filtration

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Introduction

The term "residue" (together with its formal definition) appeared for the first time in an article by A. Cauchy (1826), although one can find such a notion as implicit in Cauchy's prior work (1814) about the computation of particular integrals which were related with his research towards hydrodynamics. For the next three-four years, Cauchy developed residue calculus and applied it to the computation of integrals, the expansion of functions as series and infinite products, the analysis of differential equations, and so on.

Though it was already transparent in the pioneer work by N. Abel, a major step towards the elaboration of the residue concept was made by H. Poincaré who introduced in 1887 the notion of differential *residue* 1-*form* attached to any rational differential 2-form in $\mathbf{C}^2$ with simple poles along a smooth complex curve. Such rational form can be considered as the simplest prototype of differential forms called *logarithmic* in the modern terminology. Subsequently É. Picard (1901), G. de Rham (1932/36),

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A. Weil (1947) obtained a series of important results about residues of meromorphic forms of degree 1 and 2 on complex manifolds; such developments were associated with cohomological ideas, leading to the formulation of a cohomological residue formula, and therefore to explicit computations of integrals of rational forms (in the spirit of Cauchy or Abel) along cycles.

Among other things Poincaré has also proved that the residue is, in fact, a *holomorphic* 1-form. A simple generalization of his construction to the case of a complex analytical variety M of dimension $m \geq 2$ leads to the following exact sequence of $\mathcal{O}_M$-modules

$$0 \longrightarrow \Omega^m_M \longrightarrow \Omega^m_M(D) \xrightarrow{\text{rés}} \Omega^{m-1}_D \longrightarrow 0,$$

where $\Omega^m_M(D)$ is a sheaf of meromorphic differential forms of degree m on M with poles of the first order on the *smooth* divisor $D \subset M$, and Ω^m_M and Ω^{m-1}_D are sheaves of regular holomorphic forms on M and D of degrees m and $m-1$, respectively.

In the fifties further cohomological ideas were pursued by G. de Rham (1954) and J. Leray (1959) who defined and studied residues of *d-closed* C^∞-regular differential forms on the complement $M \setminus D$ with poles of the first order along a smooth hypersurface D in complex manifold M. Thus, for any such q-form ω there exists locally the following decomposition

$$\omega = \frac{dh}{h} \wedge \xi + \eta, \tag{1}$$

where h is the germ of a holomorphic function determining the smooth hypersurface D, and ξ, η are germs of regular forms. Moreover, the restriction of ξ to D does not depend on a local equation of the hypersurface; it is globally and uniquely determined and closed on D. The differential form $\xi\big|_D$ is called the *residue-form* denoted by rés(ω). If ω is holomorphic on $M \setminus D$, then the differential form $\xi\big|_D$ is holomorphic on D.

In 1972, J.-B. Poly showed that Leray decomposition (1) as well as the residue form are determined correctly for the so-called *semi-meromorphic* forms (not necessarily closed) if both they and their total differentials have poles of the first order along D (see [16]). Such *meromorphic* forms were called by P. Deligne (1969) differential forms with *logarithmic* poles along D; in fact, he considered the case of divisors with normal crossings. The corresponding coherent sheaves of $\mathcal{O}_M$-modules are denoted by $\Omega^q_M(\log D)$, $q \geq 1$. It is not difficult to see that in these notations there are exact sequences of $\mathcal{O}_M$-modules

$$0 \longrightarrow \Omega^q_M \longrightarrow \Omega^q_M(\log D) \xrightarrow{\text{rés}} \Omega^{q-1}_D \longrightarrow 0,$$

where Ω^{q-1}_D, $q \geq 1$, are sheaves of regular *holomorphic* differential forms on D.

In 1977, making use of decomposition (1) with a multiplier, K.Saito introduced the notion of residue res.(ω) for a meromorphic form ω on M with logarithmic poles along a reduced divisor D with arbitrary singularities (see [19]). Somewhat later the author proved (see [1], [2]) that in this case for all $q \geq 1$ there are exact sequences

$$0 \longrightarrow \Omega^q_M \longrightarrow \Omega^q_M(\log D) \xrightarrow{\text{res.}} \omega^{q-1}_D \longrightarrow 0, \tag{2}$$

where ω^q_D, $q \geq 0$, are sheaves of *regular meromorphic* q-forms on D. Further generalizations of these results are investigated in [3], [4].

For completeness it should be remarked that the original concept of residue is, in fact, a local notion; the classical local residue is given by a variant of Cauchy formula for several complex variables. In the focus of the *global theory* of residue is the *residue formula*. For rational differential 1-forms defined on a compact complex algebraic curve it is one of the fundamental results in the classical analytic and algebraic geometry (see [21]). In the multidimensional case, that is, for meromorphic differential m-forms given on an m-dimensional complex manifold many variants of the residue formula in various situations and different contexts are known (see, for example, [8]). Such a form ω is closed, $d\omega = 0$, by reason of dimension. In this case only meromorphic forms with *polar* singularities, namely *logarithmic* differential m-forms, enter non-trivial contributions in the residue formula.

The paper is organized as follows. In the first two sections some elementary properties of logarithmic differential forms with simple poles along a divisor are considered. Then in the third and fourth sections we discuss properties of *multiple residues* of logarithmic differential forms with poles along reducible hypersurfaces. In particular, it is proved that the residue map determines exact sequences similarly to the above (2) for divisors whose components are defined locally by regular sequences of function germs. The proof is based essentially on the theory of logarithmic and multi-logarithmic differential forms and some properties of the multiple residue studied earlier in [1], [2], [3]. In the next two sections the kernel and image of the multiple residue map are described. Some applications are considered in two final sections, then the obtained results adapt for computing residues of logarithmic differential forms of principal type and for description of the weight filtration on the logarithmic de Rham complex for divisors whose components are defined by a regular sequence of functions. In particular, this allows to compute the mixed Hodge structure on cohomology of the complement of divisors of certain types without using theorems on resolution of singularities or the standard reduction to the case of normal crossings. Among other things in Section 7 we also give an easy proof of the well-known theorem goes back to de Rham (1954) which asserts that the residues of *closed* meromorphic differential forms whose polar divisor has *rational quadratic* singularities are holomorphic on the divisor, and correct also some inaccuracies in its original formulation and later citations.

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1. The logarithmic de Rham complex

Let S be a complex analytical variety of dimension $m \geq 1$, and $z = (z_1, \dots, z_m)$ be a local coordinate system in a neighborhood U of the distinguished point $x \in U \subset S$. Further, suppose that a hypersurface $D \subset S$ is defined by a function $h \in \mathcal{O}_U$. We will also assume that h has no multiple factors so that the hypersurface D is *reduced*, that is, the divisor D does not contain multiple components.

Let ω be a meromorphic differential q-form on U with poles along D. Then ω is called *logarithmic* or q-form with *logarithmic poles* along D if $h\omega$ and $hd\omega$ are holomorphic on U.

Let us also denote by $S = (S, x) \cong (\mathbb{C}^m, 0)$ the germ of S at the distinguished point x. For simplification in the record identical notations for the spaces and their germs at this point are often used without additional comments when the sense is clear from the context. Throughout the paper we also use the term divisor for (locally principal) Cartier divisors D in a manifold.

The localization of the concept of logarithmic forms leads to the definition of $\mathcal{O}_{S,x}$-module $\Omega^q_{S,x}(\log D)$ which consists of the germs of meromorphic q-forms on S with poles along D such that $h\omega$ and $hd\omega$ are holomorphic at the point x, that is, $h \cdot \Omega^q_{S,x}(\log D) \subseteq \Omega^q_{S,x}$ and $h \cdot d\Omega^q_{S,x}(\log D) \subseteq \Omega^{q+1}_{S,x}$. Evidently, the second condition is equivalent to the inclusion $dh \wedge \Omega^q_{S,x}(\log D) \subseteq \Omega^{q+1}_{S,x}$. The corresponding coherent analytic sheaves of *logarithmic differential forms* are denoted by $\Omega^q_S(\log D)$, $q \geq 0$. It should be remarked that $\Omega^m_{S,x}(\log D) \cong \mathcal{O}_{S,x}(dz_1 \wedge \ldots \wedge dz_m/h)$. By definition, $\Omega^0_S(\log D) \cong \mathcal{O}_S$, and there are natural inclusions $\Omega^q_S \subseteq \Omega^q_S(\log D)$ for all $q \geq 1$ which are, in fact, isomorphisms $\Omega^q_{S,x} \cong \Omega^q_{S,x}(\log D)$ for all $x \notin D$.

The family $\Omega^q_S(\log D), q \geq 0$, endowed with differential induced by the de Rham differentiation d of $\Omega^\bullet_S$ defines an *increasing* complex called the *logarithmic* de Rham complex. Further, the sheaves of logarithmic differential forms are $\mathcal{O}_S$-modules of finite type, and their direct sum $\oplus_{q=0}^m \Omega^q_S(\log D)$ forms an $\mathcal{O}_S$-exterior algebra closed under the action of d.

Recall that $\mathcal{O}_S$-module of vector fields *logarithmic* along $D \subset S$ consists of germs of holomorphic vector fields $\mathcal{V} \in \mathrm{Der}(\mathcal{O}_S)$ on S such that $\mathcal{V}(h)$ belongs to the principal ideal $(h)\cdot\mathcal{O}_S$. In particular, $\mathcal{V}$ is tangent to D at its non-singular points. This module is denoted by $\mathrm{Der}_S(\log D)$. There is a perfect pairing

$$\mathrm{Der}_S(\log D) \times \Omega^1_S(\log D) \to \mathcal{O}_S$$

induced by the contraction of differential forms along vector fields (see [20]).

Let us also remark that in general $\Omega^q_S(\log D) \not\cong \bigwedge^q \Omega^1_S(\log D)$. However, for all $q > 0$ there exist natural inclusions $\bigwedge^q \Omega^1_S(\log D) \to \Omega^q_S(\log D)$. All these inclusions are isomorphisms if $\Omega^1_S(\log D)$ or, equivalently, $\mathrm{Der}_S(\log D)$ is *locally free*. In this case D is called the *free hypersurface* or *Saito free divisor*.

2. Logarithmic forms with poles along reducible hypersurfaces

Let $D = D_1 \cup \ldots \cup D_k$ be any irredundant (not necessarily irreducible) decomposition of a reduced divisor D. It is clear that there are natural inclusions

$$\sum_{i=1}^{k} \Omega_S^q(\log D_i) \hookrightarrow \Omega_S^q(\log D),\ q \geq 0.$$

Analogously, if $\widehat{D_i}$ is the union of all elements of the decomposition excluding D_i, that is, $\widehat{D_i} = D_1 \cup \ldots \cup D_{i-1} \cup D_{i+1} \cup \ldots \cup D_k$, then

$$\sum_{i=1}^{k} \Omega_S^q(\log \widehat{D_i}) \hookrightarrow \Omega_S^q(\log D),$$

and $\Omega_{S,x}^q(\log D_i) \cong \Omega_{S,x}^q(\log D)$ are isomorphisms for all $x \in D_i \setminus (D_i \cap \widehat{D_i})$, and so on.

Claim 1. *Assume that divisors D_i defined by function germs h_i, $i = 1, \ldots, k$, are components of a locally irredundant decomposition of D. Then there is a natural isomorphism*

$$\mathrm{Der}_S(\log D_1) \cap \ldots \cap \mathrm{Der}_S(\log D_k) \cong \mathrm{Der}_S(\log D).$$

Proof. It is clear, that the left side of the relation is contained in the rightist. Conversely, take $\mathcal{V} \in \mathrm{Der}_S(\log D)$. Then $\mathcal{V}(h) = \sum_{i=1}^{k} (h_1 \cdots \widehat{h_i} \cdots h_k)\mathcal{V}(h_i) = fh$, where $f \in \mathcal{O}_S$. After division by h_i the both part of the latter equality one obtains that the function $(h_1 \cdots \widehat{h_i} \cdots h_k)\mathcal{V}(h_i)/h_i$ is holomorphic, that is, h_i divides $\mathcal{V}(h_i)$. Hence, $\mathcal{V}(h_i) \in (h_i)\mathcal{O}_S,\ i = 1, \ldots, k$. QED.

The following assertion one may consider as a dual variant of the above statement.

Claim 2. *Under the same assumptions let us suppose that $\Omega_S^1(\log D)$ is generated by closed forms. Then one has an isomorphism*

$$\Omega_S^1(\log D_1) + \ldots + \Omega_S^1(\log D_k) \cong \Omega_S^1(\log D).$$

Proof. Due to Theorem 2.9 from [20] the conditions of closeness of generators of $\Omega_S^1(\log D)$ is equivalent to the isomorphism $\sum_{i=1}^{k} \mathcal{O}_S \frac{dh_i}{h_i} + \Omega_S^1 \cong \Omega_S^1(\log D)$. On the other side, $\frac{dh_i}{h_i} \in \Omega_S^1(\log D_i)$ and there is a natural inclusion $\sum_{i=1}^{k} \Omega_S^1(\log D_i) \hookrightarrow \Omega_S^1(\log D)$. This completes the proof. QED.

Proposition 1. *Under assumptions of Claims above there exist natural inclusions*

$$h_i \Omega_S^\bullet(\log D) \subseteq \Omega_S^\bullet(\log \widehat{D_i}), \quad dh_i \wedge \Omega_S^\bullet(\log D) \subseteq \Omega_S^{\bullet+1}(\log \widehat{D_i}), \quad i = 1, \ldots, k.$$

In other words, the external product by total differentials dh_i as well as multiplication by functions h_i "dissipates" poles of $\omega \in \Omega_S^\bullet(\log D)$ located on D_i.

Proof. Let us first examine the case $k = 2$. Let us set $i = 1$, then take $x \in D_1 \cap D_2$ and show that $h_2 \Omega_S^\bullet(\log D) \subseteq \Omega_S^\bullet(\log D_1)$. By assumptions, $h_1(h_2\omega) = h\omega \in \Omega_S^\bullet$. Further,

$$dh \wedge (h_2\omega) = h_2 dh_1 \wedge (h_2\omega) + h_1 dh_2 \wedge (h_2\omega) = h_2 dh_1 \wedge (h_2\omega) + dh_2 \wedge (h\omega).$$

Since the differential form $dh \wedge \omega$ is holomorphic then $dh \wedge (h_2\omega)$ is also a holomorphic form. Analogously, $dh_2 \wedge (h\omega) \in \Omega_S^\bullet$ and, consequently, $h_2 dh_1 \wedge (h_2\omega) = h_2^2 dh_1 \wedge \omega \in \Omega_S^\bullet$. Set $dh_1 \wedge \omega = \vartheta/h_2^2$, where $\vartheta \in \Omega_S^\bullet$. Let us note that $\frac{dh_1}{h_1} \in \Omega_S^\bullet(\log D)$, so that $\frac{dh_1}{h_1} \wedge \omega \in \Omega_S^\bullet(\log D)$ in virtue of $\wedge$-closeness. Therefore, $\frac{dh_1}{h_1} \wedge \omega = \frac{\vartheta}{h_1 h_2^2} \in \Omega_S^\bullet(\log D)$, that is, $\frac{\vartheta}{h_2} \in (h)\Omega_S^\bullet(\log D) \subseteq \Omega_S^\bullet$. Hence, $\vartheta \in (h_2)\Omega_S^\bullet$ and $dh_1 \wedge \omega = \vartheta'/h_2$, where $\vartheta' \in \Omega_S^\bullet$.

Thus, $h_2(dh_1 \wedge \omega) = dh_1 \wedge (h_2\omega) \in \Omega_S^\bullet$, that is, $dh_1 \wedge (h_2\omega)$ is a holomorphic form. It does mean that $h_2\omega \in \Omega_S^\bullet(\log D_1)$. This completes the proof of the first inclusion.

The second inclusion can be proved in the same style. Really, $h_1(dh_2 \wedge \omega)$ is a holomorphic differential form because it is equal to the difference $dh \wedge \omega - h_2 dh_1 \wedge \omega$, where the first form is holomorphic by assumptions, while the holomorphicity of the second form is established similarly to the proof of the first inclusion. Further,

$$dh_1 \wedge (dh_2 \wedge \omega) = d(h_1 dh_2 \wedge \omega) + h_1(dh_2 \wedge d\omega).$$

Since the form $h_1 dh_2 \wedge \omega$ is holomorphic then its total differential is also holomorphic. At last,

$$h_1(dh_2 \wedge d\omega) = dh \wedge d\omega - h_2(dh_1 \wedge d\omega).$$

The differential form $dh \wedge d\omega$ is holomorphic by hypothesis since the external algebra $\Omega^\bullet_S(\log D)$ is closed relative to the de Rham differentiation d, so that $d\omega \in \Omega^\bullet_S(\log D)$. As in the proof of the first inclusion one obtains that $h_2(dh_1 \wedge d\omega)$ is a holomorphic form. This implies that $h_1(dh_2 \wedge d\omega)$ as well as $h_1(dh_2 \wedge \omega)$ are holomorphic forms. Thus, $dh_2 \wedge \omega \in \Omega^\bullet_S(\log D_1)$ as required. The general case $k > 2$ is considered analogously. QED.

Remark 1. By the same reasonings one can see that for all $j = 1, \ldots, k$ there are inclusion

$$(h_1 \cdots \widehat{h_i} \cdots h_k)\Omega^\bullet_S(\log D) \subseteq \Omega^\bullet_S(\log D_i), \quad d(h_1 \cdots \widehat{h_i} \cdots h_k) \wedge \Omega^\bullet_S(\log D) \subseteq \Omega^\bullet_S(\log D_i).$$

Similar relations are also valid for divisors obtained by the exclusion of any collection of components of the decomposition.

Claim 3. *Assume that components D_i, $i = 1, \ldots, k$, of an irredundant decomposition of a reduced divisor D are defined locally by elements of a regular $\mathcal{O}_S$-sequence $(h_1, \ldots, h_k)$. Then*

$$\Omega^\bullet_S(\log \widehat{D_1}) \cap \ldots \cap \Omega^\bullet_S(\log \widehat{D_k}) = \Omega^\bullet_S,$$

and there is an exact sequence of complexes

$$0 \longrightarrow \Omega^\bullet_S \longrightarrow \oplus\, \Omega^\bullet_S(\log \widehat{D_i}) \longrightarrow \sum \Omega^\bullet_S(\log \widehat{D_i}) \longrightarrow 0.$$

Proof. It is sufficient to prove the first relation. Clearly, the right side of the relation is contained in the leftist. Conversely, let us take a differential p-form ω from the left side. Then $(h_1 \cdots \widehat{h_i} \cdots h_k)\omega \in \Omega^p_S$, $i = 1, \ldots, k$. Hence, $\omega \in \bigcap \frac{1}{(h_1 \cdots \widehat{h_i} \cdots h_k)}\Omega^p_S$, or, equivalently, $h\omega \in (h_1)\Omega^p_S \cap \ldots \cap (h_k)\Omega^p_S$. Elementary properties of regular sequences imply that the latter intersection is equal to $(h_1 \cdots h_k)\Omega^p_S$, that is, $\omega \in \Omega^p_S$. QED.

3. A decomposition of meromorphic forms along complete intersections

Let $D = D_1 \cup \ldots \cup D_k$ be a reduced reducible hypersurface. We will denote the $\mathcal{O}_S$-modules of meromorphic differential q-forms, $q \geq 1$, formed by differential q-forms with simple poles and with poles of any order on the divisor $\widehat{D_i} = D_1 \cup \ldots \cup D_{i-1} \cup D_{i+1} \cup \ldots \cup D_k$, by $\Omega^q_S(\widehat{D_i})$ and by $\Omega^q_S(\star\widehat{D_i})$, $i = 1, \ldots, k$, respectively. When $k = 1$ we set $\widehat{D_1} = \emptyset$, so that $\Omega^q_S(\widehat{D_1}) = \Omega^q_S(\star\widehat{D_1}) = \Omega^q_S$.

Let us further assume that the complex analytical space $C = D_1 \cap \ldots \cap D_k$ is a *complete intersection.* This means that the ideal $\mathcal{I}$ defining $C \subset U$ is locally generated by a *regular* $\mathcal{O}_U$-sequence $(h_1, \ldots, h_k)$ and $\dim C = m - k \geq 0$. We also suppose that $C = C_{\text{red}}$ is a *reduced* space when $\dim C > 0$. In other words, the ideal $\mathcal{I} = \sqrt{\mathcal{I}}$ is *radical.* In particular, these conditions imply that the differential k-form $dh_1 \wedge \ldots \wedge dh_k$ is not identically zero on every irreducible component of C. The following statement and its proof are slightly changed versions of considerations from [3], [4].

Theorem 1. *Suppose that in a neighborhood U of $x \in C$ all irreducible components D_i, $i = 1, \ldots, k$, of D are defined by elements of a regular $\mathcal{O}_U$-sequence $(h_1, \ldots, h_k)$. Assume also that a meromorphic differential form $\omega \in \Omega^q_U(D)$ satisfies the following conditions*

$$dh_j \wedge \omega \in \sum_{i=1}^{k} \Omega^{q+1}_U(\widehat{D_i}), \quad j = 1, \ldots, k. \tag{3}$$

Then there is a holomorphic function g, which is not identically zero on every irreducible component of the complete intersection C, a holomorphic differential form $\xi \in \Omega_U^{q-k}$ and a meromorphic q-form $\eta \in \sum_{i=1}^{k} \Omega_U^q(\widehat{D_i})$ such that there exists the following representation

$$g\omega = \frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \xi + \eta. \tag{4}$$

Proof. In a neighborhood of $x \in U$ the differential form ω is represented as follows:

$$\omega = \frac{1}{h_1 \cdots h_k} \sum_{|I|=q} a_I(z) \cdot dz_I,$$

where $I := I^q = (i_1, \ldots, i_q)$, $1 \leq i_1, \ldots, i_q \leq m$, is a multiple index, $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_q}$, and $a_I(z) \in \mathcal{O}_U$ is the set of coefficients, skew-symmetric relative to I. It is clear that conditions (3) are equivalent to inclusions

$$dh_j \wedge \sum_I a_I(z) \cdot dz_I \in \sum_{\ell=1}^{k} h_\ell \, \Omega_U^{q+1}, \quad j = 1, \ldots, k.$$

These inclusions give us the following system of relations between the coefficients a_I and the partial derivatives of h_j :

$$\sum_{\ell=1}^{q} (-1)^{\ell-1} \frac{\partial h_j}{\partial z_{i_\ell}} a_{I \setminus i_\ell} = b_{jI}^1 h_1 + \ldots + b_{jI}^k h_k, \quad j = 1, \ldots, k, \tag{5}$$

with holomorphic functions $b_{jI}^1, \ldots, b_{jI}^k \in \mathcal{O}_U$.

Let us fix a multi-index $J^p = (j_1, \ldots, j_p)$, $1 \leq j_1, \ldots, j_p \leq m$, $1 \leq p \leq k$, and denote the corresponding minor of Jacobian matrix $\operatorname{Jac}(h_1, \ldots, h_k) = \|\partial h_i / \partial z_j\|$ by

$$\Delta_{J^p} = \delta_{j_1 \ldots j_p} = \det \left\| \frac{\partial h_i}{\partial z_{j_r}} \right\|_{1 \leq i,\, r \leq p}$$

We will prove by induction on index p that the following relations are valid:

$$\Delta_{J^p} a_{I^q} \equiv \sum_{K \subset I^q,\, |K|=p} \operatorname{sgn} \begin{pmatrix} I^q \\ K, I^q \setminus K \end{pmatrix} \Delta_K \, a_{(J^p, I^q \setminus K)} \ \big(\operatorname{mod}(\mathcal{I})\big), \quad p = 1, \ldots, k, \tag{6}$$

where $\mathcal{I} \subseteq \mathcal{O}_U$ is generated by the regular sequence $(h_1, \ldots, h_k)$.

First let us assume that $p = 1$ and set $J^1 = j_1 = j$, $I = (j, I^q) = (j, i_1, \ldots, i_q)$ in formula (5). Then one gets the following relation

$$\frac{\partial h_1}{\partial z_j} a_{I^q} \equiv \sum_{\ell=1}^{q} (-1)^{\ell-1} \frac{\partial h_1}{\partial z_{i_\ell}} a_{I \setminus i_\ell} \ \big(\operatorname{mod}(\mathcal{I})\big),$$

which coincides with relation (6) for $p = 1$.

Let us suppose that (6) is true for $p - 1$ and prove it for p as follows. The cofactor expansion of determinant Δ_{J^p} along the p-th row gives the identity:

$$\Delta_{J^p} a_{I^q} = \sum_{\ell=1}^{p} (-1)^{p-\ell} \frac{\partial h_p}{\partial z_{j_\ell}} \Delta^{p-1}_{j_1 \ldots \widehat{j_\ell} \ldots j_p} a_{I^q}.$$

By the induction hypothesis there is the congruence

$$\Delta_{j_1 \ldots \hat{j}_\ell \ldots j_p} a_{I^q} \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \operatorname{sgn} \begin{pmatrix} I^q \\ K', I^q \setminus K' \end{pmatrix} \Delta_{K'} a_{(j_1 \ldots \widehat{j_\ell} \ldots j_p,\, I^q \setminus K')} \ \big(\operatorname{mod}(\mathcal{I})\big).$$

Let us substitute this expression in the previous identity. Changing then the order of summation, one obtains

$$\Delta_{J^p} a_{I^q} \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \operatorname{sgn}\begin{pmatrix} I^q \\ K', I^q \setminus K' \end{pmatrix} \Delta_{K'} \sum_{\ell=1}^{p} (-1)^{p-\ell} \frac{\partial h_p}{\partial z_{j_\ell}} a_{(j_1 \ldots \widehat{j_\ell} \ldots j_p,\, I^q \setminus K')} \pmod{(\mathcal{I})}.$$

The second sum consists of p terms containing in formula (5) with $j = p$, $I = (j_1, \ldots, j_p,\, I^q \setminus K')$.

It is not difficult to rewrite this expression in the form of the sum which contains the remaining $q - p + 1$ terms with opposite signs and an element from the ideal $(h_1, \ldots, h_k)\mathcal{O}_U$. Hence, one obtains the congruence modulo $\mathcal{I}$:

$$\Delta_{J^p} a_{I^q} \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \operatorname{sgn}\begin{pmatrix} I^q \\ K', I^q \setminus K' \end{pmatrix} \Delta_{K'} (-1)^{p-1} \sum_{i \in I \setminus K'} (-1)^{\#(i;\, I \setminus K')} \frac{\partial h_p}{\partial z_i} a_{(j_1 \ldots j_p,\, I^q \setminus K' \setminus i)}, \tag{7}$$

where $\#(i;\, I \setminus K')$ is equal to the number of occurrences of the index i in the set $I \setminus K'$. At last, let us put in order all pairs (K', i) in such a way that the multi-index $K' \cup \{i\}$ coincides with the given one $K \subset I$. For any such pair the corresponding coefficient $a_{(j_1 \ldots j_p,\, I \setminus K' \setminus i)}$ is equal to $a_{(J,\, I \setminus K)}$. Then the contribution of the above ordered set to relation (7) is equal to the following:

$$a_{(J^p,\, I^q \setminus K)} (-1)^{p-1} \sum_{i \in K} \operatorname{sgn}\begin{pmatrix} I^q \\ K \setminus i,\, I^q \setminus K,\, i \end{pmatrix} (-1)^{\#(i;\, I \setminus (K \setminus i))} \frac{\partial h_p}{\partial z_i} \Delta_{K \setminus i} =$$

$$= \operatorname{sgn}\begin{pmatrix} I^q \\ K,\, I^q \setminus K \end{pmatrix} a_{J^p,\, I^q \setminus K} \Delta_K.$$

This completes the proof of relation (6) for $p \geq 1$.

It remains to show that it is possible to choose the function g in such a way that $g \not\equiv 0$ on each irreducible component of the complete intersection C. For this we examine ideal $\mathcal{G}$ of the ring $\mathcal{O}_U$ generated by all minors $\Delta_{i_1 \ldots i_k}$ of the maximal order of Jacobian matrix $\operatorname{Jac}(h_1, \ldots, h_k)$. Since $dh_1 \wedge \ldots \wedge dh_k$ does not vanish identically on each irreducible component of the complete intersection C, then the image $\widetilde{\mathcal{G}}$ of the ideal $\mathcal{G}$ in the ring $\mathcal{O}_{C,0}$ is not equal to $\operatorname{Ann} \mathcal{O}_{C,0}$. Thus, it is possible to use Theorem 2.4. (1) from [6] which yields that $\mathcal{O}_{C,0}$-depth of the ideal $\widetilde{\mathcal{G}}$ is not less than one. Hence, there is an element $g \in \mathcal{O}_{C,0}$ with the property required by Theorem 1. QED.

Remark 2. It is not difficult to verify that formula (6) implies the following identity

$$\Delta_{i_1 \ldots i_k} \cdot \sum_{|I|=q} a_I dz_I = dh_1 \wedge \ldots \wedge dh_k \wedge \big(\sum_{|I'|=q-k} a_{i_1 \ldots i_k I'}\, dz_{I'} \big) + \nu,$$

where $\nu \in \sum_{j=1}^{k} h_j \Omega_U^{q-k}$. Therefore, by analogy with the case of hypersurface (see [20], Lemma (2.8)) the maximal minors of Jacobian matrix $\operatorname{Jac}(h_1, \ldots, h_k)$ can be considered as *universal denominators* for the complete intersection C.

If $m = k$, that is, $\dim C = 0$ and C is *non-reduced* then the latter formula implies that there exists representation (4) with a function g equal to an element of the one-dimensional socle of the local algebra $\mathcal{O}_{C,0}$ generated over the ground field by the determinant of the Jacobian matrix $\operatorname{Jac}(h)$ (see [23]). In this case the notion of multiple residue of meromorphic differential forms of degree m coincides with the so-called *multidimensional residue;* in the context of Grothendieck local duality theory it can be expressed in terms of projection of elements of a certain finite dimensional vector space to this socle (cf. [8]).

Corollary 1. *Let $\omega \in \Omega_S^q(\log D)$ be a differential form with logarithmic poles along a hypersurface D and let $C = D_1 \cap \ldots \cap D_k$ be a complete intersection. Then there exists representation* (4) *with a differential form $\eta \in \sum_{i=1}^{k} \Omega_S^{\bullet}(\log \widehat{D_i})$.*

Proof. Since for the logarithmic form ω conditions (3) are fulfilled in virtue of Proposition 1 from Section 2, then there is decomposition (4) with $\eta \in \sum_{i=1}^{k} \Omega_U^q(\widehat{D_i})$. For the sake of simplicity, let us examine the case $k = 2$. Then $\eta = \eta_1/h_1 + \eta_2/h_2$, where $\eta_1, \eta_2 \in \Omega^q_U$. Taking the external product by dh of both parts of representation (4), one concludes that the differential form

$$dh \wedge \eta = dh \wedge \Big(\frac{\eta_1}{h_1} + \frac{\eta_2}{h_2}\Big) = dh_2 \wedge \eta_1 + dh_1 \wedge \eta_2 + h_2 \frac{dh_1}{h_1} \wedge \eta_1 + h_1 \frac{dh_2}{h_2} \wedge \eta_2$$

is holomorphic. Hence, the sum of the both last terms is also holomorphic. Now let us reduce all the terms of the sum to the common denominator. This gives the inclusion

$$h_2^2(dh_1 \wedge \eta_1) + h_1^2(dh_2 \wedge \eta_2) \in (h_1h_2)\Omega^\bullet_S,$$

i.e., $h_2^2\alpha + h_1^2\beta = h_1h_2\gamma$, where $\alpha, \beta, \gamma \in \Omega^\bullet_S$. Therefore, $h_2^2\alpha + (h_1\beta - h_2\gamma)h_1 = 0$. Since (h_1, h_2) is a *regular* sequence, then, comparing the coefficients of the corresponding form for every fixed collection of differentials, one obtains that $\alpha = h_1\alpha'$, $\alpha' \in \Omega^\bullet_S$. Hence, $dh_1 \wedge \eta_1 \in (h_1)\Omega^\bullet_S$, that is, $\eta_1/h_1 \in \Omega^\bullet_S(\log D_1)$. By the same reasonings one can check that $\eta_2/h_2 \in \Omega^\bullet_S(\log D_2)$. The general case $k > 2$ is investigated analogously. QED.

Corollary 2. *Under conditions of Theorem* 1 *representation* (4) *exists if and only if there are analytical subsets $A_j \subset D_j$, $j = 1, \ldots, k$, of codimension not less than* 2 *such that the germ ω at any point $x \in \bigcup_{j=1}^k (D_j \setminus A_j)$ belongs to the space*

$$\frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \Omega^{q-k}_{U,x} + \sum_{i=1}^k \Omega^q_{U,x}(\widehat{D_i}). \tag{8}$$

Proof. Taking $A_j = D_j \cap \{g = 0\}$, $j = 1, \ldots, k$, one obtains the decomposition of Theorem 1 which implies the desired statement.

The converse is true in view of the following reasonings. If there exists representation (8) for a meromorphic form ω, then $h\omega$ is, in fact, holomorphic outside of subsets $A_i \subset D_i$, $i = 1, \ldots, k$, of codimension not less than 2. Consequently, according to Riemann extension Theorem, the differential form $h\omega$ is holomorphic everywhere so that $h_j\omega \in \Omega^q_U(\widehat{D_j})$, $j = 1, \ldots, k$.

Further, $dh_j \wedge \omega$ is represented as the sum of meromorphic forms ω_i, each of which is singular not more than on $k-1$ components of divisor $\widehat{D_i}$ and on the subset $A_i \subset D_i$ of codimension not less than 2. Again, applying Riemann Theorem to $(h_1 \cdots \widehat{h_i} \cdots h_k)\omega_i$, one obtains that the differential form ω_i has singularities *only* on $\widehat{D_i}$. As a result $dh_j \wedge \omega \in \sum_{i=1}^k \Omega^q_U(\widehat{D_i})$, $j = 1, \ldots, k$. QED.

Remark 3. If one takes a decomposition of a reducible divisor D of length $k = 1$, so that $C = D$, then representation (4) looks like this

$$g\omega = \frac{dh}{h} \wedge \xi + \eta, \quad \xi, \eta \in \Omega^\bullet_U; \tag{9}$$

it coincides with representation of the basic lemma by K.Saito (see [20], (1.1), *iii*)).

4. The multiple residue map

Let us now discuss the concept of *multiple residues* of meromorphic forms which satisfy conditions of Section 3. In notations of Theorem 1 it is not difficult to see that the function g from representation (4) is a *non-zero divisor* in $\mathcal{O}_{S,0}/(h_1, \ldots, h_k)\mathcal{O}_{S,0} \cong \mathcal{O}_{C,0}$. Therefore the restriction of the form ξ/g to the germ of complete intersection $C = D_1 \cap \ldots \cap D_k$ is well-defined.

Definition 1. The restriction of differential form ξ/g to the complete intersection C is called the *multiple residue* of the differential form ω; the corresponding map is denoted by Res_C, so that

$$\mathrm{Res}_C(\omega) = \frac{\xi}{g}\Big|_C.$$

Remark 4. The multiple residue of ω is contained in the space $\mathcal{M}_C \otimes_{\mathcal{O}_C} \Omega^{q-k}_C \cong \mathcal{M}_{\widetilde{C}} \otimes_{\mathcal{O}_{\widetilde{C}}} \Omega^{q-k}_{\widetilde{C}}$, $q \geq k$, where $\widetilde{C}$ is the normalization of C.

Proposition 2. *The multiple residue map is well-defined, that is, its values do not depend on representation* (4).

Proof. Let us assume that $q \geq k$ and a differential q-form ω have two local representations

$$g_\ell \omega = \frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \xi_\ell + \eta_\ell, \quad \ell = 1, 2.$$

Then

$$dh_1 \wedge \ldots \wedge dh_k \wedge (g_1\xi_2 - g_2\xi_1) = h_1 \cdots h_k (g_1\eta_2 - g_2\eta_1) \in (h_1, \ldots, h_k)\Omega^q_S.$$

Consequently,

$$dh_1 \wedge \ldots \wedge dh_k \wedge (g_1\xi_2 - g_2\xi_1) \equiv 0 \ \big(\mathrm{mod}\,(h_1, \ldots, h_k)\big).$$

Then the first part of the main Theorem from [18] (the generalized de Rham Lemma) with $R = \mathcal{O}_{C,0}$, $M = \Omega^1_{S,0} \otimes \mathcal{O}_{C,0}$, $e_i = z_i$, $i = 1, \ldots, m$, $\omega_j = dh_j$, $j = 1, \ldots, k$, $p = q - k \geq 0$, implies that

$$\mathcal{G}^e(g_1\xi_2 - g_2\xi_1) \subset dh_1 \wedge \Omega^{q-k-1}_{S,0} + \ldots + dh_k \wedge \Omega^{q-k-1}_{S,0} + (h_1, \ldots, h_k)\Omega^{q-k}_{S,0}, \quad e \in \mathbf{Z}_+,$$

where the ideal $\mathcal{G} \subset \mathcal{O}_{S,0}$ is generated by all minors $\Delta_{i_1 \ldots i_k}$ of maximal order of Jacobian matrix $\mathrm{Jac}(h_1, \ldots, h_k)$. As in the end of the proof of Theorem 1 we note that the image $\widetilde{\mathcal{G}}$ of the ideal $\mathcal{G}$ in the ring $\mathcal{O}_{C,0}$ is not equal to $\mathrm{Ann}\,\mathcal{O}_{C,0}$, since the germ C is *reduced*. Therefore $\mathcal{O}_{C,0}$-depth of the ideal $\widetilde{\mathcal{G}}$ is not less than 1. Consequently, there is an element $\Delta \in \widetilde{\mathcal{G}}$, a non-zero divisor in $\mathcal{O}_{C,0}$ such that

$$\Delta^e(g_1\xi_2 - g_2\xi_1) \in dh_1 \wedge \Omega^{q-k-1}_{S,0} + \ldots + dh_k \wedge \Omega^{q-k-1}_{S,0} + (h_1, \ldots, h_k)\Omega^{q-k}_{S,0}.$$

Therefore the class of the element $\Delta^e(g_1\xi_2 - g_2\xi_1)$ in $\Omega^{q-k}_{C,0}$ is equal to zero. It does mean that both elements $\dfrac{1}{g_1}\xi_1$ and $\dfrac{1}{g_2}\xi_2$ determine the same class in $\mathcal{M}_{C,0} \otimes_{\mathcal{O}_{C,0}} \Omega^{q-k}_{C,0}$. QED.

Lemma 1. *The kernel of the multiple residue map coincides with the space* $\sum_{i=1}^k \Omega^\bullet_S(\widehat{D_i})$.

Proof. It is clear that the kernel contains this sum. It remains to prove the converse inclusion. Suppose that $\mathrm{Res}_C(\omega) = 0$ for a certain q-form ω, $q \geq k$. Then there exists a function g in representation (4) of Theorem 1 such that the restriction of meromorphic form ξ/g to C vanishes. Consequently, $\xi = g(\sum h_i\xi_i + \sum dh_i \wedge \xi_i')$, where $\xi_i,\ \xi_i' \in \Omega^\bullet_S$, and

$$h\omega = dh_1 \wedge \ldots \wedge dh_k \wedge (\sum h_i\xi_i) + \frac{1}{g}(\sum h_i\eta_i),\ \eta_i \in \Omega^\bullet_S.$$

Since $h\omega$ and the first term in the right side of the identity are holomorphic, then g *divides* $\sum h_i\eta_i$ in $\Omega^\bullet_S$, that is, $g\eta_0 = \sum h_i\eta_i$, $\eta_0 \in \Omega^\bullet_S$. On the other hand, $(h_1, \ldots, h_k)$ is a regular sequence and g is a non-zero divisor in $\mathcal{O}_C = \mathcal{O}_S/(h_1, \ldots, h_k)\mathcal{O}_S$. Therefore, examining coefficients of the differentials dz_I in the coordinate representation of the holomorphic form $\sum h_i\eta_i$, one obtains that $\eta_0 \in (h_1, \ldots, h_k)\Omega^\bullet_S$. This yields $\omega \in \sum_{i=1}^k \Omega^\bullet_S(\widehat{D_i})$. QED.

5. Regular meromorphic differential forms

Let M be a complex variety, $\dim M = m$, and let $X \subset M$ be an analytical subset in a neighborhood of $x \in U \subset M$ defined by a sequence of functions $f_1, \ldots, f_k \in \mathcal{O}_U$. We denote by Ω^q_X, $q \geq 0$, the sheaves of germs of *regular holomorphic* differential q-forms on X; they are defined as restriction to X of the quotient module

$$\Omega^q_X = \Omega^q_U/((f_1, \ldots, f_k)\Omega^q_U + df_1 \wedge \Omega^{q-1}_U + \ldots + df_k \wedge \Omega^{q-1}_U)\Big|_X.$$

Then the usual differential d endows this family of sheaves with structure of a complex; it is called the de Rham complex on X and is denoted by $(\Omega^\bullet_X, d)$.

Throughout this section we assume that X is a *Cohen-Macaulay* space and $\dim X = n$. Then

$$\omega^n_X = \mathrm{Ext}^{m-n}_{\mathcal{O}_M}(\mathcal{O}_X, \Omega^m_M)$$

is called the Grothendieck dualizing module of X.

Definition 2. For any $q \geq 0$ the coherent sheaf of $\mathcal{O}_X$-modules ω_X^q is locally defined as the set of germs of meromorphic differential forms ω of degree q on X such that $\omega \wedge \eta \in \omega_X^n$ for any $\eta \in \Omega_X^{n-q}$. In other words (see [5]),

$$\omega_X^q \cong \operatorname{Hom}_{\mathcal{O}_X}(\Omega_X^{n-q}, \omega_X^n) \cong \operatorname{Ext}_{\mathcal{O}_M}^{m-n}(\Omega_X^{n-q}, \Omega_M^m).$$

Elements of ω_X^q are called *regular meromorphic* differential q-forms on X. There are also other equivalent definitions of such forms in terms of Noether normalization and trace (see [13], [5]), in terms of residual currents (see [3]), and so on. Here are some useful properties of regular meromorphic differential forms.

1) $\omega_X^q = 0$, if $q < 0$ or $q > \dim X$;

2) ω_X^q has no torsion, that is, $\operatorname{Tors} \omega_X^q = 0$, $q \geq 0$;

3) de Rham differential d acting on ω_X^q is extended on the family of modules ω_X^q, $0 \leq q \leq n$, and endows this family with structure of complex $(\omega_X^\bullet, d)$;

4) there exists an inclusion $\omega_X^q \subseteq j_*j^*\Omega_X^q$, where $j\colon X \setminus Z \longrightarrow X$ is the canonical inclusion and $Z = \operatorname{Sing} X$; moreover, if X is a *normal* space, then $\omega_X^q \cong j_*j^*\Omega_X^q$;

5) if $\pi\colon \widetilde{X} \to X$ is a finite morphism of the *normalization* of X, then the mapping of direct image $\pi_*\colon \omega_{\widetilde{X}}^\bullet \to \omega_X^\bullet$ is injective; if moreover the germ of the normalization is smooth and the codimension of the set of points, in neighborhood of which π is a local isomorphism, is not less than two, then mapping π_* is surjective (see [5]). This means that $\omega_{\widetilde{X}}^\bullet$ and $\omega_X^\bullet$ are isomorphic and, in particular, $\Omega_{\widetilde{X}}^\bullet \cong \omega_X^\bullet$.

6) if X is a *simple rational* singularity of type A_k, D_k, E_6, E_7 or E_8, then the complex $(\omega_X^\bullet, d)$ is acyclic in positive dimensions (see [11]), that is, $\omega_X^\bullet$ is a *resolution* of the constant sheaf $\mathbb{C}_X$.

Let us now assume that $X = C$ is a *complete intersection* given by a regular sequence of functions $f_1, \dots, f_k \in \mathcal{O}_U$ in a neighborhood U of a point $x \in C$. Then $n = m - k$ and

$$\omega_{C,x}^n = \operatorname{Ext}_{\mathcal{O}_{M,x}}^k(\mathcal{O}_{C,x},\ \Omega_{M,x}^m) \cong \mathcal{O}_{C,x}(\omega_0),$$

where ω_0 is the uniquely (modulo $df_1, \dots, df_k$) determined meromorphic differential n-form in $j_*j^*\Omega_{C,x}^n$ for which there is a representation $\omega_0 \wedge df_1 \wedge \ldots \wedge df_k = dz_1 \wedge \ldots \wedge dz_m$ in $j_*j^*(\Omega_{M,x}^m \otimes \mathcal{O}_{C,x})$ with local coordinates $z_1, \dots, z_m$ in U. Thus, the dualizing module ω_C^n is a *locally free* $\mathcal{O}_C$-module of rank one. Furthermore, there are isomorphisms of $\mathcal{O}_M$-modules

$$\omega_C^q \cong \operatorname{Hom}_{\mathcal{O}_C}(\Omega_C^{n-q},\ \mathcal{O}_C) \cong \operatorname{Ext}_{\mathcal{O}_M}^k(\Omega_C^{n-q},\ \mathcal{O}_M), \quad 0 \leq q \leq n.$$

Changing by places the arguments of the extension group Ext^k, one obtains another useful description of regular meromorphic forms [5].

Lemma 2. *Let a subspace $C \subset M$ be a complete intersection. Then there is an exact sequence of $\mathcal{O}_C$-modules*

$$0 \longrightarrow \omega_C^q \xrightarrow{\mathcal{C}} \operatorname{Ext}_{\mathcal{O}_M}^k\big(\mathcal{O}_C,\ \Omega_M^{q+k}\big) \xrightarrow{\mathcal{E}} \Big(\operatorname{Ext}_{\mathcal{O}_M}^k\big(\mathcal{O}_C,\ \Omega_M^{q+k+1}\big)\Big)^k, \quad q \geq 0, \tag{10}$$

where $\omega_C^q \subseteq j_*j^*\Omega_C^q$, the morphism $\mathcal{C}$ is the multiplication by the fundamental class C, and the mapping $\mathcal{E}$ is locally defined by $\mathcal{E}(e) = (e \wedge df_1,\ \dots,\ e \wedge df_k)$.

Corollary 3. *Let $C = C_1 \cup \dots \cup C_r$ be an irredundant decomposition of a complete intersection space C. Then there is a canonical inclusion of complexes of regular meromorphic forms*

$$\omega_{C_1}^\bullet \oplus \dots \oplus \omega_{C_r}^\bullet \hookrightarrow \omega_C^\bullet.$$

Proof. It is sufficient to examine the case $r = 2$. Thus, let $C = C' \cup C''$ be the union of two sets which consist of irreducible components of C and have no common elements. One can apply the functor $\operatorname{Ext}_{\mathcal{O}_M}^*$ to the short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C'} \oplus \mathcal{O}_{C''} \to \mathcal{O}_{C' \cap C''} \to 0,$$

then use Lemma 2 and standard properties of functor Ext. QED.

6. Multiple residues of logarithmic forms

As already mentioned before (cf. Corollary 1) for logarithmic differential forms with poles along a divisor satisfying assumptions of Section 3, there exists representation (4), and, consequently, the restriction of multiple residue map Res_C to the subspace of such logarithmic forms is well-defined.

Lemma 3. *Let $\omega \in \Omega^q_S(\log D)$ be a differential form with logarithmic poles along D and let $C = D_1 \cap \ldots \cap D_k$ be a complete intersection. Then the multiple residue map commutes with de Rham differentiation.*

Proof. Let us apply differentiation d to representation (4) for ω :

$$\omega = \frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \frac{\xi}{g} + \frac{\eta}{g},$$

Corollary 1 implies that the form η is logarithmic as well as its total differential $d\eta$. Thus, $\mathrm{Res}_C(d\omega) = d(\frac{\xi}{g})\big|_C$. This completes the proof. QED.

The following assertion in the case $k = 1$ has been obtained in [1] (see also [2]); we give a proof in general case $k > 1$ similarly to the proof of Theorem from [3].

Theorem 2. *In notations of Section 3 let $C = D_1 \cap \ldots \cap D_k$ be a complete intersection. Then for $p \geq k$ there is an exact sequence of $\mathcal{O}_S$-modules*

$$0 \longrightarrow \textstyle\sum_{i=1}^k \Omega^p_S(\log \widehat{D_i}) \longrightarrow \Omega^p_S(\log D) \overset{\mathrm{Res}_C}{\longrightarrow} \omega^{p-k}_C \longrightarrow 0.$$

Proof. Let us first compute the kernel of the restriction of the multiple residue morphism Res_C to $\Omega^\bullet_S(\log D)$. In view of Claim 3 from Section 2 and Lemma 1 from Section 4 it is sufficient to verify that for all $j = 1, \ldots, k$ one has

$$\Omega^\bullet_S(\log D) \cap \Omega^\bullet_S(\widehat{D_j}) = \Omega^\bullet_S(\log \widehat{D_j}).$$

Since $\Omega^\bullet_S(\log \widehat{D_j}) \subseteq \Omega^\bullet_S(\log D)$ then the right side is contained in the leftist. To prove the converse inclusion we examine in detail the case $k = 2$. Thus, take $\omega \in \Omega^\bullet_S(\widehat{D_1}) = \Omega^\bullet_S(D_2) \cong \frac{1}{h_2}\Omega^\bullet_S$, that is, $\omega = \xi/h_2$. If $\omega \in \Omega^\bullet_S(\log D)$, then $h\omega = h_1\xi \in \Omega^\bullet_S$ and

$$dh \wedge \omega = dh_1 \wedge \xi + dh_2 \wedge (h_1\omega) \in \Omega^\bullet_S.$$

This implies that $dh_2 \wedge (h_1\omega) \in \Omega^\bullet_S$, that is, $dh_2 \wedge (h_1\xi) \in (h_2)\Omega^\bullet_S$. Therefore, $h_1 dh_2 \wedge \xi = h_2\eta$, where $\eta \in \Omega^\bullet_S$. Since h_1 and h_2 form a regular sequence, then, comparing coefficients of the differential forms $dh_2 \wedge \xi$ and η, one obtains that h_1 divides η, and, therefore, $dh_2 \wedge \xi \in (h_2)\Omega^\bullet_S$, $dh_2 \wedge \omega \in \Omega^\bullet_S$, that is, $\omega \in \Omega^\bullet_S(D_2) = \Omega^\bullet_S(\widehat{D_1})$ as required. The general case $k \geq 2$ is analyzed analogously. QED.

Now we are going to describe the image of morphism Res_C, following the scheme of the proof from [1], § 4. Thus, it suffices to check everything locally. Let us first note that the image of Res_C is an $\mathcal{O}_C$-module, since in view of Proposition 1 of Section 2 there are inclusions $h_j\Omega^\bullet_S(\log D) \subseteq \Omega^\bullet_S(\log \widehat{D_j})$ for all $j = 1, \ldots, k$. In particular, the ideal $\mathcal{I} = (h_1, \ldots, h_k)$ annihilates this image. Further, Remark 2 yields that

$$\Delta_{i_1 \ldots i_k} \cdot \mathrm{Res}_C \Omega^q_S(\log D)\big|_U \subset \Omega^{q-k}_C\big|_{C \cap U},$$

for maximal minors $\Delta_{i_1 \ldots i_k}$, $(i_1, \ldots, i_k) \in [1, \ldots, m]$ of Jacobian matrix $\mathrm{Jac}(h_1, \ldots, h_k)$. Since $\omega^n_{C,0} \cong \mathcal{O}_{C,0}(dz_1 \wedge \ldots \wedge dz_{n+k}/dh_1 \wedge \ldots \wedge dh_k)$, then by definition of regular meromorphic forms one obtains that $\mathrm{Res}_C\big(\Omega^q_{S,0}(\log D)\big) \subseteq \Omega^{q-k}_{C,0}$. Let now $\mathcal{K}_\bullet(\underline{h})$ be the usual Koszul complex associated with the regular sequence $\underline{h} = (h_1, \ldots, h_k)$:

$$0 \to \mathcal{O}_{S,0}\langle e_0 \wedge \ldots \wedge e_{k-1}\rangle \overset{d_{k-1}}{\longrightarrow} \ldots \overset{d_1}{\longrightarrow} \sum_{i=0}^{k-1} \mathcal{O}_{S,0}\langle e_i\rangle \overset{d_0}{\longrightarrow} \mathcal{O}_{S,0} \overset{d_{-1}}{\longrightarrow} \mathcal{O}_{C,0} \to 0 \ ,$$

where $\mathcal{K}_k(\underline{h}) = \mathcal{O}_{S,0}\langle e_0 \wedge \ldots \wedge e_{k-1}\rangle$, $\ldots$, $\mathcal{K}_1(\underline{h}) = \mathcal{O}_{S,0}\langle e_0\rangle + \ldots + \mathcal{O}_{S,0}\langle e_{k-1}\rangle$, $\mathcal{K}_0(\underline{h}) = \mathcal{O}_{S,0}$, $d_0(e_i) = h_{i+1}$, $i = 0, \ldots, k-1$, $d_{-1}(1) = 1$.

The dual exact sequence implies an isomorphism

$$\mathrm{Ext}^k_{\mathcal{O}_{S,0}}(\mathcal{O}_{C,0},\ \Omega^{q+1}_{S,0}) \cong \mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_k(\underline{h}),\ \Omega^{q+1}_{S,0})/d^{k-1}\big(\mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_{k-1}(\underline{h}),\ \Omega^{q+1}_{S,0})\big).$$

Thus, any element from the space $\mathrm{Ext}^k_{\mathcal{O}_{S,0}}(\mathcal{O}_{C,0},\ \Omega^{q+1}_{S,0})$ can be represented as a Čzech $(k-1)$-cochain (more precisely, as a $(k-1)$-cocycle) as follows:

$$\frac{\nu}{h_1\cdots h_k}\in \mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_k(\underline{h}),\ \Omega^{q+1}_{S,0})\cong C^{k-1}_{(k)}(\Omega^{q+1}_{S,0}),$$

where $\nu\in\Omega^{q+1}_{S,0}$. Let us consider an element $\nu\in\Omega^{q+1}_{S,0}$ such that the meromorphic form

$$\frac{\nu}{h_1\cdots h_k}\wedge dh_j\in \mathrm{Ext}^k_{\mathcal{O}_{S,0}}(\mathcal{O}_{C,0},\ \Omega^{q+2}_{S,0}),\quad j=1,\dots,k,$$

corresponds to the *trivial* element.

This means that for any $j=1,\dots,k$ the differential form $\nu\wedge dh_j/h_1\cdots h_k$ is determined by a certain element from the space $d^{k-1}\big(\mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_{k-1}(\underline{h}),\ \Omega^{q+2}_{S,0})\big)$. Hence, one gets

$$\nu\wedge dh_j\in\sum_{i=1}^{k}h_i\Omega^{q+2}_S,\quad j=1,\dots,k,$$

or, equivalently,

$$\omega\wedge dh_j\in\sum_{i=1}^{k}\Omega^{q+2}_S(\widehat{D_i}),\quad j=1,\dots,k,\quad \text{where}\quad \omega=\frac{\nu}{h_1\cdots h_k}.$$

As a result, the differential form ω satisfies conditions (3) of Theorem 1. It remains to use exact sequence (10) of Lemma 2 as follows.

Let $\tilde\nu=\mathcal{C}^{-1}(\nu/h_1\cdots h_k)$. Then $\mathcal{C}(\tilde\nu)$ corresponds to Čzech cocycle $\nu/h_1\cdots h_k$ such that $\nu=\tilde\nu\wedge dh_1\wedge\cdots\wedge dh_k$. Making use of the description for ω^q_C in terms of multiplication by the fundamental class $C\subset S$ in exact sequence (10), one can take $v=\tilde\nu$, $w=\nu$, since $\mathcal{C}(v)$ corresponds to Čzech cocycle w/h such that $w=v\wedge dh_1\wedge\cdots\wedge dh_k$. This implies

$$\omega=\tilde\nu\wedge\frac{dh_1}{h_1}\wedge\cdots\wedge\frac{dh_k}{h_k},\qquad \mathrm{Res}_C(\omega)=\mathrm{Res}_C\left(\frac{\nu}{h_1\cdots h_k}\right)=\tilde\nu.$$

Thus, for any element $\tilde\nu\in\omega^{q-k}$ there exists a preimage relatively to the residue map Res_C represented by $\omega=\nu/h_1\cdots h_k$ such that the differential form $h\omega$ is holomorphic, and $dh\wedge\omega=0$. In particular, this means that $\omega\in\Omega^\bullet_S(\log D)$ as required. QED.

Remark 5. In notations of Remark 3 let us take $k=1$, and $C=D$. Then $\mathrm{Res}_C=\mathrm{Res}_D$; it is, in fact, the residue map res. introduced by K.Saito [20]. In this case there is (see [1]) an exact sequence

$$0\longrightarrow\Omega^q_S\longrightarrow\Omega^q_S(\log D)\xrightarrow{\text{res.}}\omega^{q-1}_D\longrightarrow 0,\ q\geq 1, \tag{11}$$

supplementing diagram (2.5) of [20] from the right side. Thus, Theorem 2 can be considered as an extension of this sequence for the multiple residue map.

Corollary 4. *Under the same assumptions there is a natural isomorphism*

$$\mathcal{H}^p_{DR}(\Omega^\bullet_S(\log D))\cong\mathcal{H}^{p-1}_{DR}(\omega^\bullet_D),$$

where $\mathcal{H}^*_{DR}$ is the functor of cohomologies of complexes endowed with de Rham differentiation d. In particular, $\Omega^\bullet_S(\log D)$ is acyclic in dimensions $p>1$ when D is a simple rational singularity of type A_k, D_k, E_6, E_7 or E_8 of dimension $n\geq 2$.

Proof. The residue map is compatible with differentiation d. Hence, exact sequences (11) for all $q\geq 1$ are composed in the exact sequence of the corresponding complexes. This yields the desired isomorphism. Further, it is known (see [11], Bem. (4.8), (2)) that for rational singularities the complex $(\omega^\bullet_D, d)$ is acyclic in positive dimensions; this implies the second part of the statement. In addition, since the dimension of $\mathcal{H}^0_{DR}(\omega^\bullet_D)$ is equal to the number of irreducible components of D [loc. cite, (4.1)], then $\mathcal{H}^1_{DR}(\Omega^\bullet_S(\log D))\cong\mathbb{C}$ under our assumptions. For completeness, it should be mentioned that these results can be also obtained by direct computations (see [10]).

7. Closed differential forms and the image of the residue map

As was discussed before the image of Poincaré-Leray residue map consists of holomorphic forms on a smooth hypersurface D; in this case $\Omega^\bullet_D$ and $\omega^\bullet_D$ are naturally isomorphic. Let us prove in the context of the theory of logarithmic differential forms the following statement for singular hypersurfaces due to G. de Rham [17] (see also [14], p.83).

Theorem 3. *Let D be a hypersurface in a manifold S, $\dim S = m \geq 3$. Assume that $\operatorname{Sing} D$ consists of isolated double quadratic points and ω is a holomorphic d-closed p-form on $S \setminus D$ with a pole of the first order on D. Then the residue-form $\operatorname{res}_D(\omega)$ is holomorphic at singular points of D if and only if either $p < m$, or $p = m$ and the functional coefficient of m-form $\omega(z)h(z)$ vanishes on $\operatorname{Sing} D$.*

Proof. At first remark, that $\Omega^m_S(D) \cong \Omega^m_S(\log D)$, and $d\omega = 0$ for all $\omega \in \Omega^m_S(D)$. By assumptions, for any $p < m$ the differential p-form $\omega \in \Omega^p_S(D)$ is closed, $d\omega = 0$. Thus, $dh \wedge \omega = d(h\omega)$ is holomorphic at $x \in S$. That is, ω is *logarithmic,* $\omega \in \Omega^p_{S,x}(\log D)$.

In view of Remark 3 for such differential form ω there exists locally representation (9) with a holomorphic function g, a non-zero divisor of $\mathcal{O}_{S,x}/(h)\mathcal{O}_{S,x}$, where $x \in \operatorname{Sing} D$ and h is equal to the sum of squares of local coordinate functions, $h = z_1^2 + \ldots + z_m^2$, in a suitable neighborhood of x. Moreover, $h\omega$ is a *torsion* element of $\Omega^p_{D,x}$ and there is an exact sequence (see [1], [2])

$$0 \longrightarrow \Omega^p_{S,x} + \frac{dh}{h} \wedge \Omega^{p-1}_{S,x} \longrightarrow \Omega^p_{S,x}(\log D) \xrightarrow{\cdot h} \operatorname{Tors}\Omega^p_{D,x} \longrightarrow 0.$$

Since $m \geq 3$ and $\operatorname{Sing} D$ consists of *isolated* double quadratic points, then D is a *normal* irreducible hypersurface. Hence, $\operatorname{Tors}\Omega^p_{D,x} = 0$ for all $p < \operatorname{codim}(\operatorname{Sing} D, D) = m-1$, and for such p one has $\omega \in \frac{dh}{h} \wedge \Omega^{p-1}_{S,x} + \Omega^p_{S,x}$, that is, the function g in representation (9) is *invertible* at x. Consequently, $\operatorname{res}_D(\omega) = \xi\big|_D$ is holomorphic on D.

When $p = m$, then $\omega = \varphi\, dz_1 \wedge \ldots \wedge dz_m/h$, where φ is a holomorphic function germ. The vanishing of φ at $x \in \operatorname{Sing} D$ yields that $h\omega = dh \wedge \xi$. Hence $\operatorname{res}_D(\omega) = \xi\big|_D$, where ξ is holomorphic at $x \in S$ and vice versa.

It remains to analyze the case $p = m-1$. In this case one has $\operatorname{Tors}\Omega^{m-1}_{D,x} = \Omega^m_{D,x} \neq 0$. To be more precise, if $z_1, \ldots, z_m$ is a local coordinate system at $x \in S$, $x = 0$, then $\operatorname{Tors}\Omega^{m-1}_{D,x}$ is generated over $\mathbb{C}$ by the Euler differential form $\vartheta = \sum (-1)^{\ell-1} z_\ell\, dz_1 \wedge \ldots \wedge \widehat{dz_\ell} \wedge \ldots \wedge dz_m$, the result of contraction of the canonical generator of $\operatorname{Tors}\Omega^m_{D,x} = \Omega^m_{D,x} \cong \mathbb{C}\langle dz_1 \wedge \ldots \wedge dz_m\rangle$ along Euler vector field. The differential form ϑ/h is *not closed,* since $d(\vartheta/h) = (m-2)dz_1 \wedge \ldots \wedge dz_m/h$. Since $\mathcal{O}_{D,x}$ is a domain, then all partial derivatives $\partial h/\partial z_\ell$, $\ell = 1, \ldots, m$, are non-zero divisors. Therefore one can take the multiplier in representation (9) equal to any z_ℓ. Explicit calculations show that for $g = z_\ell$ one has

$$\begin{aligned} \xi &= \tfrac{1}{2} \sum_{j=1,\, j\neq \ell}^{m} (-1)^{\ell+j} \operatorname{sgn}(j-\ell)\, z_j\, dz_1 \wedge \ldots \wedge \widehat{dz_\ell} \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_m, \\ \eta &= (-1)^{\ell-1} dz_1 \wedge \ldots \wedge \widehat{dz_\ell} \wedge \ldots \wedge dz_m. \end{aligned}$$

It is clear that z_ℓ does not divide ξ; hence, the differential $(m-2)$-form $\operatorname{res}_D(\vartheta/h) = \frac{\xi}{g}\big|_D$ is *not holomorphic* on D. Let us describe conditions under which a logarithmic form $\omega = \eta_1 + \frac{dh}{h} \wedge \xi_1 + \varphi\frac{\vartheta}{h}$ with holomorphic η_1, ξ_1 and φ, has a holomorphic residue on D. Without loss of generality one can suppose that the above differential forms and functions are homogeneous at the distinguished point x. If φ is *invertible* at x, then $\operatorname{res}_D(\vartheta/h) = -\frac{\xi_1}{\varphi}\big|_D$ is holomorphic at x; this contradicts to the above computations. Moreover, in such a case ω is *not closed.* Otherwise, if $d\omega = 0$, then there is an identity

$$d\eta_1 - \frac{dh}{h} \wedge d\xi_1 + (m-2)\varphi\frac{dz_1 \wedge \ldots \wedge dz_m}{h} = 0,$$

or, equivalently,

$$h\eta_1 - dh \wedge d\xi_1 + (m-2)\varphi dz_1 \wedge \ldots \wedge dz_m = 0.$$

However, it is impossible, since h and dh vanish at x, while φ is invertible. Finally, let suppose that φ is *not invertible,* that is, φ is contained in the maximal ideal of $\mathcal{O}_{D,x}$. In this case $\varphi\frac{\vartheta}{h}$ is contained in $\Omega^{m-1}_{S,x} + \frac{dh}{h} \wedge \Omega^{m-2}_{S,x}$ in view of the above calculations. As a result, ω has a holomorphic residue on D. In particular, we also obtain that all closed logarithmic $(m-1)$-forms are contained in $\Omega^{m-1}_{S,x} + \frac{dh}{h} \wedge \Omega^{m-2}_{S,x}$, and, obviously, their residues are holomorphic on D. QED.

Remark 6. It is useful to examine also the case $m = 2$ separately. Thus, $h = z^2 + w^2$, that is, D is a node; it is a divisor with normal crossing in a plane. The module $\Omega^1_S(\log D)$ is generated by differential forms dh/h and ϑ/h, where $\vartheta = -wdz + zdw$. It is not difficult to verify that $d(\vartheta/h) = 0$ in contrast with the case $m \geq 3$ considered in the above Theorem. Furthermore, $\mathrm{res}_D(\vartheta/h) = -\frac{w}{z}\big|_D = \frac{z}{w}\big|_D$ is *not holomorphic* on D. Simple considerations show that this residue is, in fact, a *weakly holomorphic* function on D, that is, it is holomorphic only on the *normalization* $\widetilde{D}$ of D. Really, changing coordinate system, one gets $h = zw$, and $\Omega^1_S(\log D)$ is generated by two *closed* differential forms dz/z and dw/w whose residues are holomorphic on $\widetilde{D}$, but not on D (see [18], (2.11)). In fact, this phenomenon occurs not only for divisors with normal crossings (see [loc.cite, Th. (2.9)]). Curiously that in the original formulation of the theorem as well as in its later citations the restriction $m \geq 3$ is omitted (cf. [14], pp. 84, 103, or [9], § 5).

More generally, in a similar style one can describe the image of the multiple residue map for divisors with normal crossings. In this case this map coincides with the multidimensional Poincaré residue considered in [7], (3.1.5). To be more precise, residues of logarithmic p-forms along the union of any collection $D_{i_1} \cup \ldots \cup D_{i_\ell}$ of irreducible components of D consist of restrictions to the intersection $D_{i_1} \cap \ldots \cap D_{i_\ell}$ of differential $(p-\ell)$-forms holomorphic on the ambient space. Hence they are regular holomorphic on the intersection as well as on its normalization since the map of direct image π_* is an isomorphism in view of property 5) of Section 5.

The next example is a simple modification of the above. By definition, $\mathcal{O}_S$-modules of logarithmic differential p-forms of *principal type* $\Omega^p_S\langle D\rangle$, $p \geq 0$, are defined as follows:

$$\Omega^0_S\langle D\rangle = \mathcal{O}_S, \quad \Omega^1_S\langle D\rangle = \sum_{i=1}^{k} \mathcal{O}_S \frac{dh_i}{h_i} + \Omega^1_S, \quad \Omega^p_S\langle D\rangle = \bigwedge^p \Omega^1_S\langle D\rangle,\ p \geq 2.$$

One can easily verify that the family $\Omega^p_S\langle D\rangle$, $p \geq 0$, forms a subcomplex of the logarithmic de Rham complex $\Omega^\bullet_S(\log D)$ closed under the external differentiation and external product by dh_i/h_i, $1 \leq i \leq k$. Clearly, for divisors with normal crossings the equality $\Omega^p_S\langle D\rangle = \Omega^p_S(\log D)$ holds for all $p \geq 0$. Further, any logarithmic form of principal type has decomposition (4) of Theorem 1 with an *invertible* multiplier g. Similarly to the case of divisors with normal crossings, multiple residues of such forms are *holomorphic* on the corresponding complete intersection.

More generally, if D is a divisor such that there is an isomorphism $\Omega^p_S\langle D\rangle \cong \Omega^p_S(\log D)$ for certain $p \geq k$, then the image of the multiple residue map can be characterized as above. A special class of such divisors considered in cohomology theory of the "twisted" de Rham complex can be described as follows (another examples are also studied in [15]).

Let h_j, $j = 1, \ldots, \ell$, be non-constant homogeneous polynomials on S. Denote the ideal generated by all minors $\Delta_{i_1 \ldots i_r}$ of maximal order of Jacobian matrix $\mathrm{Jac}(h_{i_1}, \ldots, h_{i_r})$ and polynomials $h_{i_1}, \ldots, h_{i_r}$ by $\mathcal{G}_{i_1 \ldots i_r} \subset \mathcal{O}_{S,0}$.

Claim 4. *Assume that for any $1 \leq r \leq \min\{\ell, m-1\}$, the algebraic set defined by the ideal $\mathcal{G}_{i_1 \ldots i_r}$ is either empty or the origin, and $h_{i_1}, \ldots, h_{i_s}$ is a regular sequence for $1 \leq s \leq \min\{\ell, m\}$. Then any logarithmic differential p-form, $0 \leq p \leq m-2$, has decomposition* (4) *of Theorem* 1 *with an invertible multiplier g, and there are isomorphisms $\Omega^p_S\langle D\rangle \cong \Omega^p_S(\log D)$.*

Proof. It is a slightly modified version of considerations from [12] or [15]. QED.

8. The weight filtration

The concept of weight filtration on the logarithmic de Rham complex for divisors with normal crossings on manifolds was introduced by P.Deligne [7] for computation of the mixed Hodge structure on the cohomologies of the complement. The case of divisors with normal crossings on V-varieties was examined by J.Steenbrink [22]. In this section we construct the weight filtration on the logarithmic de Rham complex for divisors whose components are defined by a regular sequence of functions. In particular, this allows to compute the mixed Hodge structure on cohomology of the complement of divisors of certain types without using theorems on resolution of singularities or the standard reduction to the case of normal crossings.

Let X be an analytical manifold, $D \subset X$ be a reduced divisor with *irreducible* decomposition $D = D_1 \cup \ldots \cup D_k$. It is also assumed that D has no components with self-intersections. For any ordered collection $I = (i_1 \cdots i_n)$, $1 \le i_1 < \ldots < i_n \le k$, of length $n = \#(I)$, let us consider the following germs:

$$D_I = D_{(i_1 \cdots i_n)} = D_{i_1} \cup \ldots \cup D_{i_n}, \quad C^I = C^{(i_1 \cdots i_n)} = D_{i_1} \cap \ldots \cap D_{i_n}.$$

We denote by $C^{(n)}$ an analytical subspace of X given by the union of $C^{(i_1 \cdots i_n)}$ for all permissible collections so that $C^{(1)} = D$, $C^{(k)} = C^{(i_1 \cdots i_k)} = C$, and so on. Let us also set $D_0 = C^0 = \emptyset$.

Definition 3. The *weight filtration*, or *filtration of weights* W on the logarithmic de Rham complex $\Omega^p_X(\log D)$ is locally defined as follows:

$$W_n(\Omega^p_{X,x}(\log D)) = \begin{cases} 0, & n < 0; \\ \Omega^p_{X,x}, & n = 0; \\ \sum\limits_{\#(I)=p} \Omega^p_{X,x}(\log D_I), & n \ge p,\ 0 < p < k_x, \\ \sum\limits_{\#(I)=n} \Omega^p_{X,x}(\log D_I), & \text{otherwise}, \end{cases}$$

where k_x is the number of irreducible components of D passing through the point $x \in X$.

First non-trivial elements of the weight filtration in the case $k = 3$.

$$\begin{array}{ccccc}
W_0 & \Omega^1_X & \Omega^2_X & \Omega^3_X & \Omega^4_X \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
W_1 & \sum \Omega^1_X(\log D_i) & \sum \Omega^2_X(\log D_i) & \sum \Omega^3_X(\log D_i) & \sum \Omega^4_X(\log D_i) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
W_2 & \sum \Omega^1_X(\log D_i) & \sum \Omega^2_X(\log(D_i \cup D_j)) & \sum \Omega^3_X(\log(D_i \cup D_j)) & \sum \Omega^4_X(\log(D_i \cup D_j)) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
W_3 & \sum \Omega^1_X(\log D_i) & \sum \Omega^2_X(\log(D_i \cup D_j)) & \sum \Omega^3_X(\log D) & \sum \Omega^4_X(\log D) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Thus, $W_n(\Omega^p_{X,x}(\log D)) = \Omega^p_{X,x}(\log D)$, if $n \ge p \ge k_x$. Further, W is an increasing filtration and, in view of d- and $\wedge$-closeness of $\Omega^\bullet_X(\log D)$, there exist the following natural inclusions

$$\begin{aligned} d(W_n(\Omega^\bullet_X(\log D))) &\subset W_n(\Omega^\bullet_X(\log D)), \\ W_n(\Omega^p_X(\log D)) \bigwedge W_\ell(\Omega^q_X(\log D)) &\subset W_{n+\ell}(\Omega^{p+q}_X(\log D)) \end{aligned}$$

for all entire numbers p, q, n, ℓ. It should be remarked that for any $n \le p$ the module $W_n(\Omega^p_{X,x}(\log D))$ contains all differential forms of principal type from $\Omega^n_X\langle D\rangle$ considered in Section 7 :

$$\frac{dh_{i_1}}{h_{i_1}} \wedge \cdots \wedge \frac{dh_{i_\ell}}{h_{i_\ell}} \wedge \Omega^{p-\ell}_{X,x}, \quad 1 \le i_1 < \ldots < i_\ell \le k,\ 1 \le \ell \le n,$$

where $h_{i_1}, \ldots, h_{i_\ell}$ are local equations of the corresponding components of divisor D passing through the point $x \in X$. In general,

$$\Omega^n_{X,x}\langle D\rangle \wedge \Omega^{p-n}_{X,x} \subseteq W_n(\Omega^p_{X,x}(\log D)) \subseteq \Omega^n_{X,x}(\log D) \wedge \Omega^{p-n}_{X,x},\ n \in \mathbb{Z},\ p \ge n.$$

For divisors with normal crossings two complexes $\Omega^\bullet_X\langle D\rangle$ and $\Omega^\bullet_X(\log D)$ are equal. Therefore, both inclusions in the latter formula are, in fact, equalities and the weight filtration on the complex $\Omega^\bullet_X\langle D\rangle$ is given as follows:

$$W_n(\Omega^p_X\langle D\rangle) = \Omega^n_X\langle D\rangle \wedge \Omega^{p-n}_X,\ n \in \mathbb{Z}.$$

The following assertion can be considered as a generalization of isomorphism (3.1.5.2) from [7] valid for divisors with normal crossings to the case of divisors whose components are given by a regular sequence of functions.

Let $\pi\colon \widetilde{C}^{(n)} \to C^{(n)}$ be a morphism of normalization so that $\widetilde{C}^{(n)}$ coincides with the non-connected sum of normalizations $\widetilde{C}^{(i_1\cdots i_n)}$ for all possible collections of length $n \geq 1$. We denote by ι the projection $\widetilde{C}^{(n)}$ in X, so that $\iota = i \circ \pi$, where $i\colon C^{(n)} \to X$ is a natural inclusion.

Proposition 3. *Let us assume that a divisor D satisfies assumptions of Theorem 1 and the morphism of normalization induces an isomorphism of complexes $\pi_*\colon \omega^\bullet_{\widetilde{C}^{(n)}} \cong \omega^\bullet_{C^{(n)}}$. Then the multiple residue map*

$$\operatorname{Res}^\bullet_n\colon W_n(\Omega^\bullet_X(\log D)) \longrightarrow \iota_*\omega^\bullet_{\widetilde{C}^{(n)}}[-n],$$

induces an isomorphism of complexes of $\mathcal{O}_X$-modules

$$\operatorname{Gr}^W_n(\Omega^\bullet_X(\log D)) \cong \iota_*\omega^\bullet_{\widetilde{C}^{(n)}}[-n].$$

Proof. Let firstly note that the morphism of normalization induces the isomorphism of direct image π_* if condition 5) from Section 5 is fulfilled. Furthermore, it suffices to prove our assertion locally, for the germ (X, x) and for all $n \leq p$.

For any ordered collection $I = (i_1 \cdots i_n)$, $1 \leq i_1 < \ldots < i_n \leq k_x$, accordingly Theorem 2 with $D = D_I$ there exists an exact sequence of complexes of $\mathcal{O}_{X,x}$-modules

$$0 \longrightarrow \sum_{\ell=1}^{n} \Omega^\bullet_{X,x}(\log(\widehat{D_I})_{i_\ell}) \longrightarrow \Omega^\bullet_{X,x}(\log D_I) \overset{\operatorname{Res}_{C^I}}{\longrightarrow} \omega^\bullet_{C^I,x}[-n] \longrightarrow 0.$$

From basic properties of regular meromorphic differential forms it follows that $\omega^\bullet_{\widetilde{C}^{(n)}}$ is isomorphic to the direct sum $\omega^\bullet_{\widetilde{C}^I}$ taking through all permissible collections $I = (i_1 \cdots i_n)$. Further, any differential form $\omega \in W_n(\Omega^\bullet_X(\log D))$ is decomposed into the sum of elements $\omega_I \in \Omega^\bullet_X(\log D_I)$. Let us denote the sum of $\operatorname{Res}_{C^I}(\omega_I)$ by $\operatorname{Res}_{C^{(n)}}(\omega)$. One then obtains an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow W_{n-1}(\Omega^\bullet_X(\log D)) \longrightarrow W_n(\Omega^\bullet_X(\log D)) \overset{\operatorname{Res}_{C^{(n)}}}{\longrightarrow} \iota_*\omega^\bullet_{\widetilde{C}^{(n)}}[-n] \longrightarrow 0.$$

This yields the existence of required isomorphisms. QED.

Corollary 5. *Under conditions of Proposition 3 there are natural isomorphisms of cohomology spaces*

$$\mathcal{H}^i(\operatorname{Gr}^W_n(\Omega^\bullet_X(\log D))) \cong \mathcal{H}^i(\omega^\bullet_{\widetilde{C}^{(n)}})[-n],$$

where $i \geq 1$ and $1 \leq n \leq k$.

Proof. Since the normalization π is finite and, therefore it is an *affine* morphism then

$$\mathcal{H}^i(\pi_*\omega^\bullet_{\widetilde{C}^{(n)}}) \cong \mathcal{H}^i(\omega^\bullet_{\widetilde{C}^{(n)}}),\ i \geq 1,$$

and the desired assertion follows from Proposition above. QED.

Remark 7. Analyzing a more general situation where complexes $\omega^\bullet_{\widetilde{C}^{(n)}}$ and $\omega^\bullet_{C^{(n)}}$ are non-isomorphic, the corresponding isomorphisms in the formulation of Proposition 3 should be replaced by epimorphisms.

Remark 8. Suppose that a (finite) group G acts on a manifold X. Then it is not difficult to verify that the residue mapping Res_C is *compatible* with the action of this group in the usual sense. In this case the complex of regular meromorphic forms $\omega^\bullet_{X/G}$ on the quotient variety X/G is a *resolution* of constant sheaf [11]. Making use of simplest properties of sheaves $\Omega^p_X(\log D)$, Ω^q_C and the corresponding subsheaves invariant relative to action of G, one obtains the isomorphism of Lemma (1.19) from [22] for divisors with normal crossings on a V-variety.

Let us examine a simple application. The canonical decreasing Hodge filtration F on the logarithmic de Rham complex $\Omega^p_X(\log D)$ is defined as follows:

$$F^n(\Omega^p_X(\log D)) = \begin{cases} \Omega^p_X(\log D), & n \leq p, \\ 0, & n > p. \end{cases}$$

Suppose now that D is a reduced divisor as before and the natural inclusions

$$\sum_{\#(I)=p} \Omega^p_{X,x}(\log D_I) \longrightarrow \Omega^p_{X,x}(\log D) \qquad (12)$$

are isomorphisms for all $1 \leq p < k_x$. Then $W_n(\Omega^p_X(\log D)) \cong \Omega^p_X(\log D)$ for all $n \geq p$ similarly to the classical case of divisors with normal crossings (see (3.1.8) in [7]). Hence, under assumptions of Proposition 3 one can define a natural morphism α from the complex $\Omega^\bullet_X(\log D)$ endowed by Hodge filtration F into the same complex with *decreasing* filtration W given as $W^n = W_{-n}$.

Corollary 6. *Under the same assumptions the above morphism α is a filtered quasi-isomorphism if $\mathcal{H}^i(\omega^\bullet_{\widetilde{C}^{(n)}}) = 0$ for $i \neq 0$.*

Proof. Analogously to the proof of (3.1.8.2) in [7]. QED.

Remark 9. Of course, for divisors with normal crossings inclusions (12) are isomorphisms for all $p \geq 1$. A special class of divisors with $\sum_{i=1}^k \Omega^1_{X,x}(\log D_i) \cong \Omega^1_{X,x}(\log D)$ is considered in Theorem 2.9 by [20] (see Section 2.)

Remark 10. The vanishing condition of Corollary 6 means that the complex of regular meromorphic forms on the normalization $\widetilde{C}^{(n)}$ is *acyclic* in positive dimensions. Besides the case of divisors with normal crossings examined in [7], another types of varieties satisfied this condition are known. Among them there are rational normal complete intersections, quotient singularities of smooth varieties with action of a finite group (see [11]), V-varieties (see [22]), and so on.

Remark 11. If two complexes $\Omega^\bullet_X(\log D)$ and $\Omega^\bullet_X(\star D)$ endowed with standard Hodge filtration are quasi-isomorphic (see, for example, [10]), then the morphism β from Proposition (3.1.8) of [7] is a quasi-isomorphism. If additionally the condition of the previous Corollary 6 is satisfied, then α is also a quasi-isomorphism. This means that in all cases mentioned in Remark above there are isomorphisms (3.1.8.2) of [7]:

$$R^n j_* \mathbf{C} \cong \mathcal{H}^n(j_* \Omega^*_{X^*}) \cong \mathcal{H}^n(\Omega^\bullet_X(\log D)),$$

where X is a manifold, $X^* = X \setminus D$, $j\colon X^* \to X$ is the canonical inclusion.

Further analysis shows that under standard assumptions on the ambient manifold X (smooth, Kähler, complete) the bifiltered complex $(\Omega^\bullet_X(\log D), F, W)$ can be used (similarly to [22], p.532) for computation of the canonical mixed Hodge structure on the cohomology of complements $H^*(X \setminus D, \mathbb{C})$ as well as on the local cohomology $H^*_C(X, \Omega^\bullet_X(\log D))$ without the using of resolution theorems or a standard reduction to the case of divisors with normal crossings.

In conclusion we note that the differential d is *strictly compatible* ([7], (1.1.5)) with filtration W at degree $k+1$, that is,

$$d\Omega^k_X(\log D) \cap W_n(\Omega^{k+1}_X(\log D)) = d(W_n(\Omega^k_X(\log D)), \ n \in \mathbb{Z}.$$

Consequently, the weight filtration on the *canonically truncated* logarithmic de Rham complex

$$\tau_{\geq k} \Omega^\bullet_X(\log D)$$

is also well-defined; in its turn, it induces the weight filtration on the complex of regular meromorphic differential forms on a complete intersection with the help of the multiple residue map.

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