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HOMOTOPICALLY TRIVIAL DEFORMATIONS

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The aim of this note is to call attention to a question about deformations that are homotopically trivial. First we need a definition.

Definition 1. A proper morphism of complex spaces $f : X \to Y$ is called a *homotopy fiber bundle* if Y has an open cover $Y = \cup U_i$ such that for every i and for every $y \in U_i$ the inclusion

$$f^{-1}(y) \hookrightarrow f^{-1}(U_i) \quad \text{is a homotopy equivalence.}$$

For every $y \in Y$ there is an open neighborhood $y \in U_y$ such that $f^{-1}(y)$ is a deformation retract of $f^{-1}(U_y)$. Choose a retraction $r_y : f^{-1}(U_y) \to f^{-1}(y)$. Thus $f^{-1}(y) \hookrightarrow f^{-1}(U_y)$ is a homotopy equivalence and so f is a homotopy fiber bundle iff for every $y \in Y$ and $y' \in U_y$ the induced map

$$r_{y'\to y} : f^{-1}(y') \to f^{-1}(y) \quad \text{is a homotopy equivalence.}$$

Similarly, if R a commutative ring, then $f : X \to Y$ is called an *R-homology fiber bundle* if

$$H_*\big(f^{-1}(y), R\big) \to H_*\big(f^{-1}(U_i), R\big) \quad \text{is an isomorphism.}$$

As above, these conditions hold iff the retraction maps $r_{y'\to y}$ induce isomorphisms

$$\big(r_{y'\to y}\big)_* : H_*\big(f^{-1}(y'), R\big) \to H_*\big(f^{-1}(y), R\big).$$

We are mostly interested in cases when the fibers of f are irreducible.

If the fibers are reducible, some pathological cases are shown by Example 9. To avoid these, one should also assume that the image of the fundamental class $\big[f^{-1}(y)\big]$ in $H_*\big(f^{-1}(U_i), R\big)$ is independent of y. Equivalently, the retraction $r_{y'\to y}$ maps $\big[f^{-1}(y')\big]$ to $\big[f^{-1}(y)\big]$.

All the examples of $\mathbb{Z}$-homology fiber bundles that we know are also homotopy fiber bundles but being a $\mathbb{Q}$-homology fiber bundle is a much weaker property.

The main problem we want to consider is the following.

Question 2. Let $f : X \to Y$ be a homotopy or $\mathbb{Z}$-homology fiber bundle. Under what conditions is it a topological or differentiable fiber bundle?

If X is non-normal, it is easy to give examples of homotopy fiber bundles $f : X \to Y$ where not all fibers $f^{-1}(y)$ are homeomorphic to each other, see Example 8. In the first version of [14] it was asked whether the answer was positive for normal spaces. We show in Example 19 that this is not the case. However, we still hope that for smooth varieties the situation is as nice as possible.

Conjecture 3. *Let $f : X \to Y$ be a homotopy or $\mathbb{Z}$-homology fiber bundle such that X is smooth. Then f is smooth hence $f : X \to Y$ is a differentiable fiber bundle.*

A stronger version, more closely related to deformation theory is formulated as Conjecture 11.

4 (Origin of the conjecture)**.** We were led to this question by the study of universal covers of projective varieties. Their modern study was initiated by Shafarevich [19, Sec.IX.4]; see [11, 2, 18, 4, 3] and the references there for recent results and surveys. One aim of these investigations is to understand projective varieties whose universal cover is "simple." There are several ways to define what "simple" should mean; here we focus on a topological variant considered in [14].

Question 4.1. Describe projective varieties X whose universal cover $\tilde{X}$ is homotopic to a finite CW complex.

This seems to be a rather difficult problem in general, so here we consider a series of special cases that seem especially important for applications.

Let X be a smooth projective variety and $f : X \to Y$ a surjective morphism. Let $\tilde{Y} \to Y$ denote the universal cover. By pull-back we obtain $\tilde{f} : \tilde{X} \to \tilde{Y}$. In light of [14] the following seems quite plausible.

Question 4.2. Assume that $\tilde{Y}$ is contractible and $\tilde{X}$ is homotopic to a finite CW complex. Does this imply that f is a homotopy fiber bundle?

Conversely, if f is a homotopy fiber bundle and $\tilde{Y}$ is homotopic to a finite CW complex then most likely $\tilde{X}$ is homotopic to a finite CW complex. Thus if Conjecture 3 is true then we would have a rather complete understanding of when a variety X with a nontrivial morphism $X \to Y$ has a "simple" universal cover.

5 (First properties)**.** If $f : X \to Y$ is a $\mathbb{Q}$-homology fiber bundle then all fibers $f^{-1}(y)$ have the same dimension and the same number of irreducible components. Thus if X is normal then, by taking the Stein factorization, we may assume that all fibers are irreducible.

Assume that $g : S \to C$ is an elliptic surface such that all reduced fibers are smooth. Then g is a $\mathbb{Q}$-homology fiber bundle but it is a $\mathbb{Z}$-homology fiber bundle only if there are no multiple fibers.

We see below that there are many $\mathbb{Q}$-homology fiber bundles that are not $\mathbb{Z}$-homology fiber bundles (16, 17).

It is much harder to get nontrivial examples of $\mathbb{Z}$-homology fiber bundles. For now we note two basic results.

Proposition 6. *Let X, Y be normal spaces and $f : X \to Y$ a $\mathbb{Z}$-homology fiber bundle. Then every fiber of f is generically reduced and f is smooth at every smooth point of* $\operatorname{red} f^{-1}(y)$ *for every $y \in Y$.*

Proof. As we noted, we may assume that all fibers are irreducible. In the terminology of [12, I.3.9–10], f is a well defined family of proper algebraic cycles. Moreover, all fibers have multiplicity 1. Thus the scheme theoretic fibers are generically reduced and f is smooth at every smooth point of $\operatorname{red} f^{-1}(y)$ for every $y \in Y$ by [12, I.6.5]. □

Corollary 7. *Let $f : X \to Y$ be a $\mathbb{Z}$-homology fiber bundle where X is smooth and Y is normal. Then Y is smooth and f is flat with local complete intersection fibers.*

Proof. Pick $y \in Y$ and let $x \in \operatorname{red} f^{-1}(y)$ be a smooth point. Then f is smooth at x by Proposition 6. Since X is smooth at x, this implies that Y is smooth at y.

Thus Y is smooth, f is equidimensional and X is smooth, hence Cohen-Macaulay. These imply that f is flat [10, Exrc.III.10.9]. □

Families of curves.

We start with two examples of families of reducible curves.

Example 8. $X \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{C}_s$ is a reducible surface and $f : X \to \mathbb{C}_s$ is the coordinate projection. The fiber X_0 is the projective closure of the 3 coordinate axes in $\mathbb{C}^3$ and X_s is obtained by sliding the x_3-axis along the x_2 axis. In concrete equations

$$X := \bigl(x_1 = x_3 = 0\bigr) \cup \bigl(x_2 = x_3 = 0\bigr) \cup \bigl(x_1 = x_2 - sx_0 = 0\bigr) \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{C}_s.$$

It is easy to check that $f : X \to \mathbb{C}_s$ is a homotopy fiber bundle, all the fibers are reduced, the fibers X_s are isomorphic to each other for $s \neq 0$ but X_0 is not homeomorphic to X_s for $s \neq 0$.

It is straightforward to modify this example and obtain an irreducible (but still non-normal) surface S with a proper morphism $f : S \to \mathbb{C}$ which is a homotopy fiber bundle such that not all fibers are homeomorphic to each other.

Example 9. Here again $X \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{C}_t$ is a reducible surface and $f : X \to \mathbb{C}_t$ is the coordinate projection. The general fiber is a line L_t and a conic C_t intersecting at a point $p \in \mathbb{P}^3$. As we approach the special fiber, the conic degenerates to a pair of lines $L_0 + L_0'$ and the line L_0 is also the limit of the family L_t. In concrete equations

$$X := \bigl(x_2 = x_3 - tx_1 = 0\bigr) \cup \bigl(x_3 = x_0x_2 - tx_2^2 = 0\bigr) \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{C}_s.$$

In this example the retraction map induces isomorphisms

$$\bigl(r_{y' \to y}\bigr)_* : H_*\bigl(L_t \cup C_t, R\bigr) \to H_*\bigl(L_0 \cup L_0', R\bigr)$$

but the fundamental class $\bigl[L_t \cup C_t\bigr]$ is mapped to $2\bigl[L_0\bigr] + \bigl[L_0'\bigr]$ which is different from the fundamental class of the central fiber $\bigl[L_0 \cup L_0'\bigr]$.

For curves the following result completes the picture.

Proposition 10. *Let Y be a normal complex space and $f : X \to Y$ a $\mathbb{Z}$-homology fiber bundle of relative dimension 1 with smooth general fibers. Let $\pi : X^n \to X$ be the normalization of X. Then*

(1) $\pi : X^n \to X$ is a homeomorphism and
(2) $f \circ \pi : X^n \to Y$ is smooth hence a differentiable fiber bundle.

Proof. As we noted before, we may assume that f has irreducible fibers.

Let us start with the case when Y is a smooth curve. Let B be a general fiber and B_0 any fiber. Let B_0^n be the corresponding fiber in $X^n \to Y$. Note that the retraction $r : B \to B_0$ factors through B_0^n. It is easy to see that $H_1(B_0^n, \mathbb{Z}) \to H_1(B_0, \mathbb{Z})$ is surjective iff $B_0^n \to B_0$ is a homeomorphism. Thus if $X \to Y$ is a $\mathbb{Z}$-homology fiber bundle then $B_0^n \to B_0$ and $X^n \to X$ are homeomorphisms. We may thus assume that X is normal. In particular the fibers are reduced and $p_a(B_0) = p_a(B) = p_g(B)$ since B is smooth.

Let $B_0' \to B_0$ denote the seminormalization and $B_0'' \to B_0'$ the normalization. If m_i are the multiplicities of the points of B_0' then

$$p_a(B_0) \geq p_a(B_0') = p_a(B_0'') + \sum(m_i - 1) = p_g(B_0'') + \sum(m_i - 1).$$

We can thus estimate the topological Euler characteristic as

$$\begin{aligned} \chi^{top}(B_0) &= \chi^{top}(B_0') = \chi^{top}(B_0'') - \sum(m_i - 1) = 2 - 2p_g(B_0'') - \sum(m_i - 1) \\ &\geq 2 - 2p_a(B_0) + \sum(m_i - 1). \end{aligned}$$

On the other hand, if f is a homotopy fiber bundle then

$$\chi^{top}(B_0) = \chi^{top}(B) = 2 - 2p_g(B) = 2 - 2p_a(B_0).$$

Comparing these two we see that $\sum(m_i - 1) = 0$ and $p_a(B_0) = p_a(B_0')$ hence $B_0 \cong B_0' \cong B_0''$. Thus every fiber of f is smooth.

This implies the general case by applying Proposition 12 to the class of all smooth projective curves as $\mathcal{V}$. $\square$

Reduction to 1-parameter families.

Here we show that a variant of Conjecture 3 can be reduced to the case when $\dim Y = 1$. To make this precise, fix a class of projective varieties $\mathcal{V}$ and consider the following.

Conjecture 11. *Let $f : X \to Y$ be a projective morphism of complex spaces, Y normal. Assume that*

(1) there is a Zariski dense open subset $Y^0 \subset Y$ such that the fibers of f over Y^0 are all in $\mathcal{V}$ and

(2) f is a homotopy (resp. $\mathbb{Z}$-homology) fiber bundle.

Let $\pi : X^n \to X$ be the normalization of X. Then

(3) $\pi : X^n \to X$ is a homeomorphism and

(4) $f \circ \pi : X^n \to Y$ is smooth hence a differentiable fiber bundle.

We can now state the precise form of the dimension reduction.

Proposition 12. *Fix a class of smooth projective varieties $\mathcal{V}$ and assume that Conjecture 11 holds for $\mathcal{V}$ whenever $\dim Y = 1$.*

Then Conjecture 11 holds for $\mathcal{V}$ in general.

Proof. Let $\Delta \subset \mathbb{C}$ be the unit disc and $\phi : \Delta \to Y$ any holomorphic map whose image is not contained in $Y \setminus Y^0$. Let X^n_ϕ denote the normalization of $X \times_Y \Delta$. By assumption, $X^n_\phi \to \Delta$ is smooth hence it is the simultaneous normalization of $\phi^* f : X \times_Y \Delta \to \Delta$. In particular, the normalization of the fibers are all smooth and the normalization map is a homeomorphism.

By [13, Thm.1], there is a monomorphism $Y' \to Y$ such that $\phi^* f : X \times_Y \Delta \to \Delta$ has a simultaneous normalization iff ϕ factors through $Y' \to Y$. Thus $Y' = Y$, the composite $f \circ \pi : X^n \to Y$ is smooth and $\pi : X^n \to X$ is a homeomorphism. $\square$

Localization.

Motivated by Proposition 12, from now on we concentrate on 1-parameter families. That is, X is a normal analytic space and $f : X \to \Delta$ a proper morphism with central fiber $X_0 = f^{-1}(0)$. By shrinking Δ we may assume that $X \setminus X^0 \to \Delta^*$ is a topological fiber bundle.

We show that if X_0 has isolated singularities then $\mathbb{Z}$-homology fiber bundles can be characterized in terms of the Milnor fibers of the singular points of X_0. Subsequent examples show that there are global issues for non-isolated singularities.

Proposition 13. *Let X be a normal analytic space and $f : X \to \Delta$ a proper morphism with central fiber $X_0 = f^{-1}(0)$. Assume that X_0 has only isolated singularities $p_i \in X_0$. For each i, let B_i be a small ball around p_i and set $M_{i,t} := X_t \cap B_i$. (If X_t is smooth, this is the Milnor fiber.) The following are equivalent.*

(1) For $0 < |t| \ll 1$, the retraction map $r_t : X_t \to X_0$ is an R-homology equivalence.

(2) For $0 < |t| \ll 1$ every $M_{i,t}$ is an R-homology ball.

Proof. Choose $\Delta_\epsilon \subset \Delta$ small enough so that X_t meets ∂B_i transversely for any i and any $t \in \Delta_\epsilon$. One can choose the retraction such that r_t induces a homeomorphism

$$r_t : X_t \setminus \cup_i M_{i,t} \cong X_0 \setminus \cup_i M_{i,0}.$$

Comparing the long exact homology sequences of the pairs

$$r_t : \left(X_t, \cup_i M_{i,t}\right) \to \left(X_0, \cup_i M_{i,0}\right)$$

we see that $r_t : X_t \to X_0$ is an R-homology equivalence iff the restrictions $r_{i,t} : M_{i,t} \to M_{i,0}$ are R-homology equivalences. Since the $M_{i,0}$ are contractible, the latter holds iff the $M_{i,t}$ are R-homology balls. □

When the source X of the mapping f is smooth, the following local result for non-isolated singularities is a corollary of the work of A'Campo on monodromy of singularities. We thank A'Campo for pointing this out.

Proposition 14. *Let X be smooth and $p \in X$ a point. Let $f : (X, p) \to \Delta$ be a germ of analytic mapping. Let B be a Milnor ball around p and D a Milnor disc around $f(p)$. Set $M_{i,t} := X_t \cap B_i$ for any $t \in D$ (for $t \neq f(p)$ this is the Milnor fiber.) Let R be any ring. The following are equivalent.*

(1) For $0 < |t| \ll 1$, the retraction map $r_t : X_t \to X_0$ is an R-homology equivalence.
(2) The morphism f is smooth at p.

Proof. In [1] it is proved that under the given hypothesis the Lefschetz number of the monodromy of the Milnor fibration equals 0 if f is not smooth at 0 and it is obvious that it equals 1 if f is smooth. If the retraction map is a R-homology equivalence, then the Lefschetz number of the monodromy of the Milnor fibration equals 1. □

Milnor fibers of isolated singularities have been extensively studied. For surfaces the following result seems to have been known but not explicitly stated; see [20, 15] for closely related results. The argument below was shown to us by A. Némethi.

Proposition 15. *Let X be a normal threefold and $f : X \to \Delta$ a $\mathbb{Z}$-homology fiber bundle whose general fiber is smooth and whose central fiber X_0 is normal. Then f is smooth, X is smooth and f is a differentiable fiber bundle.*

Proof. Using Proposition 13, we need to consider the Milnor fibers of the singular points of X_0.

In general, let $(s \in S)$ be an isolated surface singularity and M the Milnor fiber of a smoothing. The link L of S is diffeomorphic to the boundary ∂M of M. Let μ_0, μ_+, μ_- denote the number of zero (resp. positive, negative) eigenvalues of the intersection form on the middle cohomology of M.

If M is a $\mathbb{Q}$-homology ball then these are all 0. By [20, 2.24], $\mu_0 + \mu_+ = 2p_g(s \in S)$ where p_g denotes the geometric genus of the singularity $(s \in S)$. For a normal surface singularity $p_g(s \in S) = \dim_s R^1 g_* \mathcal{O}_{S'}$ where $g : S' \to S$ is a resolution of singularities. Thus if M is a $\mathbb{Q}$-homology ball then $(s \in S)$ is a rational singularity.

(If $(s \in S)$ is an isolated non-normal surface singularity, then $p_g(s \in S) = \dim_s R^1 g_* \mathcal{O}_{S'} - \dim \mathcal{O}_{\bar{S}}/\mathcal{O}_S$ where $\bar{S} \to S$ is the normalization. There are many examples where M is a $\mathbb{Q}$-homology ball yet $(\bar{s} \in \bar{S})$ is not a rational singularity.)

If M is a $\mathbb{Z}$-homology ball, then $L \sim \partial M$ is a $\mathbb{Z}$-homology sphere, hence $\mathrm{Cl}(S) \cong H^2(L, \mathbb{Z})$ is trivial [17, p.240]. Thus S is rational and K_S is Cartier; this happens only if S is a Du Val singularity. For smoothings of isolated hypersurface singularities there are vanishing cycles. □

Remark 16. This suggests that Conjecture 11 may hold for $\mathcal{V} = \{\text{smooth surfaces}\}$, but there are many more cases to check. We do not even know what happens when the special fiber has isolated (but non-normal) singularities.

By contrast, there are many normal surface singularities whose Milnor fiber is a $\mathbb{Q}$-homology ball. See, for instance, [15, 5.9].

Example 17. Let $X^n \subset \mathbb{P}^N$ be a smooth variety and $Y \subset X$ a hyperplane section such that $X \setminus Y \cong \mathbb{C}^n$ The simplest examples are smooth quadrics $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ where Y is a tangent plane; for more complicated examples with $\dim X = 3$ see [7, 9, 8].

One gets a family of n–folds $f : \mathbf{X} \to \mathbb{C}$ whose general fibers X_t are isomorphic to X and whose special fiber X_0 is isomorphic to the cone over Y (possibly with some embedded points at the vertex). For quadrics an explicit example is the family

$$\big(x_0^2 + \cdots x_{n-1}^2 + tx_n^2 + tx_{n+1}^2 = 0\big) \subset \mathbb{P}^n_{\mathbf{x}} \times \mathbb{C}_t.$$

Note that the rank drops by 2 at the origin.

If n is odd, this is a $\mathbb{Q}$-homology fiber bundle but the retraction map

$$\mathbb{Z} \cong H_{n+1}\big(X_t, \mathbb{Z}\big) \to H_{n+1}\big(X_0, \mathbb{Z}\big) \cong \mathbb{Z}$$

is multiplication by 2. In all the other 3-fold examples the retraction induces

$$\mathbb{Z} \cong H_4\big(X_t, \mathbb{Z}\big) \to H_4\big(X_0, \mathbb{Z}\big) \cong \mathbb{Z}$$

which is multiplication by $\deg X > 1$.

The following lemma shows that this construction never gives interesting $\mathbb{Z}$-homology equivalences.

Proposition 18. *Let $X \subset \mathbb{P}^N$ be a smooth projective variety and $Y = H \cap X \subset X$ a hyperplane section. Let $C(Y)$ denote the cone over Y and $r_Y : X \to C(Y)$ the retraction. Assume that $X \setminus Y$ is a $\mathbb{Z}$-homology ball and r_Y is a $\mathbb{Z}$-homology equivalence. Then X is a linear subspace.*

Proof. Let $L \in H^2\big(\mathbb{P}^N, \mathbb{Z}\big)$ denote the hyperplane class. We will prove that cap product with L gives isomorphisms

$$\cap L : H_{i+2}(X, \mathbb{Z}) \cong H_i(X, \mathbb{Z}) \quad \text{for } 0 \le i \le 2 \dim Y. \tag{18.1}$$

Composing the even ones gives an isomorphism

$$\big(\cap L\big)^{\dim X} : H_{2\dim X}(X, \mathbb{Z}) \cong H_0(X, \mathbb{Z}).$$

Thus $\deg X = 1$ and so X is a linear subspace.

Since r_Y is a $\mathbb{Z}$-homology equivalence, (18.1) is equivalent to

$$\cap L : H_{i+2}\big(C(Y), \mathbb{Z}\big) \cong H_i\big(C(Y), \mathbb{Z}\big) \quad \text{for } 0 \le i \le 2 \dim Y. \tag{18.2}$$

This map can be factored as the Gysin map $H_{i+2}\big(C(Y), \mathbb{Z}\big) \to H_i\big(Y, \mathbb{Z}\big)$ followed by the inclusion map $H_i\big(Y, \mathbb{Z}\big) \to H_i\big(C(Y), \mathbb{Z}\big)$.

Taking the cone over a cycle gives a natural isomorphism $H_i(Y, \mathbb{Z}) \cong H_{i+2}\big(C(Y), \mathbb{Z}\big)$ and the Gysin map is its inverse. Again using that r_Y is a $\mathbb{Z}$-homology equivalence, $H_i\big(Y, \mathbb{Z}\big) \to H_i\big(C(Y), \mathbb{Z}\big)$ is isomorphic to the inclusion map $H_i\big(Y, \mathbb{Z}\big) \to H_i\big(X, \mathbb{Z}\big)$. The latter is an isomorphism for $i \le 2 \dim Y$ since $X \setminus Y$ is a $\mathbb{Z}$-homology ball. This shows (18.1). □

Families of cubic hypersurfaces.

In [5] several families with constant Lê numbers and non-constant topology are produced. One of them is a family of homogeneous polynomials, giving examples of homotopy fiber bundles which are not locally trivial topologically. All the examples in [5] are non-normal but here we construct a normal variant. Notice that all these examples belong to a class of non-isolated singularities that has been studied systematically in [6].

Example 19. Consider the family of homogeneous cubic polynomials

$$f_t(x_1,x_2,x_3,y_1,y_2,y_3) := (y_1,y_2,y_3)\cdot\begin{pmatrix} tx_1 & x_2 & x_3 \\ x_2 & tx_3 & x_1 \\ x_3 & x_1 & tx_2 \end{pmatrix}\cdot\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Set $F(t,\mathbf{x},\mathbf{y}) = f_t(\mathbf{x},\mathbf{y})$ and $C_0 := \mathbb{C}_t \setminus \{0,-2,-2\xi,-2\xi^2\}$ where ξ is a third root of unity. Consider the family of cubic hypersurfaces

$$X := \big(F(t,\mathbf{x},\mathbf{y}) = 0\big) \subset \mathbb{P}^5_{\mathbf{x},\mathbf{y}} \times C_0$$

and let $\pi : X \to C_0$ be the second projection. For $t \in C_0$ the fiber $\pi^{-1}(t)$ is denoted by

$$X_t = \big(f_t(\mathbf{x},\mathbf{y}) = 0\big) \subset \mathbb{P}^5_{\mathbf{x},\mathbf{y}}.$$

We claim that $\pi : X \to C_0$ has the following properties.

(1) The singular set of X_t is the 2-plane $(y_1 = y_2 = y_3 = 0)$ for every $t \in C_0$. Furthermore, X_t is normal and has only canonical singularities.
(2) $\pi : X \to C_0$ is a homotopy fiber bundle.
(3) $\pi : X \to C_0$ is not topologically locally trivial in any neighborhood of t if $\xi' t^3 - 3t + 2\xi' = 0$ for some third root of unity ξ'. (For example $t = 1$ is one such value.)

Proof. The 2-plane $P := (y_1 = y_2 = y_3 = 0)$ is clearly contained in $\operatorname{Sing} X_t$. If we project X_t from P, the fibers are linear spaces. By an explicit computation we see that C_0 was chosen such that the fibers are all 2-dimensional. So $X_t \setminus P$ is a rank 2 vector bundle over $\mathbb{P}^2$, hence smooth. This implies that X_t is smooth in codimension 1, hence normal.

The projection shows that X_t has a resolution $r_t : \bar{X}_t \to X_t$ where $\bar{X}_t$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$. The exceptional divisor $E_t \subset \bar{X}_t$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ but the restriction of r_t gives a conic bundle structure $r_t|_{E_t} : E_t \to P$. Corresponding to the fibers of this conic bundle, $P = \operatorname{Sing} X_t$ is stratified according to the rank of the matrix

$$\begin{pmatrix} tx_1 & x_2 & x_3 \\ x_2 & tx_3 & x_1 \\ x_3 & x_1 & tx_2 \end{pmatrix}.$$

The third assertion follows from this and from the proof of [5, Prop.7] almost word by word. It is not worth to reproduce it, but the key idea is that any homeomorphism between X_s and X_t carries the singular set of X_s to the singular set of X_t and preserves the stratification. For generic t the locus of non-maximal rank is a smooth cubic curve but for $t = 1$ it is a singular cubic curve.

For the second assertion we check, by a direct computation, the conditions of Lemmas 20 and 21. Alternatively, comparing the homology sequences of the pairs $\big(\bar{X}_t, E_t\big)$ and $\big(X_t, P\big)$ shows that $\pi : X \to C_0$ is a $\mathbb{Z}$-homology fiber bundle. $\square$

We follow the ideas of [5]. Let $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a family of holomorphic function germs depending holomorphically on a parameter. Define $F : \mathbb{C}^n \times \mathbb{C}$ by $F(x,t) := f_t(x)$. Consider the projection $\pi : \mathbb{C}^n \times \mathbb{C}$ to the second factor. Let B_ϵ be the closed ball of radius ϵ centered at the origin of $\mathbb{C}^n$, let $\mathbb{S}_\epsilon$ be its boundary sphere and let D_δ be the disk of radius δ centered at 0. Denote the punctured disk by D^*_δ.

Lemma 20. *Let ϵ, δ and η be radii such that for any $t \in D_\eta$ the restriction*

$$f_t : B_\epsilon \cap f_t^{-1}(D^*_\delta) \to D^*_\delta$$

is a locally trivial fibration. Then the following restrictions of the projection mapping are homotopy fiber bundles:

$$\pi : B_\epsilon \times D_\eta \cap F^{-1}(0 \times D_\eta) \to D_\eta,$$

$$\pi : \mathbb{S}_\epsilon \times D_\eta \cap F^{-1}(0 \times D_\eta) \to D_\eta.$$

Proof. The condition implies that for any $t \in D_\eta$ the inclusions of $f_t^{-1}(0) \cap B_\epsilon$ in $f_t^{-1}(D_\delta) \cap B_\epsilon$ and of $f_t^{-1}(0) \cap \mathbb{S}_\epsilon$ in $f_t^{-1}(D_\delta) \cap \mathbb{S}_\epsilon$ are homotopy equivalences. The condition also implies that for any $\xi \leq \eta$ the inclusions of $F^{-1}(0 \times D_\eta) \cap (B_\epsilon \times D_\xi)$ in $F^{-1}(D_\delta \times D_\eta) \cap (\mathbb{S}_\epsilon \times D_\xi)$ and of $F^{-1}(0 \times D_\eta) \cap (\mathbb{S}_\epsilon \times D_\xi)$ in $F^{-1}(D_\delta \times D_\eta) \cap (B_\epsilon \times D_\xi)$ are homotopy equivalences.

The condition and Ehresmann Fibration Theorem implies that the following restrictions of the projection mapping are differentiable locally trivial fibrations:

$$\pi : B_\epsilon \times D_\eta \cap F^{-1}(D_\delta \times D_\eta) \to D_\eta,$$

$$\pi : \mathbb{S}_\epsilon \times D_\eta \cap F^{-1}(D_\delta \times D_\eta) \to D_\eta. \quad \square$$

Usually one checks the condition of the previous Lemma by showing, for any $t \in D_\eta$, that in the ball B_ϵ the function f_t has no critical points outside $f_t^{-1}(0)$ and that the fibers $f_t^{-1}(s)$ are transverse to ∂B_ϵ for any $s \in D_\delta \setminus \{0\}$.

The Lemma above helps in the local case. From it one can deduce that certain projective morphisms are homotopy fiber bundles. Suppose that f_t is a family of homogeneous polynomials. Let $V(F) \subset \mathbb{P}^{n-1} \times D_\eta$ be the family of projective varieties defined by the zeros of F. Denote by π the projection of $\mathbb{P}^{n-1} \times D_\eta$ to the second factor.

Lemma 21. *Suppose that the condition of the previous lemma is satisfied, and in addition that f_t is a family of homogeneous polynomials. Then the restriction of the projection*

$$\pi : V(F) \to D_\eta$$

is a homotopy fiber bundle.

Proof. It is enough to prove that for any $\xi \leq \eta$ the inclusion of

$$V(f_0) \hookrightarrow V(F) \cap \pi^{-1}(D_\xi) \tag{1}$$

is a homotopy equivalence. By the previous lemma we know that

$$\pi : \mathbb{S}_\epsilon \times D_\eta \cap F^{-1}(0 \times D_\eta) \to D_\eta$$

is a homotopy fiber bundle. Therefore the inclusion

$$F^{-1}(0,0) \cap (\mathbb{S}_\epsilon \times \{0\}) \hookrightarrow F^{-1}(0 \times D_\eta) \cap (\mathbb{S}_\epsilon \times D_\xi)$$

is a homotopy equivalence for any $\xi \leq \eta$. Thus the induced homomorphisms of homotopy groups are isomorphisms.

There is an free action of the sphere $\mathbb{S}^1$ of complex numbers of modulus 1 which is equivariant with respect to the inclusion whose quotient is the inclusion (1). Applying the long exact sequence of homotopy groups associated to the fibrations given by the quotients of the free action we conclude that the inclusion (1) induces isomorphisms of homotopy groups. Whitehead's Theorem implies that then it is a homotopy equivalence. $\square$

Remark 22. The proof of [5, Thm.10] gives that if B_ϵ is a Milnor ball of f_0 and the Lê numbers of f_t with respect to a prepolar coordinate system (a sufficiently generic coordinate system, see [16, p.26] for a precise definition), then the condition in Lemma 20 is satisfied.

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