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A CONJECTURE ON THE ŁOJASIEWICZ EXPONENT

SZYMON BRZOSTOWSKI, TADEUSZ KRASIŃSKI AND GRZEGORZ OLEKSIK

Abstract. In this paper, we present a conjecture connecting the Łojasiewicz exponent of an isolated nondegenerate singularity with some geometrical characteristics of the Newton diagram associated with this singularity. We prove the conjecture for a class of surface singularities.

1. Introduction

Let $f = f(z_1, \ldots, z_n) \in \mathbb{C}\{z_1, \ldots, z_n\}$ be a convergent power series defining an isolated singularity at the origin $0 \in \mathbb{C}^n$. The *Łojasiewicz exponent* $\mathcal{L}_0(f)$ *of* f is by definition the smallest $\theta > 0$ such that there exist a neighbourhood U of $0 \in \mathbb{C}^n$ and a constant $c > 0$ such that

$$|\nabla f(z)| \geq c\,|z|^{\theta} \quad \text{for all } z \in U,$$

where $\nabla f = (f'_{z_1}, \ldots, f'_{z_n})$. It is an important discrete invariant of isolated singularities: it is a rational number [L-JT], it is a biholomorphic invariant, $\mathcal{L}_0(f) + 1$ is equal to the maximal polar invariant of f [T], it is attained on analytic paths centered at 0 [L-JT], $[\mathcal{L}_0(f)] + 1$ is C^0-degree of sufficiency of f [ChL, T]. In spite of its importance $\mathcal{L}_0(f)$ is not well known (in contrast to the Milnor number) even among experts in singularity theory. An interesting mathematical problem is to give formulas for $\mathcal{L}_0(f)$ (in terms of another invariants of f) or an algorithm to compute it. Almost all is known on $\mathcal{L}_0(f)$ for the plane curve singularities ($n = 2$) (see [CK1, CK2, K, GKP]). For $n \geq 3$ there are only estimations of $\mathcal{L}_0(f)$ [P1, P2]. A standard technique in singularity theory is the method of Newton diagrams, developed by the Moscow School (Kouchnirenko, Varchenko, Khovansky and others). In the paper we propose a conjecture that the Łojasiewicz exponent of a nondegenerate singularity could be read off from its Newton diagram. It is true in the case $n = 2$ (Lenarcik [L]). For general n only estimations of $\mathcal{L}_0(f)$ in terms of Newton diagrams (see [A, B, BE, F, O1, O2]) are known. On the other hand a counter-example to it would disprove the Teissier conjecture that $\mathcal{L}_0(f)$ is a topological invariant of f.

For $n = 2$ Lenarcik computes $\mathcal{L}_0(f)$ from the Newton diagram of f by removing from it some exceptional segments. The main difficulty with the extension of his method to n dimensions is to define exceptional faces appropriately. The third-named author proposed a definition in [O2] which we claim to be the right one. Using this definition we prove our conjecture for surface ($n = 3$) nondegenerate singularities that have only one unexceptional face. We also give a formula for the Łojasiewicz exponent of semi-weighted homogeneous surface singularities.

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2. Preliminaries

Let us recall that if $(w_1, \ldots, w_n)$ is a sequence of n rational positive numbers (called *weights*) then a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is called *weighted homogeneous of type* $(w_1, \ldots, w_n)$ if it is a linear combination of monomials $z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ with $\alpha_1/w_1 + \ldots + \alpha_n/w_n = 1$.

A nonzero holomorphic function $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ defined in some open neighbourhood of $0 \in \mathbb{C}^n$ is a *singularity* if $f(0) = 0$ and $\nabla f(0) = 0$. A singularity f is an *isolated singularity* if it has an isolated critical point at the origin i.e. $\nabla f(z) \neq 0$ for $z \neq 0$ near 0. Let $\sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$ be the Taylor expansion of f at 0. We define $\Gamma_+(f) := \operatorname{conv}\{\nu + \mathbb{R}^n_+ : a_\nu \neq 0\} \subset \mathbb{R}^n$ and call it the *Newton diagram* of f. Let $u \in \mathbb{R}^n_+ \setminus \{0\}$. Put $l(u, \Gamma_+(f)) := \inf\{< u, v >: v \in \Gamma_+(f)\}$ and $\Delta(u, \Gamma_+(f)) := \{v \in \Gamma_+(f) :< u, v >= l(u, \Gamma_+(f))\}$. We say that $S \subset \mathbb{R}^n$ is a *face* of $\Gamma_+(f)$, if $S = \Delta(u, \Gamma_+(f))$ for some $u \in \mathbb{R}^n_+ \setminus \{0\}$. The vector u is called a *primitive vector* of S. It is easy to see that S is a closed and convex set and $S \subset \operatorname{Fr}(\Gamma_+(f))$, where $\operatorname{Fr}(A)$ denotes the boundary of A. One can prove that a face $S \subset \Gamma_+(f)$ is compact if and only if all coordinates of its primitive vector u are positive. We call the family of all compact faces of $\Gamma_+(f)$ the *Newton boundary* of f and denote it by $\Gamma(f)$. We denote by $\Gamma^k(f)$ the set of all compact k-dimensional faces of $\Gamma(f)$, $k = 0, \ldots, n-1$. For every compact face $S \in \Gamma(f)$ we define weighted homogeneous polynomial $f_S := \sum_{\nu \in S} a_\nu z^\nu$. A singularity f is *nondegenerate on the face* $S \in \Gamma(f)$ if the system of equations $(f_S)'_{z_1} = \ldots = (f_S)'_{z_n} = 0$ has no solution in $(\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. A singularity f is *nondegenerate in the Kouchnirenko sense* (shortly *nondegenerate*) if it is nondegenerate on each face of $\Gamma(f)$. A singularity f is *semi-weighted homogeneous* if there exists a face S of $\Gamma(f)$ such that f_S is an isolated singularity.

Let $i \in \{1, \ldots, n\}$, $n \geq 2$. We say that $S \in \Gamma^{n-1}(f) \subset \mathbb{R}^n$ is an *exceptional face for f with respect to the axis OX_i* if one of its vertices is at distance 1 to the axis OX_i and the remaining vertices define $(n-2)$-dimensional face which lies in one of the coordinate hyperplanes including the axis OX_i.

Example 2.1. *Let $f(z_1, z_2, z_3) = z_1 z_3^4 + z_2^2 z_3^6 + z_2^4 z_3 + z_1^6$. It is easy to check that $\Gamma^2(f) = \{S_1, S_2\}$, where $S_1 = \operatorname{conv}\{(0,4,1), (0,2,6), (1,0,4)\}$ is an exceptional face for f with respect to OX_3 and $S_2 = \operatorname{conv}\{(0,4,1), (1,0,4), (6,0,0)\}$ is not an exceptional face. Let us notice that f_{S_2} is an isolated singularity, so f is a semi-weighted homogeneous singularity.*

A face $S \in \Gamma^{n-1}(f)$ is an *exceptional face for f* if there exists $i \in \{1, \ldots, n\}$ such that S is an exceptional face for f with respect to the axis OX_i. Denote by E_f the set of all exceptional faces for f. We call a face $S \in \Gamma^{n-1}(f)$ *unexceptional for f* if $S \notin E_f$.

A singularity f is *convenient* (resp. *nearly convenient*) if its Newton diagram has nonempty intersection with every coordinate axis (resp. its distance to every coordinate axis doesn't exceed 1).

For every $(n-1)$-dimensional compact face $S \in \Gamma(f)$ we shall denote by $x_1(S), \ldots, x_n(S)$ the coordinates of intersection of the hyperplane determined by the face S with the coordinate axes $OX_1, \ldots, OX_n$. We put $m(S) := \max\{x_1(S), x_2(S), \ldots, x_n(S)\}$. It is easy to see that

$$x_i(S) = \frac{l(u, \Gamma_+(f))}{u_i},\ i = 1, \ldots, n, \tag{1}$$

where u is a primitive vector of S.

3. Main results

An interesting problem concerning the Łojasiewicz exponent is to compute $Ł_0(f)$ for nondegenerate isolated singularities f in terms of the Newton diagram $\Gamma_+(f)$. In this paper we propose the following conjecture.

Conjecture. *Let $f:(\mathbb{C}^n,0)\longrightarrow(\mathbb{C},0)$ be an isolated nondegenerate singularity. If $\Gamma^{n-1}(f)\setminus E_f\neq\emptyset$, then*

(2) $$Ł_0(f)=\max_{S\in\Gamma^{n-1}(f)\setminus E_f} m(S)-1.$$

There are some results that confirm our conjecture:

- Lenarcik [L] improved a bound for $Ł_0(f)$ obtained by Lichtin [Lt] and proved formula (2) for $n=2$.
- The third-named author proved in [O2] the inequality

(3) $$Ł_0(f)\leq\max_{S\in\Gamma^{n-1}(f)\setminus E_f} m(S)-1$$

  for $n=3$.
- For weighted homogeneous singularities the Conjecture is true [KOP].
- Fukui [F] proved a weaker bound for $Ł_0(f)$ for any $n\geq 2$. His result was improved in [O1, O2]. Abderrahmane [A] gave another result of this type.

The main result of this note is the proof of the Conjecture in the case of nondegenerate surface singularities with one unexceptional face.

Theorem 3.1. *Let $f:(\mathbb{C}^3,0)\longrightarrow(\mathbb{C},0)$ be an isolated nondegenerate singularity such that $\#(\Gamma^2(f)\setminus E_f)=1$. Then*

$$Ł_0(f)=m(S)-1,$$

where S is the unique unexceptional face for f.

Example 3.2. *The isolated singularity in Example 2.1 satisfies the assumptions of the above theorem. We easily check that $Ł_0(f)=m(S_2)-1=5$.*

The proof of Theorem 3.1 is based on the following formula for the Łojasiewicz exponent of a semi-weighted homogeneous singularity.

Theorem 3.3. *Let $f:(\mathbb{C}^3,0)\longrightarrow(\mathbb{C},0)$ be a semi-weighted homogeneous singularity. Then*

$$Ł_0(f)=Ł_0(f_S),$$

where S is a face of $\Gamma(f)$ such that f_S is an isolated singularity.

To calculate $Ł_0(f_S)$ one can use the main result of [KOP].

Remark 3.4. *Theorem 3.3 is also true for $n=2$ (one can prove it using Cor. 4 in [KOP]). It is an open question if $Ł_0(f_S)=Ł_0(f)$ for $n>3$.*

4. Proofs of the main results

First we prove an auxiliary inequality (see Cor. 4.8 in [BE] for another proof) for any dimension.

Proposition 4.1. *Let $f:(\mathbb{C}^n,0)\longrightarrow(\mathbb{C},0)$ be a semi-weighted homogeneous singularity and let $S\in\Gamma(f)$ be a face such that f_S is an isolated singularity. Then*

(4) $$Ł_0(f_S)\leq Ł_0(f).$$

Proof. Let $v=(v_1,\ldots,v_n)$ be a primitive vector of S such that $v_i\in\mathbb{N}_+$. We expand f in the form

$$f=f^{[d]}+f^{[d+1]}+\ldots,\quad f^{[d]}\neq 0,$$

where $f^{[i]}$ are weighted homogeneous polynomials of type $(v_1, \dots, v_n)$, $\deg_v f^{[i]} = i$, $i = d, d+1, \dots$. Of course $f^{[d]} = f_S$. Take the following family of singularities

$$f_t := f(z_1 t^{v_1}, \dots, z_n t^{v_n})/t^d,\ t \in \mathbb{C} \setminus \{0\}$$

and $f_0 := f^{[d]}$. Notice that

$$f_t = f^{[d]} + t f^{[d+1]} + t^2 f^{[d+2]} + \dots, \quad t \in \mathbb{C}.$$

The family (f_t) has the following properties:

- (f_t) is a holomorphic family with respect to t,
- f_t are semi-weighted homogeneous singularities,
- $\mu_0(f_t) = \mu_0(f^{[d]})$ for $t \in \mathbb{C}$ ([AGV], Thm. in Section 12.2), where $\mu_0(f)$ is the Milnor number of a singularity f,
- $f_0 = f_S$.

By the semicontinuity of the Łojasiewicz exponent in holomorphic μ-constant families of isolated singularities [T, P3] we obtain

$$\mathcal{L}_0(f_0) \leq \mathcal{L}_0(f_t)$$

for t sufficiently close to 0. On the other hand $\mathcal{L}_0(f_t) = \mathcal{L}_0(f)$ for $t \neq 0$, because

$$f_t = \alpha \cdot (f \circ L),$$

where $\alpha \in \mathbb{C} \setminus \{0\}$ and L is a linear change of coordinates in $\mathbb{C}^n$. Hence for any sufficiently small $t \neq 0$ we have

$$\mathcal{L}_0(f_S) = \mathcal{L}_0(f_0) \leq \mathcal{L}_0(f_t) = \mathcal{L}_0(f).$$

□

Now, we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Let $L \subset \mathbb{R}^3 : \alpha_1/w_1 + \alpha_2/w_2 + \alpha_3/w_3 = 1$ be a supporting plane to $\Gamma_+(f)$ along the face S (if S is 2-dimensional then L and $w = (w_1, w_2, w_3)$ are uniquely determined). Since $\operatorname{supp}(f_S) \subset L$, f_S is a weighted homogeneous polynomial of type (w_1, w_2, w_3). Write $f = f_S + f'$, where all monomials appearing in the Taylor expansion of f' lie above the plane L. Now, by ([KOP], Thm. 3) we get

$$\mathcal{L}_0(f_S) = \min\big(\max_{i=1}^{3} w_i - 1, \prod_{i=1}^{3}(w_i - 1)\big). \tag{5}$$

Using ([P2], Prop. 2.2) we obtain $\mathcal{L}_0(f) \leq \max_{i=1}^3 w_i - 1$. By ([P1], Thm. 1), ([AGV], Thm. in Section 12.2) and the Milnor-Orlik formula [MO] we get $\mathcal{L}_0(f) \leq \mu_0(f) = \mu_0(f_S) = \prod_{i=1}^3 (w_i - 1)$. Consequently

$$\mathcal{L}_0(f) \leq \min\big(\max_{i=1}^{3} w_i - 1, \prod_{i=1}^{3}(w_i - 1)\big) \tag{6}$$

On the other hand by Proposition 4.1 we get

$$\mathcal{L}_0(f_S) \leq \mathcal{L}_0(f) \tag{7}$$

By (5), (6), (7) we obtain the assertion of the theorem. □

To prove Theorem 3.1 we give some lemmas and properties.

Property 4.2. *Every isolated singularity $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ is nearly convenient.*

Proof. It suffices to show that for every $i = 1, 2, \dots, n$ there exists $j \in \{1, 2, \dots, n\}$ and $k \geq 1$ such that monomial $z_j z_i^k$ appears in the Taylor expansion of f with a non-zero coefficient. Indeed, suppose to the contrary that for some $i \in \{1, 2, \dots, n\}$ no monomial $z_j z_i^k$ appears in the expansion of f for every $j \in \{1, 2, \dots, n\}$ and $k \geq 1$. Then one can easily check that $f'_{z_j}(0, \dots, 0, z_i, 0, \dots, 0) \equiv 0$, $j = 1, \dots, n$, which is impossible since ∇f has an isolated zero at 0. □

For a series $\phi \in \mathbb{C}\{t\}$, $\phi \neq 0$, by $\operatorname{info} \phi$ (resp. $\operatorname{inco} \phi$) we mean the initial form of ϕ (resp. the non-zero coefficient of $\operatorname{info} \phi$).

Lemma 4.3. *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 3$, *be a singularity and* $\nabla f \circ \phi = 0$ *for some* $\phi = (\phi_1, \dots, \phi_n) \in \mathbb{C}\{t\}^n$, $\phi(0) = 0$, $\phi_1, \dots, \phi_k \neq 0$, $\phi_{k+1} = \dots = \phi_n = 0, k \geq 2$, *and* $f(z_1, \dots, z_k, 0, \dots, 0) \not\equiv 0$. *Then there exists* $S \in \Gamma(f)$ *on which* f *is degenerate.*

Proof. We can represent f in the form

$$f(z_1, \dots, z_n) = g(z_1, \dots, z_k) + z_{k+1} h_{k+1}(z_1, \dots, z_n) + \dots + z_n h_n(z_1, \dots, z_n)$$

By the assumption we get $g \neq 0$, $g(0) = 0$, $\nabla g(\phi_1, \dots, \phi_k) = 0$. By [O2, Cor. 2.4] there exists $S \in \Gamma(g)$, such that $(\operatorname{ord} \phi_i)_{i=1}^k$ is a primitive vector of S and

$$\nabla g_S(\operatorname{info} \phi_1, \dots, \operatorname{info} \phi_k) = 0. \tag{8}$$

By [O2, Property 2.10] we get $S \in \Gamma(f)$. Of course $f_S = g_S$. Therefore we have

$$(f_S)'_{z_i}(\operatorname{info} \phi_1(t), \dots, \operatorname{info} \phi_k(t), t, \dots, t) \equiv 0,\ i = k+1, \dots, n$$

and by (8) we get

$$(f_S)'_{z_i}(\operatorname{info} \phi_1(t), \dots, \operatorname{info} \phi_k(t), t, \dots, t) \equiv 0,\ i = 1, \dots, k.$$

Hence

$$(f_S)'_{z_i}(\operatorname{inco} \phi_1, \dots, \operatorname{inco} \phi_k, 1, \dots, 1) = 0, \quad i = 1, \dots, n,$$

thus f is degenerate on S. □

Proposition 4.4. *Let* $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$ *be a nondegenerate nearly convenient singularity such that* $\Gamma_+(f) \cap OX_iX_j \neq \emptyset$ *for* $i \neq j$. *Then* f *is an isolated singularity.*

Proof. Suppose to the contrary that f is not an isolated singularity. Then there exists a non-zero parametization ϕ such that $\nabla f \circ \phi = 0$. It is not possible for ϕ to have two coordinates equal to zero, because if for example $\phi = (0, 0, \phi_3)$, $\phi_3 \neq 0$, then by Property 4.2 we get that monomial $z_i z_3^k$ appears in the Taylor expansion of f with a non-zero coefficient for some $i \in \{1, 2, 3\}$ and $k \geq 1$. Then one can check that $\operatorname{info} f'_{z_i}(0, 0, \phi_3(t)) = (\operatorname{info} \phi_3(t))^k \neq 0$. Hence $f'_{z_i}(0, 0, \phi_3) \neq 0$, which contradicts the hypothesis $\nabla f \circ \phi = 0$. Therefore we may assume that $\phi = (\phi_1, \phi_2, \phi_3)$ and $\phi_i \neq 0, \phi_j \neq 0$ for some $i \neq j$. Without loss of generality we may assume that $\phi_1 \neq 0, \phi_2 \neq 0$. Then by Lemma 4.3 we have that f is degenerate on some face $S \in \Gamma(f)$, which contradicts the assumption on f. □

Lemma 4.5. *Let* $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$ *be a singularity. Suppose there exists an unexceptional face* S *for* f *such that* f_S *is an isolated singularity. Put* $w_i := x_i(S)$ *for* $i = 1, 2, 3$. *Then*

$$m(S) - 1 = \min\big(\max_{i=1}^{3} w_i - 1, \prod_{i=1}^{3}(w_i - 1)\big). \tag{9}$$

Proof. Since f_S is an isolated singularity, therefore $\operatorname{ord} f_S \geq 2$ and hence $x_i(S) > 1$, $i = 1, 2, 3$. We consider two cases.

If $w_i \geq 2$, $i = 1, 2, 3$, then

$$\prod_{i=1}^{3}(w_i - 1) \geq \max_{i=1}^{3} w_i - 1 = \max_{i=1}^{3} x_i(S) - 1 = m(S) - 1,$$

which gives (9).

If $w_i < 2$ for some $i \in \{1, 2, 3\}$, say $i = 1$, then $1 < x_1(S) < 2$ and by Property 4.2 there exists a monomial z_1z_2 or z_1z_3, say z_1z_2, appearing in the Taylor expansion of f with a non-zero coefficient. Then $(1, 1, 0)$ lies on the plane $\alpha_1/w_1 + \alpha_2/w_2 + \alpha_3/w_3 = 1$. Hence $(w_1 - 1)(w_2 - 1) = 1$ and thus $\prod_{i=1}^{3}(w_i - 1) = w_3 - 1$. Since S is an unexceptional face, there exists a point $(1, 0, k) \in \operatorname{supp}(f_S)$, $k \geq 1$. Therefore $x_3(S) \geq x_2(S)$ and obviously $x_2(S) > 2$. Hence $m(S) = x_3(S) = w_3$. □

Proof of Theorem 3.1. Using the Lemma about the choice of an unexceptional face (Lemma 3.1 in [O2]) one can check that f_S is nearly convenient and $\Gamma_+(f_S) \cap OX_iX_j \neq \emptyset$ for $i \neq j$. Then by Proposition 4.4 we get that f_S has an isolated singularity. Therefore by Theorem 3.3 and by Theorem 3 in [KOP] we get

$$Ł_0(f) = Ł_0(f_S) = \min\big(\max_{i=1}^{3} w_i - 1, \prod_{i=1}^{3}(w_i - 1)\big),$$

where $w_i = x_i(S)$, $i = 1, 2, 3$. Since S is an unexceptional face, by Lemma 4.5 we have

$$m(S) - 1 = \min\big(\max_{i=1}^{3} w_i - 1, \prod_{i=1}^{3}(w_i - 1)\big).$$

Summing up we get

$$Ł_0(f) = m(S) - 1.$$

□

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Faculty of Mathematics and Computer Science, University of Lodz, Banacha 22, 90-238 Lodz, POLAND

E-mail address: brzosts@math.uni.lodz.pl, krasinsk@uni.lodz.pl, oleksig@math.uni.lodz.pl