Journal *of* Singularities
Volume 6 (2012), 37-53
DOI: 10.5427/jsing.2012.6e

Singularities in Geometry and Appl., Będlewo, 2011

CLASSIFICATION OF CURVES ON SURFACES AND FREE LINKS VIA HOMOTOPY THEORY OF WORDS AND PHRASES

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Abstract. In this paper, we introduce Turaev's homotopy theory of words and phrases. As new results, we give the classification of oriented ordered pointed irreducible multi-component curves on surfaces which is called monoliteral type with at most six crossings up to stably equivalence using Turaev's homotopy theory of words and phrases. Moreover we also give the classification of (oriented) ordered pointed irreducible free links of monoliteral type with at most six crossings.

1. Introduction.

A *knot* is the image of a smooth embedding of S^1 into $\mathbb{R}^3$. Further, a k-components *link* is the image of a smooth embedding of the disjoint union of k circles into $\mathbb{R}^3$. When we study knots and links, we often use link diagrams of links. A *knot diagram* is a smooth immersion of S^1 into $\mathbb{R}^2$ with transversal double points such that the two paths at each double point are assigned to be the over path and the under path respectively (we call a double point of such immersion a *crossing*). If a knot diagram D is obtained as the image of a knot by a projection of $\mathbb{R}^3$ to $\mathbb{R}^2$, then we call D a diagram of the knot. A *link diagram* is defined similarly as a smooth immersion of the disjoint union of circles to $\mathbb{R}^2$.

In the paper [14], L. Kauffman introduced the theory of virtual knots and links using combinatorially extended link diagrams which are called virtual link diagrams. A *virtual knot diagram* is a planar graph of valency four endowed with the following structure: Each vertex either has an overcrossing and undercrossing (in other words, *real crossing*) or is marked by a *virtual crossing* (See Figure 3). A *virtual link diagram* is defined similarly. Then, we define *virtual links* by the set of virtual link diagrams quotiented by an equivalence relation generated by the virtual Reidemeister moves (see [14] for more details).

We call a virtual link diagram *pointed* if each component is endowed with a base point distinct from the crossing points. Further, we call a virtual link diagram *ordered* if its components are numerated. We also call a virtual link diagram *flat* when we ignore over/under at real crossings. A *pointed ordered flat virtual link* is defined by a set of pointed ordered flat virtual link diagrams quotiented by an equivalence relation generated by the flat virtual Reidemeister moves which are applied away from the base points.

The theory of flat virtual links is closely related to the theory of curves on surfaces. In fact, for all positive integer k, oriented ordered pointed k-components flat virtual links are in one to one correspondency to stably equivalent classes of oriented ordered pointed k components curves on surfaces (see [13] for example).

In this paper, we introduce the classification of oriented ordered pointed multi-components curves on surfaces up to stable equivalence with some conditions. To do this, we use the theory of nanowords which are introduced by Vladimir Turaev in [18] and [19]. Turaev defined generalized

2010 *Mathematics Subject Classification.* Primary 57M99; Secondary 68R15.
Key words and phrases. étale phrases, nanophrases, curves on surfaces, stably equivalent.

words and phrases which are called nanowords and nanophrases. Moreover he introduced an equivalence relation which is called S-homotopy on a set of generalized words and he proved that if we consider some special cases of homotopy of words and phrases, then we obtain the theory of curves on surfaces and link diagrams on surfaces. Therefore we can use the homotopy theory of words and phrases to study curves on surfaces and link diagrams on surfaces. Another applications of the theory of words can be found in [11] and [12]. N. Ito studies curves on a plane and wave fronts on a plane by using Turaev's theory of words. See [11] and [12] for more details.

This paper is organized as follows. In section 2, we review the theory of topology of words. We introduce some important notions to obtain the main result. In section 3, we introduce geometric meanings of the theory of words and phrases. We describe how to construct a bijection from the set of stable equivalence classes of curves on surfaces to the set of homotopy classes of nanophrases. Moreover we introduce flat virtual links. This leads us to a simple presentation of curves on surfaces. In section 4, we introduce the classification of nanowords, nanophrases and monoliteral phrases with some conditions on the length of words and on the number of component of phrases, which is proved by Turaev in [18] and the author in [1], [3] and [4]. In section 5, we introduce some homotopy invariants of nanophrases which was used to classify nanophrases. In section 6, we introduce application of the classification theorems. As a new result, we classify oriented ordered pointed irreducible curves on surfaces of monoliteral type with at most six crossings up to stably equivalent. Moreover we make the list of a complete representative system of oriented ordered pointed irreducible curves on surfaces of monoliteral type with at most six crossings. Moreover in section 8, we give the classification of (oriented) ordered pointed irreducible free links with at most two crossings and the classification of (oriented) ordered pointed irreducible free links of monoliteral type with at most six crossings.

2. Turaev's Homotopy Theory of Words and Phrases

In this section we introduce Turaev's homotopy theory of words and phrases which was introduced by V. Turaev in papers [18] and [19]. We can find a survey of Turaev's theory of words in the paper [20].

2.1. **Étale words and nanowords.** In this paper an *alphabet* means a finite set and a *letter* means an element of an alphabet. For an alphabet $\mathcal{A}$ and $n \in \mathbb{N}$, a *word* on $\mathcal{A}$ of length n is a map $w : \hat{n} \to \mathcal{A}$ where $\hat{n}$ is $\{1, 2, \cdots, n\}$. We denote a word of length n by $w(1)w(2)\cdots w(n)$. Roughly speaking, a word is a finite sequence of elements of an alphabet. We regard the map from empty set to empty set as the word of length 0 and denote it by $\emptyset$. A *phrase* of length k on $\mathcal{A}$ is a sequence of words $w_1, w_2, \cdots, w_k$ on $\mathcal{A}$. We denote this sequence by $(w_1|w_2|\cdots|w_k)$. We call the number $\sum_{i=1}^{k}(\text{length of } w_i)$ *number of letters* of the phrase. Especially if each letter in $\mathcal{A}$ appear exactly twice in a word w on $\mathcal{A}$, then we call this word w a *Gauss word*. Similarly for a phrase P on $\mathcal{A}$ if each letter in $\mathcal{A}$ appear exactly twice in P, then we call P a *Gauss phrase* (C. F. Gauss studied topology of plane curves using Gauss words. See [6] for more details).

In [18] and [19], Turaev introduced generalized words and phrases. Let α be an alphabet endowed with an involution $\tau : \alpha \to \alpha$. Then an *α-alphabet* is a pair of an alphabet $\mathcal{A}$ and a map $|\cdot| : \mathcal{A} \to \alpha$. We call this map $|\cdot|$ *projection* and we denote the image of a letter $A \in \mathcal{A}$ under the projection $|A|$. We also call $|A|$ a *projection of A*. Now we define generalized words (respectively Gauss words) which are called étale words (respectively nanowords). An *étale word* over α is a pair (an α-alphabet $\mathcal{A}$, a word w on $\mathcal{A}$). We call the length of w *length of étale word* $(\mathcal{A}, w)$. Especially if w is a Gauss word on $\mathcal{A}$, then we call a pair $(\mathcal{A}, w)$ a *nanoword* over α. Next we define generalized phrases (respectively Gauss phrases) which are called étale phrases (respectively nanophrases). An *étale phrase* over α is a pair (an α-alphabet $\mathcal{A}$, a phrase P on $\mathcal{A}$). We call length of P the *length of étale phrase* $(\mathcal{A}, P)$. Especially if P is a Gauss phrase on $\mathcal{A}$, then we call a pair $(\mathcal{A}, P)$ a *nanophrase* over α.

Example 2.1. Let α be an alphabet given by $\{a, b\}$ with an involution $\tau : \alpha \to \alpha$ given by $\tau(a)$ is equal to b. Let $\mathcal{A}$ be a α-alphabet given by $\{A, B, C\}$ with a projection given by $|A|$ is equal to a and $|B|$ and $|C|$ are equal to b. Then a pair $(\mathcal{A}, ABCABC)$ is a nanoword over α of length six. Furthermore, a pair $(\mathcal{A}, (AB|AC|BC))$ is a nanophrase over α of length three with six letters. On the other hand, a pair $(\mathcal{A}, ABCBC)$ is an étale word over α of length five, but not nanoword over α since the letter A appear only once in the word $ABCBC$. A pair $(\mathcal{A}, ABAB)$ is not nanoword, since the letter C does not appear. A pair $(\mathcal{A} - \{C\}, ABAB)$ is a nanoword over α of length four.

2.2. **S-homotopy of nanophrases and étale phrases.** In the paper [18] Turaev defined an equivalence relation on nanophrases which is called S-homotopy. This is suggested by the Reidemeister moves in the theory on knots. In this subsection, we introduce S-homotopy theory of words and phrases.

To define S-homotopy of nanophrases we prepare some definitions. First we define isomorphism of nanophrases.

Definition 2.1. Let $(\mathcal{A}_1, (w_1|\cdots|w_k))$ and $(\mathcal{A}_2, (v_1|\cdots|v_k))$ be nanophrases of length k over an alphabet α. Then $(\mathcal{A}_1, (w_1|\cdots|w_k))$ and $(\mathcal{A}_2, (v_1|\cdots|v_k))$ are *isomorphic* if there exist a bijection φ between $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $|A| = |\varphi(A)|$ for all $A \in \mathcal{A}_1$ and $v_j = \varphi(w_j)$ for each $j \in \hat{k}$.

Next we define S-homotopy moves of nanophrases.

Definition 2.2. Let S be a subset of $\alpha \times \alpha \times \alpha$. Then we define *$S$-homotopy moves* (H1) - (H3) of nanophrases as follows:

(H1) $(\mathcal{A}, (xAAy)) \longrightarrow (\mathcal{A} \setminus \{A\}, (xy))$
for all $A \in \mathcal{A}$ and x, y are sequences of letters in $\mathcal{A} \setminus \{A\}$, possibly including the | character.

(H2) $(\mathcal{A}, (xAByBAz)) \longrightarrow (\mathcal{A} \setminus \{A, B\}, (xyz))$
if $A, B \in \mathcal{A}$ satisfy $|B| = \tau(|A|)$. x, y, z are sequences of letters in $\mathcal{A} \setminus \{A, B\}$, possibly including the | character.

(H3) $(\mathcal{A}, (xAByACzBCt)) \longrightarrow (\mathcal{A}, (xBAyCAzCBt))$
if $A, B, C \in \mathcal{A}$ satisfy $(|A|, |B|, |C|) \in S$. x, y, z, t are sequences of letters in $\mathcal{A}$, possibly including the | character.

We call this S *homotopy data*.

Now we define *S-homotopy* of nanophrases.

Definition 2.3. Let $(\mathcal{A}_1, P_1)$ and $(\mathcal{A}_2, P_2)$ be nanophrases over α. Then $(\mathcal{A}_1, P_1)$ and $(\mathcal{A}_2, P_2)$ are *S-homotopic* (denote $\simeq_S$) if they are related by a finite sequence of isomorphism, S-homotopy moves (H1) - (H3) and inverse of (H1) - (H3).

Remark 2.1. S-homotopy moves and isomorphism of nanophrases do not change length of nanophrases. Thus for two different integers k_1 and k_2, a nanophrase of length k_1 and a nanophrase of length k_2 are not homotopic to each other.

Especially if S is the diagonal set of $\alpha \times \alpha \times \alpha$, then we call S-homotopy *homotopy*.

We denote the set $\{$Nanophrases of length k over $\alpha\}/(\text{S}-\text{homotopy})$ by $\mathcal{P}_k(\alpha, S)$ and $\mathcal{P}_1(\alpha, S)$ by $\mathcal{N}(\alpha, S)$.

Example 2.2. Nanophrases $(AB|ADDCBC)$ and $(BA|CACB)$ with $|A| = |B| = |C| \in \alpha$ over α are homotopic. Indeed

$$(AB|A\underline{DD}CBC) \simeq (\underline{AB}|\underline{ACBC}) \simeq (BA|CACB).$$

Next we define S-homotopy of étale phrases. To do so, we define desingularization of étale phrases.

For a nanophrase $(\mathcal{A}, P)$ and a letter A in $\mathcal{A}$, we define *multiplicity* of the letter A by the number of A in the phrase P. We denote multiplicity of A by $m_P(A)$. Let $\mathcal{A}^d$ be an α-alphabet $\{A_{i,j} := (A, i, j) | A \in \mathcal{A}, 1 \leq i < j \leq m_P(A)\}$ with the projection $|A_{i,j}| := |A|$ for all $A_{i,j}$. The phrase P^d is obtained from P by first deleting all $A \in \mathcal{A}$ for which $m_P(A)$ is less than or equal to one. Then for each $A \in \mathcal{A}$ for which $m_P(A)$ is grater than or equal to two and each $i = 1, 2, \ldots m_P(A)$, we replace the i-th entry of A in P by

$$A_{1,i}A_{2,i}\ldots A_{i-1,i}A_{i,i+1}A_{i,i+2}\ldots A_{i,m_P(A)}.$$

The resulting $(\mathcal{A}^d, P^d)$ is a nanophrase with $\sum m_P(A)(m_P(A) - 1)$ letters and called a *desingularization of* $(\mathcal{A}, P)$. Note that if $(\mathcal{A}, P)$ is a nanophrase, then desingularization of $(\mathcal{A}, P)$ is isomorphic to itself.

Example 2.3. Let α be an alphabet. Let $\mathcal{A}$ be an α-alphabet given by $\{A, B, C\}$ and P be a phrase given by $(AA|BB|A|C)$. Then desingularization of an étale phrase $(\mathcal{A}, P)$ is given by

$$(\{A_{12}, A_{13}, A_{23}, B_{12}\}, (A_{12}A_{13}A_{12}A_{23}|B_{12}B_{12}|A_{13}A_{23}|\emptyset)),$$

with $|A_{12}| = |A_{13}| = |A_{23}| = |A|$ and $|B_{12}| = |B|$.

Now we define S-homotopy of étale phrases.

Definition 2.4. Two étale phrases $(\mathcal{A}_1, P_1)$ and $(\mathcal{A}_2, P_2)$ over α are *S-homotopic* (denoted $(\mathcal{A}_1, P_1) \simeq (\mathcal{A}_2, P_2)$) if $((\mathcal{A}_2)^d, (P_2)^d)$ can be obtained from $((\mathcal{A}_1)^d, (P_1)^d)$ by a finite sequence of isomorphism, S-homotopy moves (H1) - (H3) and the inverse of moves (H1) - (H3).

Remark 2.2. By the definition of homotopy of étale phrases, every homotopy invariant I of nanophrases extends to a homotopy invariant I of étale phrases by $I(P) := I(P^d)$.

We recall two lemmas from [18] and [19].

Lemma 2.1 (Turaev [18], [19])**.** *Let (α, S) be homotopy data and $\mathcal{A}$ be an α-alphabet. Let A, B, C be distinct letters in $\mathcal{A}$ and let x, y, z, t be words in $\mathcal{A} \setminus \{A, B, C\}$ such that $xyzt$ is a Gauss phrase. Then the following (i)-(iii) hold :*

(i) $(\mathcal{A}, (xAByCAzBCt)) \simeq_S (\mathcal{A}, (xBAyACzCBt))$
if $(|A|, \tau(|B|), |C|) \in S$,
(ii) $(\mathcal{A}, (xAByCAzCBt)) \simeq_S (\mathcal{A}, (xBAyACzBCt))$
if $(\tau(|A|), \tau(|B|), |C|) \in S$,
(iii)$(\mathcal{A}, (xAByACzCBt)) \simeq_S (\mathcal{A}, (xBAyCAzBCt))$
if $(|A|, \tau(|B|), \tau(|C|)) \in S$.

Lemma 2.2 (Turaev [18], [19])**.** *Suppose that $S \cap (\alpha \times b \times b) \neq \emptyset$ for all $b \in \alpha$. Let $(\mathcal{A}, (xAByABz))$ be a nanoword over α with $|B| = \tau(|A|)$ where x, y, z are words in $\mathcal{A} \setminus \{A, B\}$ such that xyz is a Gauss phrase. Then*

$$(\mathcal{A}, (xAByABz)) \simeq_S (\mathcal{A} \setminus \{A, B\}, (xyz)).$$

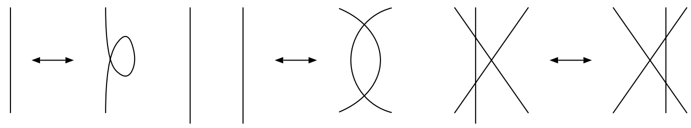

FIGURE 1. The flat Reidemeister moves.

3. Geometric Interpretation of Homotopy of Nanophrases.

In this section we explain geometric interpretation of S-homotopy of nanophrases which was introduced in the paper [19].

3.1. **Stable equivalence of curves on surfaces.** In this subsection we introduce stable equivalence of curves on surfaces. First we define some terminologies. Through this paper a *curve* means the image of a generic immersion of an oriented circle into an oriented surface. The word "generic" means that the curve has only a finite set of self-intersections which are all double and transversal. A k*-component curve* is defined in the same way as a curve with the difference that they may be formed by k curves. These curves are called *components* of the k-component curve. A k-component curve is *pointed* if each component is endowed with a base point (the origin) distinct from the crossing points of the k-component curve. A k-component curve is *ordered* if its components are numerated. Next we introduce an equivalence relation which is called stably equivalence. Two ordered, pointed curves are *stably homeomorphic* if there is an orientation preserving homeomorphism of their regular neighborhoods in the ambient surfaces mapping the first multi-component curve onto the second one and preserving the order, the origins and the orientations of the components.

Now we define stable equivalence of ordered, oriented, pointed multi-component curves [14]: Two ordered, pointed multi-component curves are *stably equivalent* if they can be related by a finite sequence of the following transformations: (i) a move replacing an ordered, pointed multi-component curve with a stably homeomorphic one; (ii) the flat Reidemeister moves away from the origin as in Figure 1.

We denote the set of stable equivalence classes of ordered, oriented, pointed k-component curves by $\mathcal{C}_k$.

Remark 3.1. The theory of stable equivalence of curves is closely related to the theory of virtual strings. See [17], [21] and Section 3.3 in this paper for more details.

3.2. **Geometric interpretation of S-homotopy of nanophrases.** In the paper [19] Turaev gave geometric meanings of S-homotopy of nanophrases over α with an involution τ for some α, S and τ. More precisely, Turaev proved the following theorem.

Theorem 3.1 (Turaev [19])**.** *There is a canonical bijection between $\mathcal{C}_k$ and $\mathcal{P}_k(\alpha_0, S_0)$ where α_0 is equal to $\{a, b\}$ with an involution τ_0 where $\tau_0(a)$ is equal to b and S_0 is equal to $\{(a, a, a), (b, b, b)\}$.*

The way of making a nanophrase $P(C)$ from an ordered, oriented, pointed k-component curve C is as follows. Let us label the double points of the curve C by distinct letters $A_1, \cdots, A_n$. Starting at the origin of first component of C and following along C in the positive direction, we write down the labels of double points which we passes until return to the origin. Then we obtain a word w_1. Similarly we obtain words $w_2, \cdots, w_k$ on the alphabet $\mathcal{A} = \{A_1, \cdots, A_n\}$ from second component, $\cdots$, k-th component. Let t_i^1 (respectively, t_i^2) be the tangent vector to C at the double point labeled A_i appearing at the first (respectively, second) passage through this point. Set $|A_i|$ is equal to a, if the pair (t_i^1, t_i^2) is positively oriented, and $|A_i|$ is equal to b otherwise. Then we obtain a required nanophrase $P(C) := (\mathcal{A}, (w_1|\cdots|w_k))$.

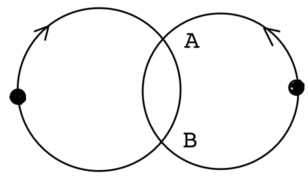

FIGURE 2. An example

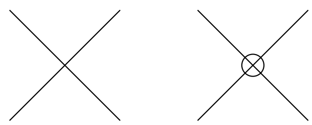

FIGURE 3. A real crossing and a virtual crossing.

Remark 3.2. By the above theorem if we classify the homotopy classes of nanophrases, then we obtain the classification of ordered, pointed multi-component curves under the stable equivalence as a corollary.

Example 3.1. Consider a two-component pointed ordered curve shown in Fig. 2. Assume that a left circle is first component of this curve and a right circle is second component of this curve. Then a nanophrase which corresponds to this curve is $(\{A, B\}, (AB|AB))$ with $|A|$ is equal to b and $|B|$ is equal to a.

Moreover let $\mathcal{L}_k$ be the set of *stable equivalence classes of k-component pointed ordered oriented link diagrams* (definition of the stable equivalence of link diagrams is given in [19] for example). Then Turaev proved following theorem.

Theorem 3.2 (Turaev [19])**.** *There is a canonical bijection between $\mathcal{L}_k$ and $\mathcal{P}_k(\alpha_*, S_*)$ where α_* is equal to $\{a_+, a_-, b_+, b_-\}$ with an involution $\tau_*(a_\pm)$ is equal to $b_\mp$ and S_* is equal to $\{(a_\pm, a_\pm, a_\pm)$, $(a_\pm, a_\pm, a_\mp)$, $(a_\mp, a_\pm, a_\pm)$, $(b_\pm, b_\pm, b_\pm)$, $(b_\pm, b_\pm, b_\mp)$, $(b_\mp, b_\pm, b_\pm)\}$.*

The method of making nanophrase $P(L)$ from ordered, pointed k-component link L is similar to the case Theorem 3.1. See [19] for more details.

Remark 3.3. We can find another applications of the theory of nanowords and étale words to geometry and topology in papers [11] and [12]. N. Ito used the theory of nanowords to study planar curves and wave fronts on $\mathbb{R}^2$.

3.3. **Presentation of curves on surfaces by virtual strings.** In this subsection, we introduce useful method to illustrate curves on surfaces. To do so, we introduce virtual string diagrams and virtual strings.

A *virtual string diagram* is a planar graph of valency four endowed with the following structure: each vertex either is an unmarked crossing (in other words, *real crossing*) or is marked by a *virtual crossing* (see Figure 3). Then we define a *virtual string* by a virtual string diagram modulo *flat virtual Reidemeister moves* which are illustrated in Figure 4. We also use terminologies *pointed*, *ordered* and *oriented* same as in the case of curves on surfaces.

It is known the stable equivalence theory of pointed ordered curves on surfaces is equivalent to the theory of pointed ordered virtual strings by the correspondence illustrated in Figure 5 (see [13], [19] for example). Therefore in the rest of this paper, we illustrate curves on surfaces as virtual strings diagrams.

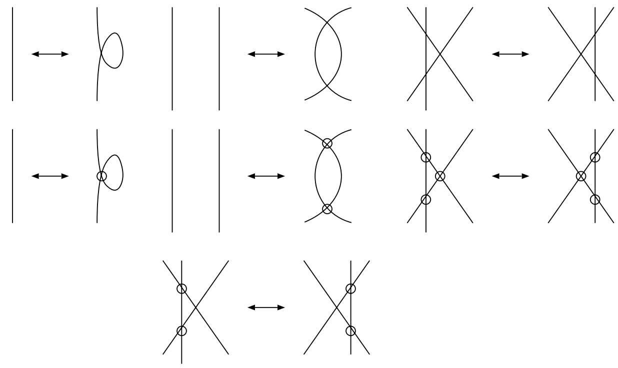

FIGURE 4. Flat virtual Reidemeister moves.

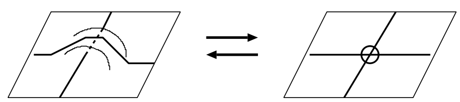

FIGURE 5. The correspondence of virtual strings and curves on surfaces.

4. Classification of Nanophrases and Étale Phrases up to Homotopy.

In this section, we introduce classification theorems of nanowords, nanophrases, étale words and étale phrases up to homotopy which were proved in [1], [3], [4] and [18].

4.1. Classification of nanowords and étale words.

First, we introduce classification of nanowords with at most six letters (see [18]). Note that an arbitrary nanoword of length two is homotopic to an empty nanoword $\emptyset$ by a first homotopy move.

Theorem 4.1 (Turaev [18]). *Let w be a nanoword of length four over α. Then w is either homotopic to the empty nanoword or isomorphic to the nanoword $w_{a,b} := (\mathcal{A} = \{A, B\}, ABAB)$ where $|A| = a, |B| = b \in \alpha$ with $a \neq \tau(b)$. Moreover for $a \neq \tau(b)$, the nanoword $w_{a,b}$ is non-contractible and two nanowords $w_{a,b}$ and $w_{a',b'}$ are homotopic if and only if $a = a'$ and $b = b'$.*

Next we introduce homotopy classification of nanowords with length less than or equal to six. Pick three letters $a, b, c \in \alpha$ (possibly coinciding). Let $\mathcal{A}$ be an α-alphabet consisting of three letters A, B and C where $|A|$ is a, $|B|$ is b and $|C|$ is c. Consider nanowords over α, $w^1_{a,b,c} = ABCABC$, $w^2_{a,b,c} = ABCACB$, $w^3_{a,b,c} = ABCBAC$, $w^4_{a,b,c} = ABCBCA$, and $w^5_{a,b,c} = ABACBC$. It is easily checked that a nanoword of length six is either homotopic to a nanoword with length less than or equal to four or isomorphic to $w^i_{a,b,c}$ for some $i \in \{1, 2, 3, 4, 5\}$. We now point out obvious sufficient conditions for $w^i_{a,b,c}$ to be isomorphic to the empty word.

If $a = \tau(b)$ or $c = \tau(b)$, then $w^1_{a,b,c} \simeq \emptyset$. We say that an ordered triple $a, b, c \in \alpha$ is 1-*regular* if $a \neq \tau(b) \neq c$.

If $c = \tau(b)$, then $w^2_{a,b,c} \simeq \emptyset$. We say that an ordered triple $a, b, c \in \alpha$ is 2-*regular* if $c \neq \tau(b)$.

If $a = \tau(b)$, then $w^3_{a,b,c} \simeq \emptyset$. We say that an ordered triple $a, b, c \in \alpha$ is 3-*regular* if $a \neq \tau(b)$.

If $c = \tau(b)$, then $w^4_{a,b,c} \simeq \emptyset$. We say that an ordered triple $a, b, c \in \alpha$ is 4-*regular* if $c \neq \tau(b)$. (This coincides with the 2-regularity).

If $a=b=c=\tau(a)$, then $w^5_{a,b,c}\simeq\emptyset$. We say that an ordered triple $a,b,c\in\alpha$ is *singular* if $a=b=c=\tau(a)$ and 5-*regular* otherwise.

The following theorem gives the homotopy classification of nanowords of length six.

Theorem 4.2 (Turaev [18])**.** *For $i\in\{1,2,3,4,5\}$ and any i-regular triple $a,b,c\in\alpha$, the nanoword $w^i_{a,b,c}$ is neither contractible nor homotopic to a nanoword of length 4. The nanowords w^i corresponding to i-regular triples a,b,c and a',b',c' are homotopic if and only if (a,b,c) is equal to (a',b',c'). For $i\neq j$, the nanowords w^i corresponding to i-regular triples are not homotopic to nanowords w^j corresponding to j-regular triples with one exception: $w^4_{a,b,c}$ is homotopic to $w^5_{a,b,c}$ for $a=b=c\neq\tau(a)$.*

Turaev constructed some homotopy invariants of nanowords, and proved the above classification theorems in [18].

Moreover, Turaev classified words with at most five letters.

Theorem 4.3 (Turaev [18])**.** *A multiplicity-one-free word of length less than or equal to four in the alphabet α has one of the following forms: aa, aaa, aaaa, aabb, abba, abab with distinct $a,b\in\alpha$ The words aa, aabb, abba are contractible. The words aaa and aaaa are contractible if and only if $\tau(a)$ is equal to a. The word abab is contractible if and only if $\tau(a)$ is equal to b. Non-contractible words of type aaa, aaaa and abab are homotopic if and only if they are equal.*

4.2. **Classification of nanophrases and étale phrases.** Next we introduce classification theorems of nanophrases and étale phrases which were proved by the author in [1], [3] and [4].

First, we introduce the homotopy classification of nanophrases with at most four letters without condition on length. Set $P^{1,1;p,q}_a := (\emptyset|\cdots|\emptyset|\ \overset{p}{\check{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\check{A}}\ |\emptyset|\cdots|\emptyset)$ with $|A|=a$ for $1\leq p<q\leq k$. Classification of nanophrases with at most two letters is described as follows.

Theorem 4.4 ([3])**.** *Let P be a nanophrase of length k with 2 letters. Then P is either homotopic to $(\emptyset|\cdots|\emptyset)$ or isomorphic to $P^{1,1;p,q}_a$ for some $p,q\in\{1,\cdots k\}$, $a\in\alpha$. Moreover $P^{1,1;p,q}_a$ and $P^{1,1;p',q'}_{a'}$ are homotopic if and only if p is equal to p', q is equal to q' and a is equal to a'.*

To describe the classification theorem of nanophrases with four letters without condition on length, we use following notations.

$P^{4;p}_{a,b} := (\emptyset|\cdots|\emptyset|\ A\overset{p}{\check{B}}AB\ |\emptyset|\cdots|\emptyset)$,

$P^{3,1;p,q}_{a,b} := (\emptyset|\cdots|\emptyset|\ A\overset{p}{\check{B}}A\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\check{B}}\ |\emptyset|\cdots|\emptyset)$,

$P^{2,2I;p,q}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\check{A}}B\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\check{A}}B\ |\emptyset|\cdots|\emptyset)$,

$P^{2,2II;p,q}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\check{A}}B\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\check{B}}A\ |\emptyset|\cdots|\emptyset)$,

$P^{1,3;p,q}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\check{A}}\ |\emptyset|\cdots|\emptyset|\ B\overset{q}{\check{A}}B\ |\emptyset|\cdots|\emptyset)$,

$P^{2,1,1I;p,q,r}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\check{A}}B\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\check{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{r}{\check{B}}\ |\emptyset|\cdots|\emptyset)$,

$P^{2,1,1II;p,q,r}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\check{B}}A\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\check{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{r}{\check{B}}\ |\emptyset|\cdots|\emptyset)$,

$P^{1,2,1I;p,q,r}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\check{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\check{A}}B\ |\emptyset|\cdots|\emptyset|\ \overset{r}{\check{B}}\ |\emptyset|\cdots|\emptyset)$,

$P^{1,2,1II;p,q,r}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\check{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\check{B}}A\ |\emptyset|\cdots|\emptyset|\ \overset{r}{\check{B}}\ |\emptyset|\cdots|\emptyset)$,

$P^{1,1,2I;p,q,r}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\check{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\check{B}}\ |\emptyset|\cdots|\emptyset|\ \overset{r}{\check{A}}B\ |\emptyset|\cdots|\emptyset)$,

$P_{a,b}^{1,1,2II;p,q,r} := (\emptyset|\cdots|\emptyset|\overset{p}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{q}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{r}{\check{B}A}|\emptyset|\cdots|\emptyset)$,
$P_{a,b}^{1,1,1,1I;p,q,r,s} := (\emptyset|\cdots|\emptyset|\overset{p}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{q}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{r}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{s}{\check{B}}|\emptyset|\cdots|\emptyset)$,
$P_{a,b}^{1,1,1,1II;p,q,r,s} := (\emptyset|\cdots|\emptyset|\overset{p}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{q}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{r}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{s}{\check{B}}|\emptyset|\cdots|\emptyset)$,
$P_{a,b}^{1,1,1,1III;p,q,r,s} := (\emptyset|\cdots|\emptyset|\overset{p}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{q}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{r}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{s}{\check{A}}|\emptyset|\cdots|\emptyset)$,
with $|A|$ is equal to a and $|B|$ is equal to b. If a is equal to $\tau(b)$, then nanophrases $P_{a,b}^{4;p}$, $P_{a,b}^{2,2I;p,q}$ and $P_{a,b}^{2,2II;p,q}$ are homotopic to $(\emptyset|\cdots|\emptyset)$. So when we write $P_{a,b}^{4;p}$, $P_{a,b}^{2,2I;p,q}$ and $P_{a,b}^{2,2II;p,q}$ we always assume that a is not equal to $\tau(b)$.

Under the above notations the classification of nanophrases with four letter is described as follows.

Theorem 4.5 ([3])**.** *Let P be a nanophrase of length k with four letters. Then P is either homotopic to nanophrase with less than or equal to two letters or isomorphic to $P_{a,b}^{X;Y}$ for some $X \in \{4,(3,1),\cdots,(1,1,1,1III)\}$, $Y \in \{1,\cdots,k,(1,2),\cdots,(k-3,k-2,k-1,k)\}$. Moreover $P_{a,b}^{X;Y}$ and $P_{a',b'}^{X';Y'}$ are homotopic if and only if $X = X'$, $Y = Y'$, $a = a'$ and $b = b'$.*

Finally we introduce the classification of étale phrases with at most four letters which are called monoliteral type.

An étale phrases P is called *monoliteral* if P has only empty word as its components or consists of a single letter. For example, étale phrases $(AAA|AAAA|\emptyset|AA)$, $(AA|A)$ and $(\emptyset|\emptyset|\emptyset)$ are monoliteral phrases. Now we consider the following étale phrases: $P_a^{1,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{3;l} := (\emptyset|\cdots|\emptyset|\overset{l}{\check{a}^3}|\emptyset|\cdots|\emptyset)$,
$P_a^{2,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}^2}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,2;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}^2}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,1,1;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{4;l} := (\emptyset|\cdots|\emptyset|\overset{l}{\check{a}^4}|\emptyset|\cdots|\emptyset)$,
$P_a^{3,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}^3}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{2,2;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}^2}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}^2}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,3;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}^3}|\emptyset|\cdots|\emptyset)$,
$P_a^{2,1,1;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}^2}|\emptyset|\cdots|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,2,1;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\overset{l_2}{\check{a}^2}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,1,2;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}^2}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,1,1,1;l_1,l_2,l_3,l_4} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_4}{\check{a}}|\emptyset|\cdots|\emptyset)$,
where $a \in \alpha$ and l, l_1,l_2,l_3,$l_4 \in \hat{k}$ with $l_1 < l_2 < l_3 < l_4$.

Note that if a is equal to $\tau(a)$, then $P_a^{4;l}$ and $P_a^{3;l}$ are homotopic to the empty phrase. So when we write $P_a^{4;l}$ or $P_a^{3;l}$ we always assume that a is not equal to $\tau(a)$. Now we describe the classification theorem of monoliteral étale phrases with less than or equal to four letters.

Theorem 4.6 ([4])**.** *Let P be a multiplicity-one-free monoliteral étale phrase over α with less than or equal to four letters. Then P is either homotopic to $(\emptyset)_k$ or isomorphic to one of the following étale phrases: $P_a^{1,1;l_1,l_2}$, $P_a^{4;l}$, $P_a^{3,1;l_1,l_2}$, $P_a^{1,3;l_1,l_2}$, $P_a^{2,1,1;l_1,l_2,l_3}$, $P_a^{1,2,1;l_1,l_2,l_3}$, $P_a^{1,1,3;l_1,l_2,l_3}$, $P_a^{1,1,1,1;l_1,l_2,l_3,l_4}$, $P_a^{3;l}$, $P_a^{2,1;l_1,l_2}$, $P_a^{1,2;l_1,l_2}$ and $P_a^{1,1,1;l_1,l_2,l_3}$ for some l_1, l_2, l_3, $l_4 \in \hat{k}$ and $a \in \alpha$. Moreover they are homotopic if and only if they are equal with one exception : $P_a^{3,1;l_1,l_2}$ and $P_a^{1,3;l_1,l_2}$ are homotopic to $P_a^{1,1;l_1,l_2}$ if a is equal to $\tau(a)$.*

Remark 4.1. A finite sequence of homotopy moves from $P_a^{3,1;l_1,l_2}$ to $P_a^{1,1;l_1,l_2}$ is realized as follows:

$$\begin{aligned}
(P_a^{3,1;l_1,l_2})^d &= (\emptyset|\cdots|\emptyset|A_{12}A_{13}A_{14}A_{12}A_{23}A_{24}A_{13}A_{23}A_{34}|\emptyset|\cdots|\emptyset|A_{14}A_{24}A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{13}A_{12}A_{14}A_{23}A_{12}A_{24}A_{23}A_{13}A_{34}|\emptyset|\cdots|\emptyset|A_{14}A_{24}A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{13}A_{12}A_{23}A_{14}A_{12}A_{23}A_{24}A_{13}A_{34}|\emptyset|\cdots|\emptyset|A_{24}A_{14}A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{13}A_{14}A_{24}A_{13}A_{34}|\emptyset|\cdots|\emptyset|A_{24}A_{14}A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{13}A_{13}A_{34}|\emptyset|\cdots|\emptyset|A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{34}|\emptyset|\cdots|\emptyset|A_{34}|\emptyset|\cdots|\emptyset) \\
&= (P_a^{1,1;l_1,l_2})^d.
\end{aligned}$$

Similarly a finite sequence of homotopy moves from $P_a^{1,3;l_1,l_2}$ to $P_a^{1,1;l_1,l_2}$ is realized as follows:

$$\begin{aligned}
(P_a^{1,3;l_1,l_2})^d &= (\emptyset|\cdots|\emptyset|A_{12}A_{13}A_{14}|\emptyset|\cdots|\emptyset|A_{12}A_{23}A_{24}A_{13}A_{23}A_{34}A_{14}A_{24}A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{12}A_{13}A_{14}|\emptyset|\cdots|\emptyset|A_{12}A_{24}A_{23}A_{13}A_{34}A_{23}A_{14}A_{34}A_{24}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{12}A_{14}A_{13}|\emptyset|\cdots|\emptyset|A_{12}A_{24}A_{23}A_{34}A_{13}A_{23}A_{34}A_{14}A_{24}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{12}A_{14}A_{13}|\emptyset|\cdots|\emptyset|A_{12}A_{24}A_{13}A_{14}A_{24}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{12}|\emptyset|\cdots|\emptyset|A_{12}A_{24}A_{24}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{12}|\emptyset|\cdots|\emptyset|A_{12}|\emptyset|\cdots|\emptyset) \\
&= (P_a^{1,1;l_1,l_2})^d.
\end{aligned}$$

Proof of classification theorems of nanophrases and monoliteral phrases are described in [1], [3] and [4]. In the next section we introduce some invariants and show examples of classifications.

5. Homotopy Invariants of Nanophrases.

In this section we introduce some homotopy invariants for nanophrases which we used to prove the classification theorems.

5.1. **Component length vector.**

In this sub-subsection, we define the component length vector of nanophrases (see [1], [3] and [8]).

Let $P = (w_1|w_2|\cdots|w_k)$ be a nanophrase over α. For $l \in \hat{k}$, we define $w(l) \in \mathbb{Z}/2\mathbb{Z}$ by the length of w_l. We call the vector

$$w(P) := (w(1), \cdots, w(k)) \in (\mathbb{Z}/2\mathbb{Z})^k$$

the *component length vector*.

Proposition 5.1 ([3])**.** *The component length vector is a homotopy invariant of nanophrases.*

Remark 5.1. Note that the component length vector is a S-homotopy invariant of nanophrases for all S.

Example 5.1. Consider nanophrases $(A|A)$ and $(\emptyset|\emptyset)$. Then $w((A|A))$ is equal to $(1,1)$. On the other hand, $w((\emptyset|\emptyset))$ is equal to $(0,0)$. Therefore $(A|A)$ and $(\emptyset|\emptyset)$ are not homotopic each other.

5.2. **Linking vector.** In this sub-section we introduce the linking vector of nanophrases (See [3] and [8]). Let π be the group which is defined as follows:

$$\pi := (a \in \alpha | a\tau(a) = 1, ab = ba \ for \ all \ a, b \in \alpha \ ).$$

Let P be a nanophrases $(w_1|w_2|\cdots|w_k)$ of length k over α. We define $l_P(i,j) \in \pi$ for $i < j$ by

$$l_P(i,j) := \prod_{A \in Im(w_i) \cap Im(w_j)} |A|.$$

We call a vector $lk(P) := (l_P(1,2), l_P(1,3), \cdots, l_P(1,k), l_P(2,3), \cdots, l_P(k-1,k)) \in \pi^{\frac{1}{2}k(k-1)}$ the *linking vector* of P.

Proposition 5.2 ([3])**.** *The linking vector of nanophrases is a homotopy invariant of nanophrases.*

Remark 5.2. This invariant is also S-homotopy invariant for all S.

Example 5.2. Consider nanophrases $(A|A)$ and $(B|B)$ over α where $|A|$ is equal to a and $|B|$ is equal to b. Then $lk((A|A))$ is equal to $a \in \pi$ and $lk((B|B))$ is equal to $b \in \pi$. Therefore $(A|A)$ and $(B|B)$ are homotopic if and only if a is equal to b.

6. Gibson's S_o invariant.

In the paper [8], A.Gibson defined a homotopy invariant of nanophrases over the one-element set. First we define some notations. Let $(\mathcal{A}, P = (w_1|\cdots|w_k))$ be a nanophrase over the one-element set. For a letter $A \in \mathcal{A}_i := \{A \in \mathcal{A} | Card(w_i^{-1}(A)) = 2\}$, we define $l_j(A) \in \mathbb{Z}/2\mathbb{Z}$ as follows : When we write P as $xAyAz$ where x, y and z are words in $\mathcal{A}$ possibly including "|" character, $l_j(A)$ is modulo 2 of the number of letters which appear exactly once in y and once in the j-th component of the phrase P. Then we define $l(A) \in (\mathbb{Z}/2\mathbb{Z})^k$ by

$$l(A) := (l_1(A), l_2(A), \cdots, l_k(A)).$$

Let v be a vector in $(\mathbb{Z}/2\mathbb{Z})^k$. Then we define $d_j(v) \in \mathbb{Z}$ by

$$d_j(v) := Card(\{A \in \mathcal{A}_j | l(A) = v\}),$$

and we define $B_j(P) \in 2^{(\mathbb{Z}/2\mathbb{Z})^k}$ by

$$B_j(P) := \{v \in (\mathbb{Z}/2\mathbb{Z})^k \setminus \{0\} | d_j(v) = 1 \ mod \ 2\}.$$

Then we define the $S_o(P) \in (2^{(\mathbb{Z}/2\mathbb{Z})^k})^k$ by

$$S_o(P) := (B_1(P), B_2(P), \cdots, B_k(P)).$$

Theorem 6.1 (Gibson [8])**.** *S_o is a homotopy invariant of nanophrases over the one-element set.*

Example 6.1. Consider nanophrases $(P^{2,1;l_1,l_2})^d$ and $(P^{2,1;l_1,l_2})^d$. Then

$$S_o((P_a^{2,1;l_1,l_2})^d) = (\emptyset, \cdots, \emptyset, \{\overset{l_1}{\check{\mathbf{e}}_{l_2}}\}, \emptyset, \cdots, \emptyset),$$

and

$$S_o((P_a^{2,1;m_1,m_2})^d) = (\emptyset, \cdots, \emptyset, \{\overset{m_1}{\check{\mathbf{e}}_{m_2}}\}, \emptyset, \cdots, \emptyset),$$

where $\mathbf{e}_i$ is equal to $(0, \cdots, 0, \overset{i}{\check{1}}, 0, \cdots, 0)$. Therefore we obtain that $P_a^{2,1;l_1,l_2}$ is not homotopic to $P_a^{2,1;m_1,m_2}$ if (l_1, l_2) is not equal to (m_1, m_2).

Remark 6.1. The author and A.Gibson generalized the S_o invariant for nanophrases over the one element set to a homotopy invariant over an arbitrary α in papers [5] and [10] independently. These two generalizations are equivalent. See [5] and [10] for more details.

6.1. **Invariant R_o for nanophrases over the one-element set.** In this subsection, we introduce an invariant of nanophrases over the one-element set which was defined in [4]. Let $(\mathcal{A}, P)$ be a nanophrase over the one-element set. For two letters $X \in \mathcal{A}_{l_1}$ and $Y \in \mathcal{A}_{l_2}$, we define $dl_P(X, Y) \in \mathbb{Z}/2\mathbb{Z}$ by

$$dl_P(X, Y) = Card\{Z \in \mathcal{A}_{l_1 l_2} | n(X, Z) = 1, n(Y, Z) = -1\} \ mod\ 2,$$

and for integers l_1 and l_2, we define $de_P(l_1, l_2) \in \mathbb{Z}/2\mathbb{Z}$ by

$$de_P(l_1, l_2) = Card\{(X, Y) \in \mathcal{A}_{l_1} \times \mathcal{A}_{l_2} | dl(X, Y) = 1\} \ mod\ 2.$$

Then we define $R_o(P)$ by

$$R_o(P) = (de(l_1, l_2))_{l_1 < l_2}.$$

Proposition 6.1 ([4])**.** *The R_o is a homotopy invariant for nanophrases over the one-element set.*

Example 6.2. Consider the étale phrase $P_a^{2,2;l_1,l_2}$. Then

$$(P_a^{2,2;l_1,l_2})^d = (\emptyset| \cdots |\emptyset| A_{12}A_{13}A_{14}A_{12}A_{23}A_{24}|\emptyset| \cdots |\emptyset| A_{13}A_{23}A_{34}A_{14}A_{24}A_{34}|\emptyset| \cdots |\emptyset).$$

We denote $(P_a^{2,2;l_1,l_2})^d$ by P. In this caseãĂĂ

$$dl_P(A_{12}, A_{34}) = Card\{A_{14}\} = 1$$

and

$$de_P(i, j) = \begin{cases} 1 \text{ if } (i, j) = (l_1, l_2), \\ 0 \text{ otherwise.} \end{cases}$$

Therefore $R_o(P)$ is equal to $\mathbf{e}_{(l_1,l_2)}$. On the other hand $R_o((\emptyset| \cdots |\emptyset))$ is equal to $\mathbf{0}$. Therefore, this example shows that $P_a^{2,2;l_1,l_2}$ is not homotopic to the empty phrase.

Using the above invariants and some properties on nanophrases and étale phrases, we can classify nanophrases and monoliteral phrases at most four letters without condition on length.

7. An Application to Curves on Surfaces.

By the theorems in Section 3, if we put α that is equal to α_0 and τ is equal to τ_o, then we obtain the classification of pointed ordered curves on surfaces up to stable equivalence.

7.1. **Applications of the classification of nanophrases.** In the papers [1], [2] and [4], the author proved the following corollaries.

Corollary 7.1 ([1])**.** *There are exactly* 19 *stable equivalence classes of two-component pointed, ordered, oriented, curves on surfaces with minimum crossing number less than or equal to* 2*.*

More generally we can prove a following statement.

Corollary 7.2 ([2])**.** *Let k be an positive integer. Then there are exactly*

$$1 + \frac{1}{2}k^2 + k^3 + \frac{1}{2}k^4$$

stable equivalence classes of ordered, pointed, k-component surface curves with minimal crossing number less than or equal to two.

An ordered, pointed multi-component surface-curve is called *irreducible* if it is not stably equivalent to a surface-curve with a simply closed component.

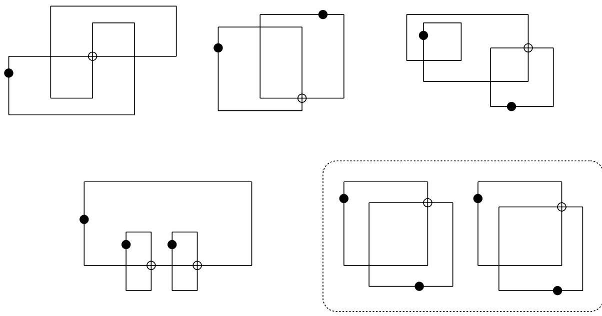

FIGURE 6. The list of irreducible curves. See also Remark 7.1.

Corollary 7.3 ([3]). *Any irreducible ordered, pointed multi-component surface-curve with minimal crossing number less than or equal to two is stably equivalent to one of the ordered, pointed multi-component curves arising from the following list (see also Remark 7.1). There are exactly* 52 *stable equivalence classes of irreducible ordered, pointed, multi-component surface-curves.*

Remark 7.1. We would like to list up the all stable equivalence classes of irreducible ordered, pointed multi-component surface-curves with minimal crossing number less than or equal to two. However there are too many curves to list up. Therefore we make just the list of multi-component curves without order and orientation of the components in Figure 6. If we choose order and orientation of components, then we obtain a ordered, pointed multi-component curve on surface. Two different pictures from Figure 6 never produce equivalent ordered, pointed multi-component curves on surfaces. On the other hand it is possible that two different additional structures (orientation and the order) on the same picture yield equivalent ordered, pointed multi-component curves on surfaces. More precisely, 2 (respectively 2, 8, 4, 24, 12) different ordered, pointed multi-component surface-curves arise from the upper left (respectively upper middle, upper right, lower left, lower middle, lower right) picture. By Theorem 4.5, ordered, pointed multi-component surface-curves arising from pictures in Figure 6 are stably equivalent if and only if nanophrases associated to these curves are homotopic, and we can obtain all of the stable equivalent classes of irreducible ordered, pointed multi-component curves on surfaces with minimal crossing number less than or equal to two by specifying order and orientation for multi-component curves in Figure 6.

7.2. **An application of the classification of monoliteral phrases.** In this sub-section we introduce an application of the classification of monoliteral phrases with at most four letters. To do so, we introduce a notion of monoliteral type curves. A curve on a surface is called of *monoliteral type* if the curve is stably equivalent to a curve which corresponds to a nanophrase obtained by desingularization of a monoliteral phrase. Now we describe the classification of irreducible monoliteral ordered pointed multi-component curves on surfaces with minimal crossing number less than or equal to six.

Corollary 7.4. *Any irreducible monoliteral ordered pointed multi-component curve on a surface with minimal crossing number less than or equal to six is stably equivalent to one of the ordered, pointed multi-component curves in Figures 7 and 8. Therefore there are exactly* 26 *stable equivalence classes of irreducible ordered, pointed, multi-component surface-curves.*

Remark 7.2. Curves in Figure 7 correspond to monoliteral phrases of type $P_a^{X;Y}$ and curves in Figure 8 correspond to monoliteral phrases of type $P_b^{X;Y}$.

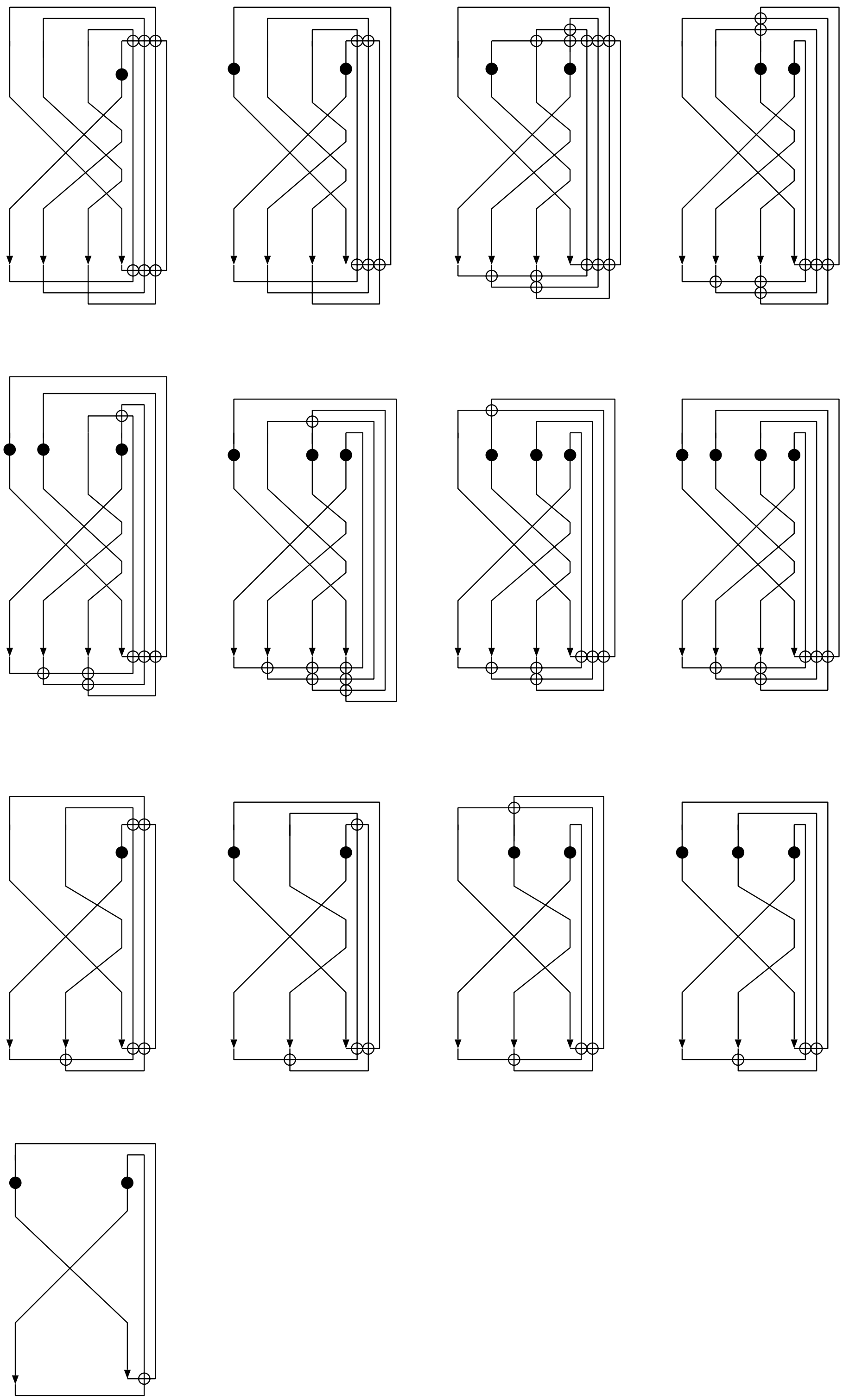

FIGURE 7. The half of list of monoliteral curves. Each component is numerated from right to left.

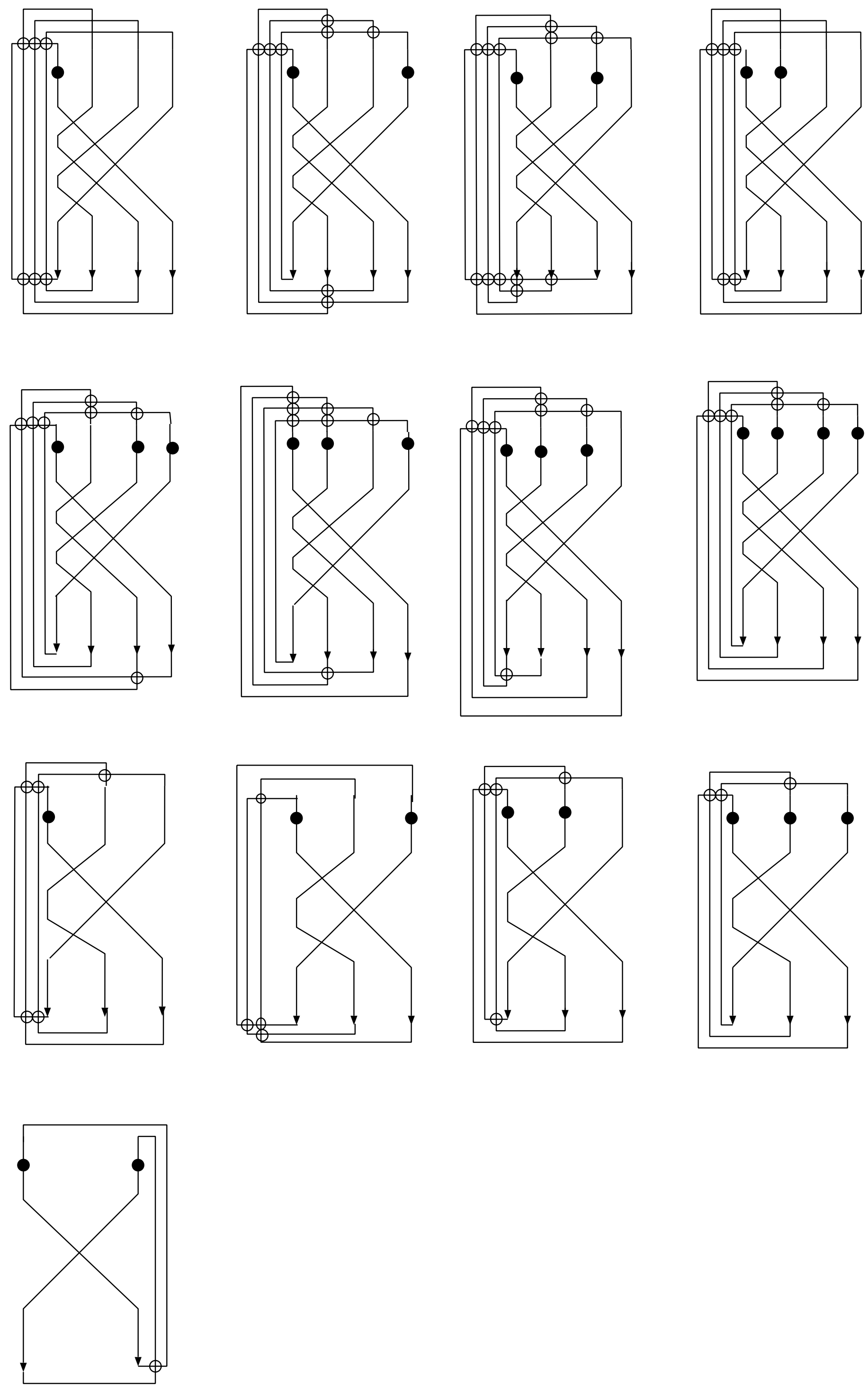

FIGURE 8. The half of list of monoliteral curves. Each component is numerated from left to right.

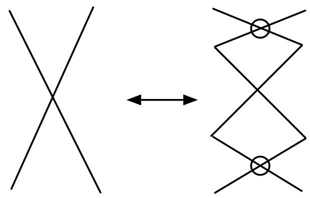

FIGURE 9. A flat virtualization move

Proof. We put that α is equal to α_0, and τ is equal to τ_0, then by Theorem 4.6 we obtain the list of a complete representable system of homotopy class of nanophrases which does not contain empty words as components of phrase as follows: $(aaaa)$, $(aaa|a)$, $(aa|aa)$, $(a|aaa)$, $(aa|a|a)$, $(a|aa|a)$, $(a|a|aa)$, $(a|a|a|a)$, (aaa), $(aa|a)$, $(a|aa)$, $(a|a|a)$, $(a|a)$, $(bbbb)$, $(bbb|b)$, $(bb|bb)$, $(b|bbb)$, $(bb|b|b)$, $(b|bb|b)$, $(b|b|bb)$, $(b|b|b|b)$, (bbb), $(bb|b)$, $(b|bb)$, $(b|b|b)$ and $(b|b)$. Note that in this case $\tau_0(a)$ is not equal to a and $\tau_0(b)$ is not equal to b, therefore $(ccc|c)$, $(c|ccc)$ and $(c|c)$ are not homotopic each other for each $c \in \{a, b\}$. Therefore by Theorem 3.1 there are exactly 26 stable equivalence classes of pointed ordered irreducible curves on surfaces of monoliteral type with at most six crossings.

Moreover by the correspondence of curves and phrases, we obtain the list of curves on surfaces in Figures 7 and 8. □

Remark 7.3. In the paper [8], A. Gibson classified un-pointed oriented flat virtual virtual knots with at most four crossings using the theory of nanowords. See [8] for more details.

8. An application to free links.

In this subsection, we give the classification of ordered pointed free links with some conditions using the classification of nanophrases and monoliteral phrases.

The theory of free knots and links was introduced by V . O. Manturov in [15] and [16]. A *free link* is an equivalence class of flat virtual link diagrams modulo flat virtual Reidemeister moves and *flat virtualization move* which is illustrated in Figure 9. We can define *ordered, pointed, irreducible* and *monoliteral* for free links similarly as in the case for flat virtual links.

It is known that there is a canonical bijection between the set of ordered pointed k-component free links and the set of homotopy classes of nanophrases over the one element set $\{a\}$ with the involution $a \mapsto a$. See [9] and [15] for example.

Now we apply the classification of nanophrases and monoliteral phrases to the classification of ordered pointed irreducible free links.

Corollary 8.1. *There are exactly* 12 *irreducible ordered pointed free links with at most two real crossings.*

Proof. We put α is equal to $\{a\}$, and τ is equal to the identity map on $\{a\}$, then by the Theorem 4.5 we obtain the list of a complete representable system of homotopy classes of nanophrases which does not contain empty words as components of phrase as follows: $(ABA|B)$, $(A|BAB)$, $(AB|A|B)$, $(BA|A|B)$, $(A|AB|B)$, $(A|BA|B)$, $(A|B|AB)$, $(A|B|BA)$, $(A|B|A|B)$, $(A|B|B|A)$, $(A|A|B|B)$ and $(A|A)$ where $|A|$ and $|B|$ are equal to a. Therefore there are 12 irreducible ordered pointed free links with at most two real crossings. □

Corollary 8.2. *There are exactly nine irreducible ordered pointed free links of monoliteral type with at most six real crossings.*

Proof. We put α is equal to $\{a\}$, and τ is equal to the identity map on $\{a\}$, then by Theorem 4.6 we obtain the list of a complete representable system of homotopy classes of nanophrases which does not contain empty words as components of phrase as follows: $(aa|aa)$, $(aa|a|a)$, $(a|aa|a)$, $(a|a|aa)$, $(a|a|a|a)$, $(aa|a)$, $(a|aa)$, $(a|a|a)$ and $(a|a)$. Therefore there are nine irreducible ordered pointed free links of monoliteral type with at most six real crossings. □

Remark 8.1. We can construct the table of irreducible ordered pointed free links of monoliteral type with at most six crossings similarly as in the case of curves on surfaces. It is similar to the Figure 7. If we delete curves which correspond to $(aaaa)$, $(aaa|a)$, $(a|aaa)$ and (aaa), then we obtain the required table. Therefore we avoid drawing the table.

Remark 8.2. From Corollary 8.1, there are no pointed free knots with at most two crossings. Examples of non trivial (pointed) free knots were found by V. O. Manturov and A. Gibson independently. See [15] and [9] for more details. On the other hand, by Corollary 8.2, there are no pointed free knots with at most six crossings. More generally, in the paper [18] Turaev proved there is no pointed free knot of monoliteral type in terms of the theory of nanowords. See [18] for more details.

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