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LOCAL CLASSIFICATION OF CONSTRAINED HAMILTONIAN SYSTEMS ON 2-MANIFOLDS

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Abstract. We give local classification results for Constrained Hamiltonian Systems, i.e., for differential systems of the form $X \lrcorner \omega = df$, where ω is a singular 2-form and f is a function, both defined and smooth (analytic) on a 2-dimensional manifold M.

1. Introduction-Main Results

All the objects considered in this paper belong in any fixed category (that is smooth, real or complex analytic). For convenience we fix real objects in the smooth (C^∞)-category, unless otherwise stated. There exist many definitions of Constrained Hamiltonian Systems (CHS) most of them extrinsic, as restrictions of Hamiltonian systems on submanifolds of their symplectic phase spaces representing the constraints (c.f. [8], [9], [11], [14], [17], [18]). In our case the following intrinsic definition of CHS (without reference to an ambient symplectic manifold) is more convenient (see [14], [15] for the Hamiltonian case and [13], [16], [20] for the general, not necessarily Hamiltonian, case):

Definition 1.1. *A CHS on a manifold M is a differential system of the form:*

$$X \lrcorner \omega = df \tag{1.1}$$

defined by a pair $\gamma = (f, \omega) \in C^\infty(M) \times \wedge^2(M)$, consisting of a function f (or generally a closed 1-form α) and a closed 2-form ω.

If we view the 1-form df as a section of the cotangent bundle $T^*M \to M$, any such pair $\gamma = (f, \omega)$ determines the diagram:

$$\begin{array}{ccc} TM & \xrightarrow{\Omega := \lrcorner\omega} & T^*M \\ \big\downarrow \pi & & \big\uparrow df \\ M & \xrightarrow{Id} & M \end{array}$$

where Ω is the skew-symmetric vector bundle morphism[1] between the tangent and cotangent bundles over M (equivalently, a morphism between the modules of vector fields and 1-forms), defined by interior multiplication $\lrcorner$ with ω. A solution of the CHS is then a vector field $X : M \to TM$ that makes the above diagram commutative. In local coordinates the 1-form df is the gradient covector $\nabla f : x \mapsto \partial f / \partial x$ and if we denote by $t \mapsto x(t)$ the phase curve of X, then equation (1.1) above is written in coordinate form as:

$$\Omega(x)\dot{x} = \nabla f(x) \tag{1.2}$$

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[1] i.e., a vector bundle map, skew-symmetric linear in the fibers and inducing the identity on the base

where $\Omega(x) = (\omega_{ij}(x))_{1\le i,j\le n}$ is the smooth skew-symmetric matrix-valued map (a $n \times n$ skew-symmetric matrix with coefficients ω_{ij} smooth functions of x) associated to the 2-form ω in the fixed basis of $\mathbb{R}^n$.

Notice that smooth solutions X of the CHS (1.1) might not exist, even locally. The obstruction is the singular locus $\Sigma(\omega)$ of the 2-form ω, i.e., the locus of points where the rank of ω is smaller than the dimension of M. For this reason we call the set $\Sigma(\omega)$ impasse set[2] and any of its points, impasse point.

In this paper we give initial results for the classification problem of generic (typical) singularities of CHS $\gamma = (f, \omega)$ at impasse points $\Sigma(\omega)$. We restrict our study on 2-dimensional manifolds M and the first occuring singularities of 2-forms (Martinet singularities). The impasse set in this case is a smooth curve $\Sigma(\omega) \subset M$, also called the Martinet curve.

By equivalence of (germs of) CHS we mean equivalence of (germs of) pairs $\gamma = (f, \omega)$ and $\gamma' = (f', \omega')$ by (germs of) diffeomorphisms Φ (changes of coordinates preserving the point of application) acting on the space of (germs of) pairs $C^\infty(M) \times \wedge^2(M)$:

$$\Phi^*\gamma' := (\Phi^* f', \Phi^*\omega') = (f, \omega) =: \gamma.$$

The topology of the space of CHS is the usual Whitney C^∞-topology.

In the nondegenerate case where ω defines a symplectic structure, the CHS reduces to a Hamiltonian system with Hamiltonian f and the local classification problem reduces to the well known Hamiltonian normal forms (c.f. [1], [4]): there exist coordinates (Darboux coordinates) such that the germ of the Hamiltonian system $\gamma = (f, \omega)$ is equivalent to the normal form

$$\gamma = (x_1(\text{or} \;\; x_2), \quad dx_1 \wedge dx_2) \tag{1.3}$$

at the regular points of f and to

$$\gamma = (\lambda(f_2), \quad dx_1 \wedge dx_2) \tag{1.4}$$

at its nondegenerate critical points, where $f_2 = f_2(x_1, x_2)$ is a nondegenerate quadratic form (the first term in the Taylor expansion of f at the origin) and λ is a function of 1-variable such that $\lambda(0) = 0$, $\lambda'(0) \neq 0$. The proof of the normal form (1.3) follows easily from the Darboux theorem (c.f. [1]). Normal form (1.4) may be obtained also from the Morse-Darboux lemma (c.f. [6]). The result holds in the smooth and analytic (real or complex) categories. The germ λ in the normal form (1.4) is a functional modulus, characteristic for the pair $\gamma = (f, \omega)$ (c.f. [6] and references therein).

In the degenerate case where ω vanishes along the points of the smooth line $\Sigma(\omega)$, the singularities of functions are defined (for the 2-dimensional case) by the relative positions of the curve $f^{-1}(0)$ with the characteristic vector field X_ω of ω:

$$span\{X_\omega\}(x) = T_x\Sigma(\omega) \cap Ker_x(\omega) = T_x\Sigma(\omega),$$

(i.e., by the relative positions of $f^{-1}(0)$ with the Martinet curve $\Sigma(\omega)$). We fix germs at the origin $0 \in \Sigma(\omega)$. The following cases (singularity classes) occur typically:

(i) $f^{-1}(0)$ is smooth and transversal to the Martinet curve at the origin:

$$X_\omega(f)(0) \neq 0$$

(ii) the germ $f^{-1}(0)$ is smooth and tangent to the Martinet curve at the origin, with 1st-order (nondegenerate) tangency:

$$X_\omega(f)(0) = 0, \quad X^2_\omega(f)(0) \neq 0,$$

where $X^2_\omega(f) = X_\omega(X_\omega(f))$.

[2]see [16] and [20] for a survey of impasse singularities for the case of general constrained systems, defined by tangent bundle endomorphisms.

Martinet ([12]) has shown that a generic germ of a singular 2-form ω at a point on the curve $\Sigma(\omega)$ is equivalent to the normal form

$$\omega = x_1 dx_1 \wedge dx_2. \tag{1.5}$$

In these coordinates the germ of the Martinet curve is given by $\Sigma(\omega) = \{x_1 = 0\}$ and the characteristic vector field by $X_\omega = \partial/\partial x_2$.

Let now f be a function germ at a generic point $0 \in \Sigma(\omega)$, i.e., satisfying the transversality condition (i). The following theorem implies that it is possible to reduce f to a simple normal form by a diffeomorphism leaving the Martinet 2-form $\omega =$ (1.5) fixed.

Theorem 1.2. *All germs of CHS $\gamma = (f, \omega)$ at impasse points of type (i) are equivalent to the normal form*

$$\gamma = (\pm x_2, \quad x_1 dx_1 \wedge dx_2). \tag{1.6}$$

Moreover, the diffeomorphism bringing γ to its normal form is unique.

Remark 1.3. The theorem holds in both smooth and real analytic categories. The existence of the "$\pm$" sign is related to the canonical orientation of the Martinet curve $\Sigma(\omega) = \{x_1 = 0\}$ induced by the opposite orientations of the two symplectic half spaces $\Sigma_0^+ = \{x_1 > 0\}$, $\Sigma_0^- = \{x_1 < 0\}$, defined by the restriction of ω on each one of them. In particular there does not exist a germ of a diffeomorphism preserving the Martinet germ $\omega =$(1.5) and sending x_2 to $-x_2$. In the complex analytic category such an orientation is not defined and the theorem still holds true after we drop the "$\pm$" sign from the normal form (1.6); indeed, the diffeomorphism $(x_1, x_2) \mapsto (ix_1, -x_2)$ conjugates x_2 to $-x_2$ and leaves ω invariant.

Consider now a germ f at a point $0 \in \Sigma(\omega)$ of type (ii). The condition of 1st-order tangency of the pair of curves $(\Sigma(\omega), f^{-1}(0))$ implies that the restriction of f on $\Sigma(\omega)$ has a nodegenerate (Morse) critical point at the origin. Notice that any such singularity is reducible by a diffeomorphism preserving $\Sigma(\omega) = \{x_1 = 0\}$ to the classical normal form $f = x_1 \pm x_2^2$. The next theorem implies that it is impossible to achieve this normal form (or any normal form with a finite number of parameters) under the action of diffeomorphisms preserving also the Martinet 2-form $\omega =$(1.5).

Theorem 1.4. *Germs of CHS $\gamma = (f, \omega)$ at impasse points of type (ii) are not finitely determined. In particular, for any germ γ there exists a function germ λ of 1-variable and with vanishing 1-jet:*

$$\lambda(t) = \sum_{i \geq 2} \lambda_i t^i, \quad \lambda_2 \neq 0,$$

such that γ is equivalent to the invariant normal form

$$\gamma = (x_1 + \lambda(x_2), \quad x_1 dx_1 \wedge dx_2). \tag{1.7}$$

Moreover, the diffeomorphism bringing γ to its normal form is unique.

Remark 1.5. The theorem holds in both smooth and analytic (real or complex) categories. Invariance of the normal form (1.7) means that it cannot contain (as a singularity class) equivalent germs γ and γ' with different λ and λ'. In the analytic category this means that: *two germs f and f' will be equivalent by an analytic diffeomorphism preserving ω if and only if the corresponding series λ and λ' are exactly the same.* It is convenient to express λ invariantly, in terms of the pair $\gamma = (f, \omega)$ as:

$$\lambda(t) = \sum_{i \geq 2} X_\omega^i(f)(0) t^i, \quad X_\omega^2(f)(0) \neq 0.$$

It follows that for any $l \geq 2$ the system of coefficients $\{X_\omega^2(f)(0), ..., X_\omega^l(f)(0)\}$ is a complete invariant for the classification of l-jets of germs f under diffeomorphisms preserving ω. The existence of the modulus λ and in particular of its first order term $\lambda_2 = X_\omega^2(f)(0)$ admits a nice geometric description in terms of action integrals for $\omega = d\alpha$ (for some primitive 1-form α):

$$A(c) = \int_c \alpha,$$

along "half-cycles" c, i.e., along smooth curves with at least two points of intersection with $\Sigma(\omega)$ (in a sufficiently small neighborhood of the origin).

Theorems 1.2 and 1.4 along with the Hamiltonian normal forms (1.3) and (1.4) give a complete classification of generic singularities (of codimension ≤ 2) of pairs $\gamma = (f, \omega)$ on 2-manifolds. Germs (1.3) (of codimension 0) and germs (1.6) (of codimension 1) are both stable, $(1,0)$ and $(1,1)$-determined respectively[3]. The isolated singularities (1.4) and (1.7) (of codimension 2) are unstable and not finitely determined[4].

For the proofs of the theorems we use the homotopy method. We review some basic facts in Section 2 and we prove Theorems 1.2 and 1.4. In Section 3 we give the geometric description of the first term of the modulus λ. In the last Section 4 we discuss the weaker classification problem of germs of phase portraits (orbital equivalence) of CHS and we show how to get a list of simple normal forms, even for non-generic (degenerate) singularities.

2. The Homotopy Method-Proofs of Theorems

Fix $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$) and consider germs of pairs $\gamma = (f, \omega)$ at the origin $0 \in \Sigma(\omega)$ of the plane $(\mathbb{K}^2, 0)$, where $f : (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ vanishes at the origin. We will say that $\gamma_0 = (f_0, \omega_0)$ and $\gamma_1 = (f_1, \omega_1)$ are equivalent if there exists a germ of a diffeomorphism $\Phi : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ fixing the origin ($\Phi(0) = 0$) and such that $\Phi^*\gamma_1 = \gamma_0$. To find such equivalences we connect γ_0 and γ_1 by a curve

$$\gamma_t = (f_t, \omega_t), \quad t \in [0, 1],$$

where we may write:

$$f_t = f_0 + t\phi, \quad \omega_t = \omega_0 + td\alpha,$$

for the pair $(\phi, d\alpha) = (f_1 - f_0, \omega_1 - \omega_0)$. We seek for 1-parameter families of diffeomorphisms Φ_t depending smoothly (analytically) on $t \in [0, 1]$, fixing the origin ($\Phi_t(0) = 0$ for all $t \in [0, 1]$), inducing the identity on $(\mathbb{K}^2, 0)$ for $t = 0$ ($\Phi_0 = Id$) and such that $\Phi_t^*\gamma_t = \gamma_0$. Let v_t be the 1-parameter family of vector fields generated by any such diffeomorphism Φ_t:

$$\frac{d\Phi_t(x)}{dt} = v_t(\Phi_t(x)), \quad \Phi_0(x) = x.$$

This family depends smoothly (analytically, e.t.c.) on $t \in [0, 1]$ and it has a singular point at the origin $v_t(0) = 0$, for all $t \in [0, 1]$ (since the origin is a fixed point for Φ_t). Then the following proposition is well known:

Proposition 2.1. *If there exists a solution v_t, $v_t(0) = 0$ of the equations*

$$v_t \lrcorner df_t = -\phi, \tag{2.1}$$

$$v_t \lrcorner \omega_t = -\alpha + dh, \tag{2.2}$$

for some function germ $h = h(x_1, x_2)$ vanishing at the origin, then the pairs γ_0 and γ_1 are equivalent.

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[3] (i,j)-determinacy means that the pair (f, ω) is equivalent to its (i,j)-jet $(j^i f, j^j \omega)$.

[4] some authors call these singularity classes "wild" (c.f. [19]).

Proof. The time 1-map of the flow Φ_t generated by v_t is the desired diffeomorphism. □

In problems of classification of pairs it is convenient to fix a singularity of one object and classify the other object relative to the symmetries of the first. Here[5] we fix the singularity f and we normalise ω relative to the symmetries of f. This simplifies calculations due to the following simple:

Lemma 2.2. *Let f be a generic function germ at the origin $0 \in \Sigma(\omega)$ of the plane (of type (i) or (ii)). Then the pair $\gamma = (f, \omega)$ is equivalent to the preliminary normal form*

$$\gamma = (\pm x_2, \quad \phi(x_1, x_2) dx_1 \wedge dx_2) \tag{2.3}$$

in the (i) case and to

$$\gamma = (x_1 \pm x_2^2, \quad \phi(x_1, x_2) dx_1 \wedge dx_2) \tag{2.4}$$

in the (ii) case, where ϕ is a nonsingular function germ at the origin, vanishing along the Martinet curve: $\phi|_{x_1=0} = 0$.

Proof. The diffeomorphism $\Phi = (\Phi_1, \Phi_2)$ bringing f to normal form preserves $\Sigma(\omega) = \{x_1 = 0\}$ and sends the Martinet normal form (1.5) to $\phi(x_1, x_2) dx_1 \wedge dx_2$, where $\phi = \Phi_1 det \Phi_*$ vanishes on $x_1 = 0$ as desired. □

2.1. Case (i).

Proof of Theorem 1.2. We consider the "+"-sign case. The "-"-sign case follows the same lines. We fix the singularity $(\Sigma(\omega) = \{x_1 = 0\}, f = x_2)$ and we consider 1-parameter families of 2-forms $\omega_t = \omega + t da$, where $\omega_0 = (1.5)$ is the Martinet germ and $\omega_1 = da$ is a 2-form which can be chosen to vanish on $x_1 = 0$ by the previous lemma. Write $d\alpha = \phi_1(x_1, x_2) dx_1 \wedge dx_2$. Since $\omega_1 = d(\alpha + d\xi)$ for any function ξ, we may always choose the primitive α in the form

$$\alpha = \alpha_1(x_1, x_2) dx_2,$$

where α_1 is a function germ vanishing to second order on $x_1 = 0$, i.e., such that

$$\alpha_1|_{x_1=0} = \frac{\partial \alpha_1}{\partial x_1}|_{x_1=0} = \phi_1|_{x_1=0} = 0. \tag{2.5}$$

It follows that the 1-parameter family of 2-forms $\omega_t = \phi_t(x_1, x_2) dx_1 \wedge dx_2$ may be always chosen so that

$$\phi_t(x_1, x_2) = x_1(1 + t\phi_{11}(x_1, x_2)),$$

where ϕ_{11} is the smooth germ defined by division $\phi_1 = x_1 \phi_{11}$. Consider now a 1-parameter family $v_t = (v_{t,1}, v_{t,2})$ of germs of vector fields at the origin preserving the pair $(\{x_1 = 0\}, x_2)$. Since v_t preserves x_2 we have that $v_{t,2} = 0$. Since v_t is also tangent to $x_1 = 0$ we have that the first coordinate $v_{t,1}$ vanishes on $x_1 = 0$. After the substitution $v_t = (v_{t,1}, 0)$ in the homological equation (2.2) we arrive to the system:

$$\begin{array}{rcl} \phi_t v_{t,1} & = & \alpha_1 - \frac{\partial h}{\partial x_2} \\ 0 & = & \frac{\partial h}{\partial x_1}, \end{array}$$

where h is some arbitrary germ. We differentiate along the x_1-axes and we arrive to the simplest Cauchy problem for the unknown function $\psi = \phi_t v_{t,1}$:

$$\left\{ \begin{array}{l} \frac{\partial \psi}{\partial x_1} = \phi_1, \\ \psi|_{x_1=0} = 0. \end{array} \right. \tag{2.6}$$

[5]after the kind suggestion of the reviewer.

This admits a unique smooth (analytic) solution:

$$\psi(x_1,x_2)=\int_0^{x_1}\phi_1(s,x_2)ds.$$

By the fact that ϕ_t vanishes on $x_1=0$ for all $t\in[0,1]$ and that ψ vanishes on $x_1=0$ to second order (since $\phi_1|_{x_1=0}=0$), it follows that there exists a unique smooth (analytic) solution $v_{t,1}$ of the homological equation (2.2), vanishing on $x_1=0$ and represented by:

$$v_{t,1}(x_1,x_2)=\frac{\int_0^{x_1}\phi_1(s,x_2)ds}{\phi_t(x_1,x_2)}.$$

This finishes the proof. □

2.2. **Case (ii).** The proof of the previous theorem relies on the existence and uniqueness of solutions of the simplest Cauchy problem (2.6), i.e., on the fact that the origin is a non-characteristic point of the initial manifold $x_1=0$ for the characteristic vector field[6] $E_f=\pm\partial/\partial x_1$. In the case (ii) where $f^{-1}(0)$ has 1st-order tangency with $x_1=0$ at the origin (and so does the characteristic vector field $E_f=(\pm 2x_2,-1)$) existence and uniqueness of solutions of the analogous Cauchy problem is not guaranteed[7]. The following lemma gives the necessary and sufficient conditions for existence and uniqueness of smooth (resp. single-valued analytic) solutions.

Lemma 2.3. *Let $\mu=\mu(x_1,x_2)$ be an arbitrary smooth (resp. analytic) function germ vanishing at the origin. Then the Cauchy problem*

$$\left\{\begin{array}{c} 2x_2\frac{\partial\xi}{\partial x_1}-\frac{\partial\xi}{\partial x_2}=\mu,\\ \xi|_{x_1=0}=0,\end{array}\right. \tag{2.7}$$

admits a smooth (resp. analytic) solution ξ if and only if μ vanishes on $x_1=0$. Moreover, the solution is unique.

Proof. Notice first that the origin is an isolated characteristic point and thus if a solution ξ of the Cauchy problem exists, then it will be unique. We consider the associated Cauchy problem for ξ with initial conditions along the transversal $x_2=0$:

$$2x_2\frac{\partial\xi}{\partial x_1}-\frac{\partial\xi}{\partial x_2}=\mu,$$
$$\xi|_{x_2=0}=\xi_1(x_1),$$

where ξ_1 is a an arbitrary function germ vanishing at the origin. For this Cauchy problem the origin is a non-characteristic point and thus a unique smooth (resp. analytic) solution ξ exists for any function germ μ in the right-hand side. We seek necessary and sufficient conditions on μ such that ξ vanishes on $x_1=0$. If we parametrise the x_1-axis by τ, then the characteristic curves $t\mapsto(-t^2+x_1(0),-t+x_2(0))$ of the characteristic vector field E_f emanating from this axis define a map $F(t,\tau)=(x_1(t,\tau),x_2(t,\tau))$ given by

$$F(t,\tau)=(-t^2+\tau,-t).$$

This map is obviously a diffeomorphism germ at the origin with inverse

$$F^{-1}(x_1,x_2)=(-x_2,x_1+x_2^2).$$

In the (t,τ)-plane the solution is given by

$$\tilde{\xi}(t,\tau)=\int_0^t\tilde{\mu}(s,\tau)ds+\tilde{\xi}_1(0),$$

[6] i.e., the characteristic direction is transversal to the initial manifold at that point c.f. [7]

[7] see [10] for a solution of the problem in the complex analytic case, in terms of multivalued functions.

and it projects on the (x_1, x_2)-plane to the single valued (smooth, analytic) solution

$$\xi(x_1, x_2) = \tilde{\xi}(F^{-1}(x_1, x_2)).$$

The preimage of the curve $x_1 = 0$ on the (t, τ)-plane consists of the two branches $t = \pm\sqrt{\tau}$ of the parabola (in the real case $\tau \geq 0$). It follows that the solution ξ vanishes on $x_1 = 0$ if and only if

$$\tilde{\xi}(\pm\sqrt{\tau}, \tau) = \int_0^{\pm\sqrt{\tau}} \tilde{\mu}(s, \tau) ds = 0.$$

We view this expression symbolically as a function of $\epsilon = \pm\sqrt{\tau}$:

$$\zeta(\epsilon) = \int_0^{\epsilon} \tilde{\mu}(s, \epsilon^2) ds.$$

Since $\zeta(0) = 0$ we have that $\zeta(\epsilon) = 0$ if and only if $\zeta'(\epsilon) = \tilde{\mu}(\epsilon, \epsilon^2) = 0$, which in turn is equivalent to $\mu(0, x_2) = 0$. Thus we have determined a unique smooth solution ξ for the specific choice of the transversal. Now we show that the solution does not depend on the choice of the transversal, which means that the solution ξ is also unique. If we choose another transversal for initial manifold of the associated Cauchy problem, say $x_2 = g(x_1)$, with new initial condition $\xi|_{x_2=g(x_1)} = \xi_2(x_1)$, then, we arrive as above to a unique solution ξ' which will vanish on $x_1 = 0$ if and only if μ does. Thus we have specified two solutions ξ and ξ' of the same Cauchy problem (2.7). Write $\Xi = \xi' - \xi$ for their difference. Then Ξ satisfies the homogeneous Cauchy problem

$$2x_2 \frac{\partial \Xi}{\partial x_1} - \frac{\partial \Xi}{\partial x_2} = 0,$$

$$\Xi|_{x_1=0} = 0,$$

which obviously, does not admit any nonzero solutions. Thus $\Xi = \xi' - \xi = 0$ and the lemma is proved. $\square$

Proof of Theorem 1.4. Fix the pair $(\Sigma(\omega) = \{x_1 = 0\}, f = x_1 + x_2^2)$ (the "-"-sign case follows again the same lines) and consider 1-parameter families of vector fields $v_t = (v_{t,1}, v_{t,2})$ preserving this pair and having a singular point at the origin. Such a general family can be represented as $v_t = (-2x_2 v_{t,2}, v_{t,2})$, where $v_{t,2}$ is a function germ, vanishing again on $x_1 = 0$. The homological equation (2.2) reduces in that case to the system

$$\begin{array}{rcl} -2x_2\phi_t v_{t,2} & = & \alpha_1 - \frac{\partial h}{\partial x_2} \\ \phi_t v_{t,2} & = - & \frac{\partial h}{\partial x_1}, \end{array}$$

for some arbitrary germ h. Write $\psi = \phi_t v_{t,2}$. It has second order vanishing on $x_1 = 0$ (since both ϕ_t and the unknown $v_{t,2}$ must vanish on $x_1 = 0$.) Then by the integrability condition for h we have that the unknown function ψ must be a solution of the following Cauchy problem:

$$\left\{ \begin{array}{c} 2x_2 \frac{\partial \psi}{\partial x_1} - \frac{\partial \psi}{\partial x_2} = \phi_1, \\ \psi|_{x_1=0} = \frac{\partial \psi}{\partial x_1}|_{x_1=0} = 0, \end{array} \right. \tag{2.8}$$

Since ϕ_1 vanishes on $x_1 = 0$ we have from the previous lemma that there exists a unique solution ψ vanishing on $x_1 = 0$. Since we want our solution to vanish on $x_1 = 0$ to second order, we differentiate the equation along the x_1-axis and we put $\xi = \partial\psi/\partial x_1$. Then ξ must be a solution of the Cauchy problem (2.7) of the above lemma, where we put $\mu = \partial\phi_1/\partial x_1$ in the right hand-side. It follows that the Cauchy problem (2.8) admits a unique smooth solution ψ if and only if ϕ_1 vanishes to second order on $x_1 = 0$. If this is the case, then from the initial substitution $\psi = \phi_t v_{t,2}$, we have have determined a unique smooth (resp. analytic) solution $v_{t,2}$ of the homological equation (2.2), given by the formula

$$v_{t,2}(x_1, x_2) = \frac{\psi(x_1, x_2)}{\phi_t(x_1, x_2)}.$$

Since ψ vanishes to second order on $x_1 = 0$ we have that $v_{t,2}$ vanishes also on $x_1 = 0$ as required. In particular, any 1-parameter family of 2-forms of the form $\omega_t = \omega_0 + tda$ can be reduced by a unique diffeomorphism preserving the pair $(x_1 = 0, x_1 + x_2^2)$ to the normal form:

$$\Phi_t^* \omega_t|_{t=1} = x_1 \tilde{\lambda}(x_2) dx_1 \wedge dx_2, \tag{2.9}$$

where $\tilde{\lambda}$ is an arbitrary function germ of 1-variable, $\tilde{\lambda}(0) \neq 0$. The invariance of this normal form is implied also by the previous lemma; indeed if $\omega_t = x_1 \tilde{\lambda}_t(x_2) dx_1 \wedge dx_2$ is any 1-parameter family of 2-forms, then any 1-parameter family of diffeomorphisms preserving f and realising equivalences between them, whould generate a 1-parameter family of vector fields v_t which should satisfy the homological equation (2.2) and in particular the Cauchy problem (2.8) with a righ hand-side of the form $\phi_1 = x_1 \tilde{\phi}(x_2)$, where $\tilde{\phi}$ is an arbitrary germ. Non-existence of smooth (resp. single-valued analytic) solutions $\psi = \phi_t v_{t,2}$ of the latter Cauchy problem is guaranteed by the previous lemma. Thus the germs ω_0 and ω_1 will be equivalent if and only if $\lambda_0 = \lambda_1$.

To obtain the initial normal form (1.7) of the theorem we consider the diffeomorphism $(x_1, x_2) \mapsto (x_1, \int_0^{x_2} \tilde{\lambda}(s) ds)$ which sends $\omega =$(2.9) to the Martinet 2-form $\omega =$(1.5) and the germ $f = x_1 + x_2^2$ to:

$$f = x_1 + \lambda(x_2),$$

where

$$\lambda(x_2) = (\int_0^{x_2} \tilde{\lambda}(s) ds)^2. \tag{2.10}$$

Obviously $\lambda(0) = \lambda'(0) = 0$ and $\lambda''(0) = 2(\tilde{\lambda}(0))^2 \neq 0$. The theorem is proved. □

2.3. **Geometric Description of the First Modulus λ_2.** Fix the Martinet germ ω_0 and write $\Sigma_0(\omega) = \Sigma_0^+ \sqcup \Sigma_0^-$ for the germs of the symplectic half-spaces so that $\Sigma_0^+ = \{x_1 > 0\}$ and $\Sigma_0^- = \{x_1 < 0\}$. Let $c = c(t)$ be any "half-cycle", i.e., a curve lying in the neighborhood of the origin with two points of intersection with the Martinet curve, such as for example $c = f^{-1}(\epsilon) = \{x_1 + x_2^2 = \epsilon\}$ for some $\epsilon > 0$ fixed (say $\epsilon = 1$). Let $\bar{c}$ be the closed curve obtained by the union of c and the segment of the Martinet curve $x_1 = 0$ lying between the two intersection points. Write $D \subset \Sigma_0^+$ (or $D \subset \Sigma_0^-$) for the closed region whose boundary is $\bar{c}$ and $A_0(D)$ for the signed integral:

$$A_0(D) = \int_D \omega_0.$$

Since ω_0 vanishes on $x_1 = 0$, this integral will be equal to the action integral along the half-cycle c:

$$A_0(D) = \int_c \alpha_0,$$

where α_0 is a primitive of ω_0. We may choose $\alpha_0 = x_1^2 dx_2/2$ and for the specific choice of $c = f^{-1}(1)$ we compute the action to be $A_0(D) = 8/15$.

Consider now a 2-form germ ω_1 with the same Martinet curve $x_1 = 0$ and let Φ be a germ of diffeomorphism sending ω_0 to ω_1. Since Φ preserves $x_1 = 0$ then the following formula holds:

$$A_0(D) = \int_D \omega_0 = \int_D \Phi^* \omega_1 = \int_{\Phi(D)} \omega_1 = A_1(\Phi(D)) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \int_c \alpha_0 = \int_{\Phi(c)} \alpha_1.$$

It follows that if the diffeomorphism Φ may be chosen to preserve also the germ $f = x_1 + x_2^2$, then $\Phi(c) = c$ and thus the signed action integrals have to be equal:

$$A_0(D) = A_1(D).$$

For the 1-jet $\omega_1 = \tilde{\lambda}_0\omega_0$ in particular, we have $A_1(D) = \tilde{\lambda}_0 8/15$ and hence, ω_1 cannot be equivalent to ω_0 for any $\tilde{\lambda}_0 \neq 1$ (the orbits of ω_0 and ω_1 under the action of symmetries of $f = x_1 + x_2^2$, belong to different cohomology classes for any $\tilde{\lambda}_0 \neq 1$).

Remark 2.4. The same result can be obtained if we fix ω_0 and vary the half-cycles $c(\lambda_2) = \{x_1 + \lambda_2 x_2^2 = 1\}$, $\lambda_2 > 0$; it suffices to substitute $\tilde{\lambda_0} = \sqrt{\lambda_2/2}$ in the calculation of $A_0(c(\lambda_2))$.

3. Orbital Equivalence of CHS

We fix real objects in C^∞-category. The results of the previous section show that the classification problem of CHS $\gamma = (f, \omega)$ at impasse points $\Sigma(\omega)$ becomes wild for all singularities of codimension ≥ 1. Despite this fact, if we are interested in the configuration of phase curves in a neighborhood of an impasse point (orbital equivalence), then the classification problem admits simple normal forms (without moduli) even for arbitrary deep singularities. Notice that for any germ f at the origin, there is a well defined Hamiltonian vector field $X^\pm$ in any of the symplectic half-spaces $\Sigma_0^\pm$ with Hamiltonian $f^\pm = f|_{\Sigma_0^\pm}$ and symplectic form the restriction $\omega^\pm = \omega|_{\Sigma_0^\pm}$ of the Martinet 2-form ω on each one of them. If f is a generic function germ (or any germ whose differential does not vanish on $\Sigma(\omega) = \{x_1 = 0\}$) then there does not exist a smooth extension of $X^\pm$ along $\Sigma(\omega)$ to a smooth vector field $X = X_f$ such that

$$X_f \lrcorner \omega = df. \tag{3.1}$$

The singularities of this type are called impasse singularities in the literature of constrained systems (c.f. [16], [20] and references therein). The notion of orbital (phase) equivalence for constrained (not necessarilly Hamiltonian) systems has been also introduced in these references. For the Hamiltonian case we need the following modifications.

Definition 3.1. *Let $\gamma = (f, \omega)$ and $\gamma' = (f', \omega')$ be two germs of CHS at the origin of the plane. Then γ will be called orbitally (or phase) equivalent with γ' if there exists a germ of a diffeomorphism Φ fixing the origin, sending the impasse curve $\Sigma(\omega)$ of γ to the impasse curve $\Sigma(\omega')$ of γ' and the oriented phase curves of γ in Σ_0 to the oriented phase curves of γ' in Σ'_0*

Remark 3.2. The definition implies that orbital equivalence of CHS is exactly orbital equivalence of the Hamiltonian vector fields X and X' defined on the symplectic components Σ_0 and Σ'_0 respectively. Since the diffeomorphism Φ sends oriented phase curves of X to oriented phase curves of X' it sends the germ of the foliation by level curves $\{f = c\}$ to the foliation $\{f' = c'\}$. The diffeomorphism Φ is not required to preserve the symplectic structures $\omega^\pm$ and $\omega'^\pm$ of the components. In particular the following cases are possible:

(a)
$$\Phi(\Sigma_0^\pm(\omega)) = \Sigma_0^\pm(\omega')$$
and Φ sends oriented phase curves of $X^\pm$ to oriented phase curves of $X'^\pm$, or

(b)
$$\Phi(\Sigma_0^\pm(\omega)) = \Sigma_0^\mp(\omega')$$
and Φ sends oriented phase curves of $X^\pm$ to oriented phase curves of $X'^\mp$.

3.1. **The Extended Vector Field.** Despite the fact that the Hamiltonian vector fields $X^{\pm}$ do not extend to a smooth vector field X_f satisfying equation (3.1), there exist many smooth extensions E with the following property: oriented phase curves of E coincide with the oriented phase curves of X^+ in the symplectic half-space Σ_0^+ and with the oriented phase curves of $-X^-$ in Σ_0^-. Following [20]:

Definition 3.3. *Let $\omega = \sigma(x)v$ be any singular 2-form with smooth impasse curve $\sigma^{-1}(0)$, where v is an area form of the plane. The Extended Vector Field E_f of the CHS $\gamma = (f, \omega)$ is the smooth vector field defined by the equation:*

$$E_f \lrcorner \omega = \sigma df, \tag{3.2}$$

or equivalently, by the Hamiltonian system

$$E_f \lrcorner v = df, \tag{3.3}$$

The fact that E_f is indeed an extension of the CHS γ as defined above, follows from the relation (in Martinet coordinates)

$$X = \frac{1}{x_1} E_f, \quad x_1 \neq 0,$$

that is, multiplication by the positive (resp. negative) function x_1 at points of the half-space Σ_0^+ (resp. Σ_0^-).

Let now $\Gamma = (E_f, \Sigma)$ and $\Gamma' = (E_{f'}, \Sigma')$ be two pairs consisting of the germs at the origin of the extended vector fields and the impasse curves of γ and γ' respectively.

Definition 3.4. *The pairs Γ and Γ' will be called orbitally equivalent if there exists a germ of a diffeomorphism Φ fixing the origin, sending Σ to Σ' and sending oriented phase curves of E_f to oriented phase curves of $E_{f'}$, i.e., there exists a nonvanishing function germ Q at the origin such that:*

$$\Phi_* E_f = Q E_{f'}. \tag{3.4}$$

The following proposition allows us to reduce the problem of orbital equivalence of CHS γ to the orbital classification of the corresponding pairs Γ (as in [20]):

Proposition 3.5. *The germs of the CHS γ and γ' are phase equivalent iff the germs of the pairs Γ and Γ' are phase equivalent.*

Proof. Let γ and γ' be phase equivalent and suppose that the diffeomorphism Φ satisfies (a). Let $x(t)$ be an oriented phase curve of the extended vector field E_f in $\Sigma_0^+(\omega)$. Then it is also an oriented phase curve of X^+ and thus $\Phi(x(t)) \in \Sigma_0^+(\omega')$ is an oriented phase curve of X'^+ and thus of $E_{f'}$. Let now $x(t)$ be an oriented phase curve of E_f in $\Sigma_0^-(\omega)$. It is also a phase curve of $-X^-$ and thus $\Phi(x(t)) \subset \Sigma_0^-(\omega')$ is a phase curve of $-X'^-$. It follows that $\Phi(x(t))$ is an oriented phase curve of $E_{f'}$ which proves the required phase equivalence of the pairs Γ and Γ'. In the case where the diffeomorphism Φ satisfies (b), one obtains in a similar way a diffeomorphism of the oriented phase curves of E_f with those of $-E_{f'}$. The converse of the proposition is proved in a similar way. □

Write $\mathcal{G}(\Sigma)$ for the pseudogroup of symmetries of the impasse curve Σ, i.e., diffeomorphism germs preserving $\{x_1 = 0\}$ and fixing the origin. The orbital classification of pairs Γ is then equivalent to the problem of classification of germs of extended vector fields E_f relative to $\mathcal{G}(\Sigma)$-action. This problem in turn contains (for $Q = 1$ in equation (3.4)) the classification of the defining functions germs f relative to $\mathcal{G}(\Sigma)$-action. The answer to the latter problem is very well known and has been given by V. I. Arnold in [3]. For the 2-dimensional case the results may be summarised in the following list of simple singularities (see also [2], [5]):

$$\begin{array}{ll} \pm x_2 & \\ x_1 \pm x_2^{k+1}, & k \geq 1 \\ \pm x_1^k \pm x_2^2, & k \geq 2 \\ x_1 x_2 \pm x_2^k, & k \geq 2 \\ x_2^3 \pm x_1^2. & \end{array} \tag{3.5}$$

It follows:

Corollary 3.6. *Let $\gamma = (f, \omega)$ be a germ of a CHS at an impasse point where the germ of f is a simple boundary singularity (relative to $\mathcal{G}(\Sigma)$). Then γ is orbitally equivalent to the normal form*

$$\gamma = (f, \quad x_1 dx_1 \wedge dx_2),$$

where f is a germ from the list (3.5) above.

In the figures below the phase portraits for singularities for $k \leq 3$ are presented. To draw them, we draw the phase portrait of the extended vector field E_f and we change the orientation of the phase curves to one of the two half-spaces. The impasse curve is represented by the bolded vertical line. The dotted origin corresponds to the singular point of the f.

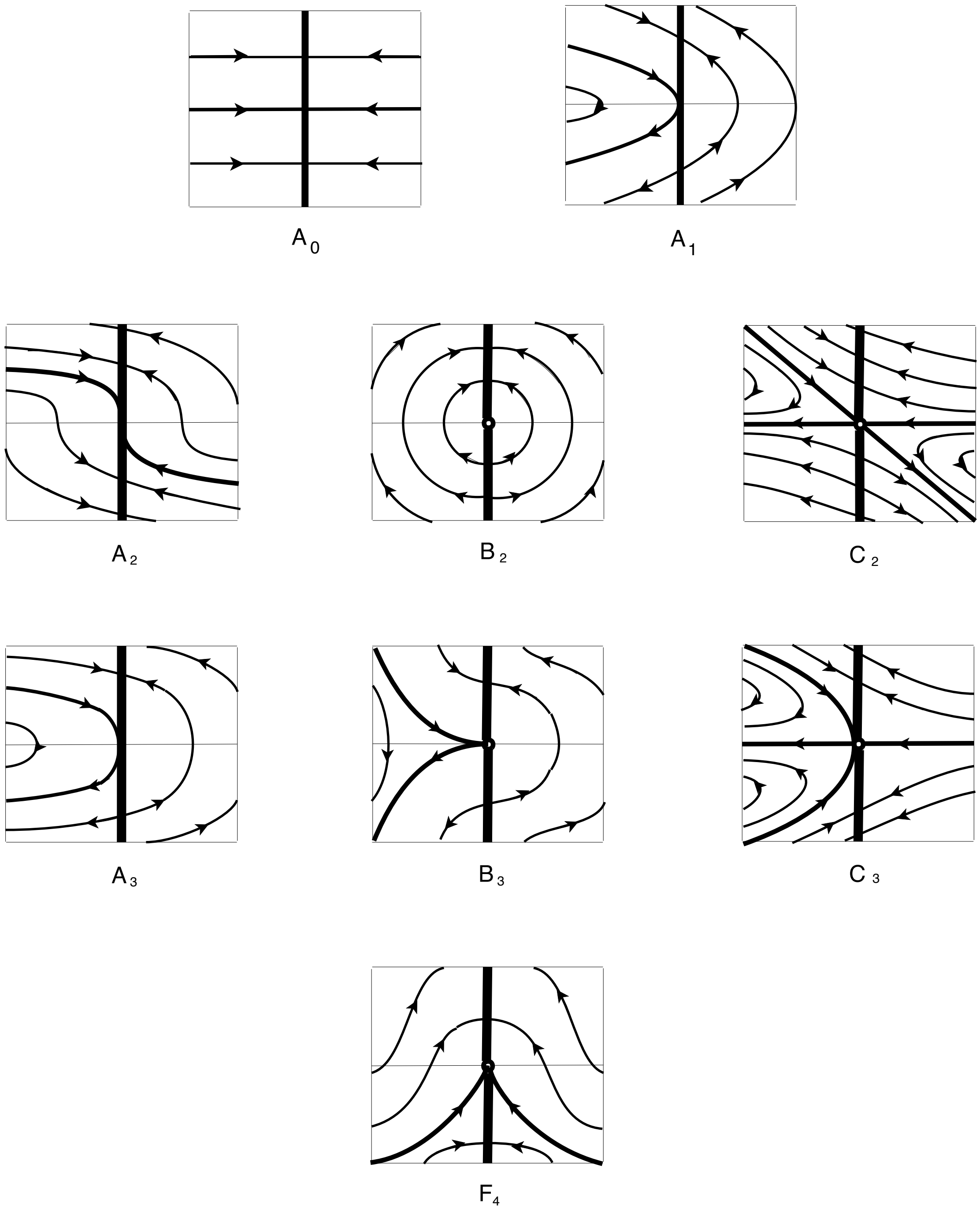

FIGURE 1. Phase portraits of simple singularities for $k \leq 3$ with the "-" sign.

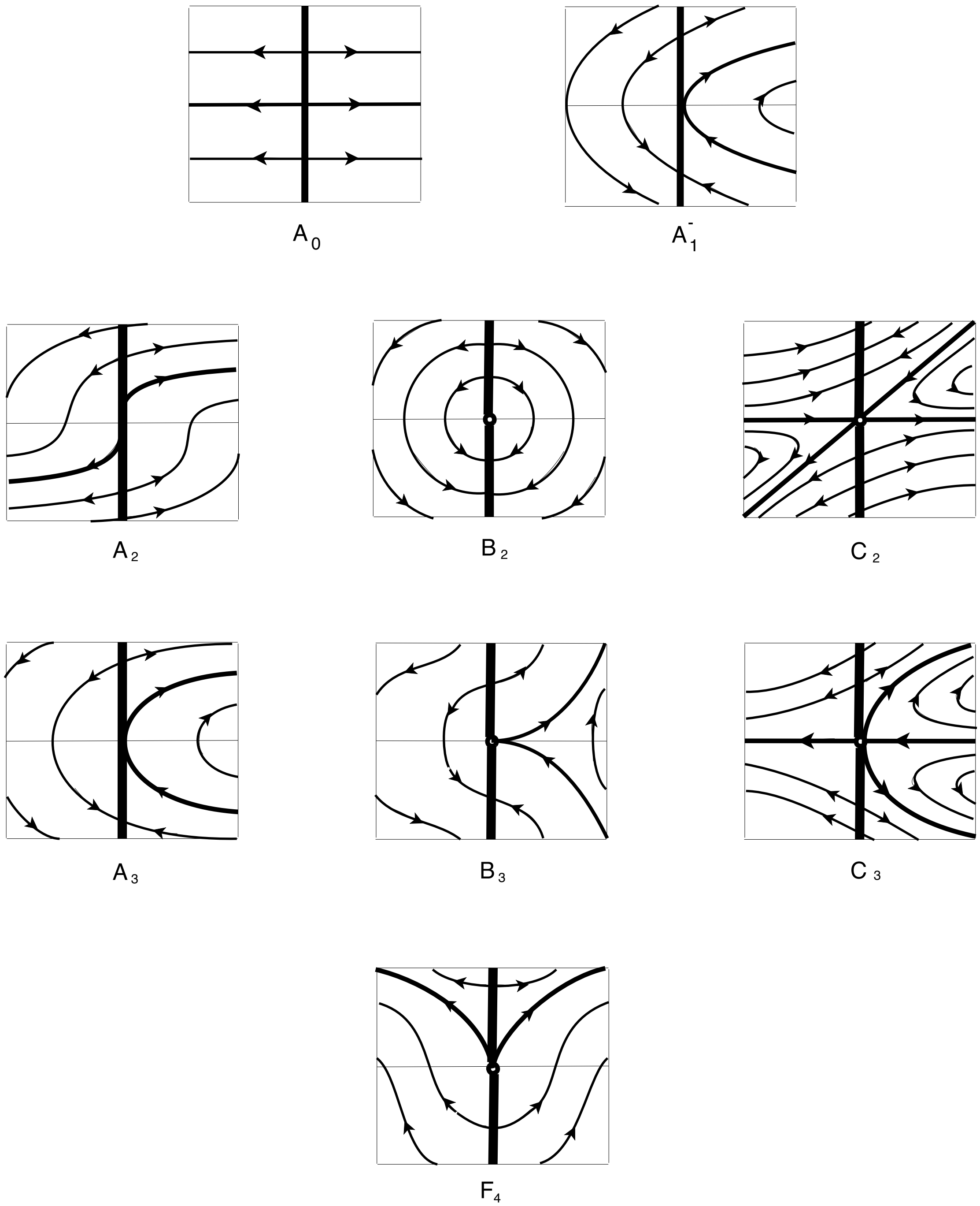

FIGURE 2. Phase portraits of simple singularities for $k \leq 3$ with the "+" sign.

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