Journal *of* Singularities
Volume 6 (2012), 112-123
DOI: 10.5427/jsing.2012.6i

Singularities in Geometry and Appl., Będlewo, 2011

TOPOLOGICAL TRIVIALITY OF FAMILIES OF MAP GERMS FROM $\mathbb{R}^2$ TO $\mathbb{R}^2$

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ABSTRACT. We show that a 1-parameter unfolding $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of a finitely determined map germ f is topologically trivial if it is excellent in the sense of Gaffney and the family of the discriminant curves $\Delta(f_t)$ is topologically trivial. We also give a formula to compute the number of cusps of 1-parameter unfoldings.

1. Introduction

In a previous paper [10], we consider the topological classification of finitely determined map germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, by means of the analysis of the associated link. The link is obtained by taking a small enough representative $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ and the restriction of f to $\tilde{S}^1_\epsilon = f^{-1}(S^1_\epsilon)$, where S^1_ϵ is a small enough sphere centered at the origin. It follows that the link is a stable map $\gamma : S^1 \to S^1$ which is well defined up to $\mathcal{A}$-equivalence and that f is topologically equivalent to the cone of its link. We also describe the topology of such links by using an adapted version of the Gauss word.

In this paper we consider a 1-parameter unfolding of f, that is, a map germ $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $F(x, t) = (f_t(x), t)$ and such that $f_0 = f$. We are interested in the topological triviality of F, which means that it is topologically equivalent as an unfolding to the constant unfolding. Our main result is that F is topologically trivial if it is excellent in the sense of Gaffney [4] and moreover, the family of the discriminant curves $\Delta(F)$ is a topologically trivial deformation of $\Delta(f)$. This can be seen as a real version of the same result obtained by Gaffney for complex analytic map germs [4, Theorem 9.9]. In fact, since $\Delta(f)$ is a plane curve, the topological triviality of F in the complex case is equivalent to the constancy of the Milnor number $\mu(\Delta(f_t))$. In the real case, we show that this is also a sufficient condition, although it is not necessary in general. In order to have a necessary and sufficient condition we should need an invariant which controls the topological triviality of a family of real plane curves. In the last section we consider unfoldings which are not topologically trivial and give a result about the number of cusps that appear in f_t.

The techniques used to prove this result have been already used by the second named author in [11], where he gets a sufficient condition for the topological triviality in the case $\mathbb{R}^2$ to $\mathbb{R}^3$. The topological triviality of plane-to-plane has been also studied by Fukuda in [3]. We also refer to the work of Ikegami and Saeki [6] for related results.

For simplicity, all map germs considered are real analytic except otherwise stated, although most of the results here are also valid for C^∞-map germs, if they are finitely determined. We adopt the notation and basic definitions that are usual in singularity theory (e.g., $\mathcal{A}$-equivalence, stability, finite determinacy, etc.), as the reader can find in Wall's survey paper [12].

2000 *Mathematics Subject Classification.* Primary 58K15; Secondary 58K40, 58K60.

Key words and phrases. stable map, link, topological triviality.

Work partially supported by DGICYT Grant MTM2009-08933. The first author is partially supported by MEC-FPU Grant no. AP2009–2646.

2. The link of a finitely determined map germ

Two smooth map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi, \psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $g = \psi \circ f \circ \phi^{-1}$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then we say that f, g are topologically equivalent.

We say that $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is k-determined if for any map germ g with the same k-jet, we have that g is $\mathcal{A}$-equivalent to f. We say that f is finitely determined if it is k-determined for some k.

Let $f : U \to V$ be a smooth proper map, where $U, V \subset \mathbb{R}^2$ are open subsets. We denote by $S(f) = \{p \in U : Jf_p = 0\}$ the singular set of f, where Jf is the Jacobian determinant. It is a consequence of the Whitney's work [13] that f is *stable* if and only if the following two conditions hold:

1. 0 is a regular value of Jf, so that $S(f)$ is a smooth curve in U.
2. The restriction $f|_{S(f)} : S(f) \to V$ is an immersion with only transverse double points, except at isolated points, where it has simple cusps.

We denote $\Delta(f) = f(S(f))$ and we define $X(f)$ as the closure of $f^{-1}(\Delta(f)) \setminus S(f)$. If f is stable, then $S(f)$ is a smooth plane curve and $\Delta(f), X(f)$ are plane curves whose only singularities are simple cusps or transverse double points. In figure 1 we present the stable map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x, xy + y^4 - y^2/2)$, which has two cusps and one transverse double fold.

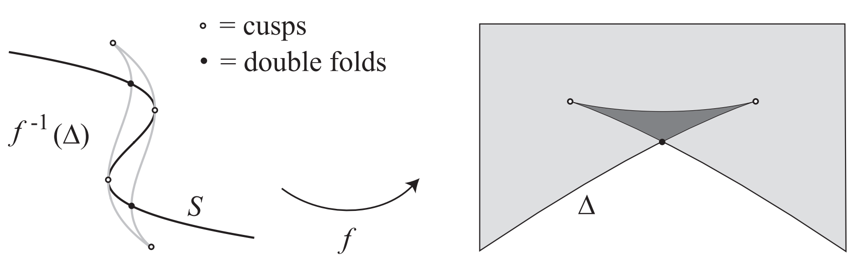

Figure 1.

Given a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, if it is real analytic, we can consider its complexification $\hat{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. It is well known that $\hat{f}$ is also finitely determined as a complex analytic map germ. Then, by the Mather-Gaffney geometric criterion [12], it has an isolated instability. In other words, we can find a small enough representative $\hat{f} : U \to V$, where U, V are open sets, such that

1. $\hat{f}^{-1}(0) = \{0\}$,
2. the restriction $\hat{f}|_{U \setminus \{0\}}$ is stable.

From the condition (2), both the cusps and the double folds are isolated points in $U \setminus \{0\}$. By the curve selection lemma [9], we deduce that they are also isolated in U. Thus, we can shrink the neighbourhood U if necessary and get a representative such that $\hat{f}|_{U \setminus \{0\}}$ is stable with only simple folds. Coming back to the real map f, we have the following immediate consequence.

Corollary 2.1. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. Then there is a representative $f : U \to V$, where $U, V \subset \mathbb{R}^2$ are open sets, such that*

1. $f^{-1}(0) = \{0\}$,
2. $f : U \to V$ *is proper,*
3. *the restriction $f|_{U \setminus \{0\}}$ is stable with only simple folds.*

Definition 2.2. We say that $f : U \to V$ is a *good representative* for a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, if the conditions (1), (2) and (3) of corollary 2.1 hold.

If f is finitely determined, then the three set germs $S(f)$, $\Delta(f)$ and $X(f)$ are plane curves with an isolated singularity at the origin and they will play an important role in the topological classification of f. In the complex case, given a plane curve $(X, 0)$ with reduced equation $h(u, v) = 0$ in $(\mathbb{C}^2, 0)$, its Milnor number is the colength of the ideal generated by the partial derivatives h_u, h_v, that is,

$$\mu(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle h_u, h_v \rangle}.$$

Definition 2.3. If $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a finitely determined map germ, we denote by $\mu(\Delta(f))$ the Milnor number of the discriminant $\Delta(\hat{f})$ of the complexification $\hat{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$.

Example 2.4. Let us consider $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ given by $f(x, y) = (x, x^2y + y^3/3)$. We have that $S(f)$ has defining equation $x^2 + y^2 = 0$ and hence, $\Delta(f)$ is given by $4u^6 + 9v^2 = 0$. Although $\Delta(f) = \{0\}$ as set germs, we have that $\mu(\Delta(f)) = 5$, which is the Milnor number of the complex curve given by this equation.

We finish this section with an important result due to Fukuda [1], which tell us that any finitely determined map germ, $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, with $n \leq p$, has a conic structure over its link. In order to simplify the notation, we only state the result in our case $n = p = 2$.

Given $\epsilon > 0$, we denote:

$$S^1_\epsilon = \{x \in \mathbb{R}^2 : \|x\|^2 = \epsilon\}, \quad D^2_\epsilon = \{x \in \mathbb{R}^2 : \|x\|^2 \leq \epsilon\}.$$

and given a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ we consider a representative $f : U \to V$ and put:

$$\widetilde{S}^1_\epsilon = f^{-1}(S^1_\epsilon), \quad \widetilde{D}^2_\epsilon = f^{-1}(D^2_\epsilon).$$

Theorem 2.5. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. Then, up to $\mathcal{A}$-equivalence, there is a representative $f : U \to V$ and $\epsilon_0 > 0$, such that, for any ϵ with $0 < \epsilon \leq \epsilon_0$ we have:*

(1) *$\widetilde{S}^1_\epsilon$ is diffeomorphic to S^1.*
(2) *The map $f|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon$ is stable, in other words, it is a Morse function all of whose critical values are distinct.*
(3) *$f|_{\widetilde{D}^2_\epsilon}$ is topologically equivalent to the cone of $f|_{\widetilde{S}^1_\epsilon}$.*

Definition 2.6. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. We say that the stable map $f|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon$ is the *link* of f, where f is a representative such that (1), (2) and (3) of theorem 2.5 hold for any ϵ with $0 < \epsilon \leq \epsilon_0$. This link is well defined, up to $\mathcal{A}$-equivalence. We also say that ϵ_0 is a *Milnor-Fukuda radius* for f.

Since any finitely determined map germ is topologically equivalent to the cone of its link, we have the following immediate consequence.

Corollary 2.7. *Two finitely determined map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ are topologically equivalent if and only if their associated links are topologically equivalent.*

Remark 2.8. If we consider a multigerm $f : (\mathbb{R}^2, S) \to (\mathbb{R}^2, 0)$, with $S = \{x_1, \ldots, x_r\}$, the construction of the link can be done in an analogous way. By reviewing carefully Fukuda's arguments, we see that the only difference is the condition (1) of theorem 2.5: now $\widetilde{S}^1_\epsilon$ is not diffeomorphic to S^1 anymore, but it is diffeomorphic to a disjoint union of r copies $S^1 \sqcup \ldots \sqcup S^1$. However, the other conditions (2) and (3) are still valid in this case.

3. Gauss words

In this section we recall briefly (for more information and examples see [10]) how we define an adapted version of the Gauss word in our particular case of study and some consequences of such definition.

Definition 3.1. Let $\gamma : S^1 \to S^1$ be a stable map, that is, such that all its singularities are of Morse type and its critical values are distinct. We fix orientations in each S^1 and we also choose base points $z_0 \in S^1$ in the source and $a_0 \in S^1$ in the target.

Suppose that γ has r critical values labeled by r letters $a_1, \dots, a_r \in S^1$ and let us denote their inverse images by $z_1, \dots, z_k \in S^1$. We assume they are ordered such that $a_0 \le a_1 < \dots < a_r$ and $z_0 \le z_1 < \dots < z_k$ and following the orientation of each S^1.

We define a map $\sigma : \{1, \dots, k\} \to \{a_1, \dots, a_r, \overline{a}_1, \dots, \overline{a}_r\}$ in the following way: given $i \in \{1, \dots, k\}$, then $\gamma(z_i) = a_j$ for some $j \in \{1, \dots, r\}$; we define $\sigma(i) = a_j$, if z_i is a regular point and $\sigma(i) = \overline{a}_j$, if z_i is a singular point (i.e., the bar $\overline{a}_j$ is used to distinguish whether the inverse image of the critical value is regular or singular). We call *Gauss word* to the sequence $\sigma(1) \dots \sigma(k)$.

For instance, the link of the cusp $f(x, y) = (x, xy+y^3)$ has two critical values with four inverse images and the associated Gauss word is $a\bar{b}\overline{a}b$ (see figure 2).

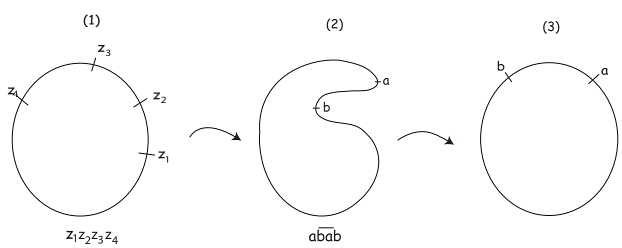

Figure 2.

It is obvious that the Gauss word is not uniquely determined, since it depends on the chosen orientations and base points in each S^1. Different choices will produce the following changes in the Gauss word:

(1) a cyclic permutation in the letters $a_1, \dots, a_r$;
(2) a cyclic permutation in the sequence $\sigma(1) \dots \sigma(k)$;
(3) a reversion in the set of the letters $a_1, \dots, a_r$;
(4) a reversion in the sequence $\sigma(1) \dots \sigma(k)$.

We say that two Gauss words are equivalent if they are related through these four operations. Under this equivalence, the Gauss word is now well defined.

In order to simplify the notation, given a stable map $\gamma : S^1 \to S^1$, we denote by $w(\gamma)$ the associated Gauss word and by $\simeq$ the equivalence relation between Gauss words. We also denote by $\deg(\gamma)$ the topological degree. Then, we can state the main result of this section (see [10]).

Theorem 3.2. *Let $\gamma, \delta : S^1 \to S^1$ be two stable maps. Then γ, δ are topologically equivalent if and only if*

$$\begin{cases} w(\gamma) \simeq w(\delta), & \text{if } \gamma, \delta \text{ are singular,} \\ \\ |\deg(\gamma)| = |\deg(\delta)|, & \text{if } \gamma, \delta \text{ are regular.} \end{cases}$$

Remark 3.3. By following step by step the proof of this theorem in [10] we can observe the following fact: if $\gamma, \delta : S^1 \to S^1$ are stable maps with $w(\gamma) \simeq w(\delta)$ and if we fix any homeomorphism in the target $\psi : S^1 \to S^1$ such that $\psi(\Delta(\gamma)) = \Delta(\delta)$, then there is a unique homeomorphism in the source $\phi : S^1 \to S^1$ such that $\psi \circ \gamma \circ \phi^{-1} = \delta$.

By combining this observation with corollary 2.7 we have an analogous result for map germs: let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs that are topologically equivalent. If we fix any homeomorphism in the target $\psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\psi(\Delta(f)) = \Delta(g)$, then there is a unique homeomorphism in the source $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\psi \circ f \circ \phi^{-1} = g$.

4. Cobordism of links

We recall that a cobordism between two smooth manifolds M_0, M_1 is a smooth manifold with boundary W such that $\partial W = M_0 \sqcup M_1$. Analogously, a cobordism between smooth maps $f_0 : M_0 \to N_0$ and $f_1 : M_1 \to N_1$ is another smooth map $F : W \to Q$ such that W, Q are cobordisms between M_0, M_1 and N_0, N_1 respectively, and for each $i = 0, 1$, $F^{-1}(N_i) = M_i$ and the restriction $F|_{M_i} : M_i \to N_i$ is equal to f_i. In the case that f_0, f_1 belong to some special class of maps (for instance, immersions, embeddings, stable maps, etc.), then we also require that the cobordism F belongs to the same class.

Definition 4.1. Given two stable maps $\gamma_0, \gamma_1 : S^1 \to S^1$, a cobordism between γ_0 and γ_1 is a stable map $\Gamma : S^1 \times I \to S^1 \times I$, where $I = [0, 1]$ and such that for $i = 0, 1$,

$$\Gamma^{-1}(S^1 \times \{i\}) = S^1 \times \{i\}, \quad \Gamma|_{S^1 \times \{i\}} = \gamma_i \times \{i\}.$$

The first condition implies that $\Gamma(S^1 \times \{0\}) \subset S^1 \times \{0\}$, $\Gamma(S^1 \times \{1\}) \subset S^1 \times \{1\}$ and $\Gamma(S^1 \times (0, 1)) \subset S^1 \times (0, 1)$, but in general, Γ is not level preserving (see figure 3).

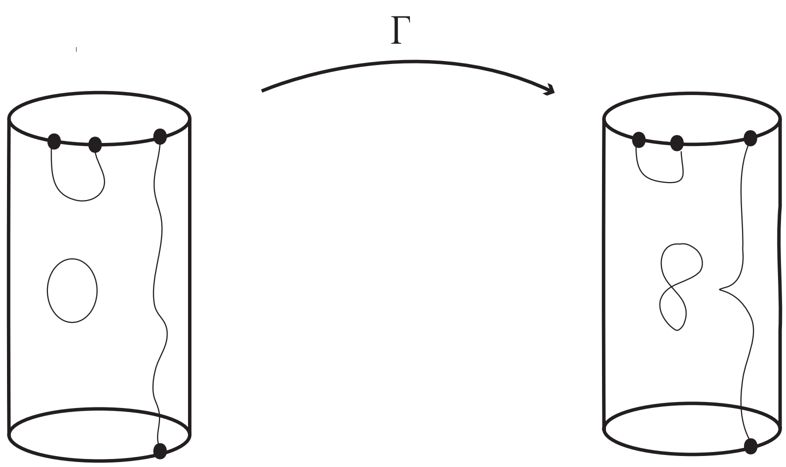

Figure 3.

Lemma 4.2. *Let Γ be a cobordism between γ_0, γ_1. If $\Delta(\Gamma)$ is diffeomorphic to $\Delta(\gamma_0) \times I$, then γ_0, γ_1 are topologically equivalent.*

Proof. Since $\Delta(\Gamma)$ is diffeomorphic to $\Delta(\gamma_0) \times I$, Γ cannot have cusps or double folds. Thus, Γ restricted to $\Gamma^{-1}(\Delta(\Gamma))$ is a local diffeomorphism and it follows that $\Gamma^{-1}(\Delta(\Gamma))$ is also diffeomorphic to $\gamma_0^{-1}(\Delta(\gamma_0)) \times I$.

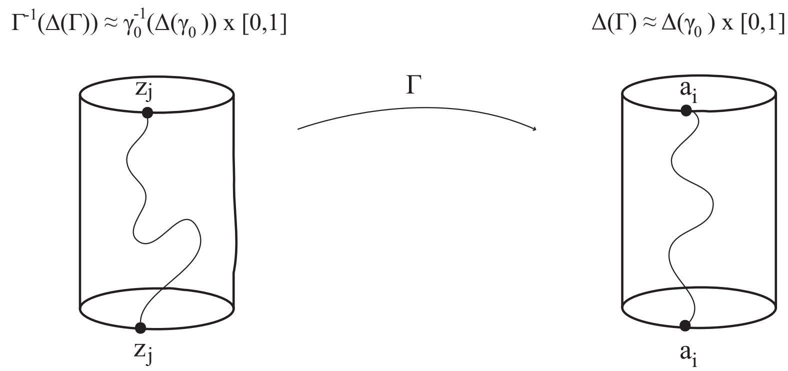

FIGURE 4.

In particular, for each critical value or each inverse image of γ_0 there is a unique arc joining the point in $S^1 \times \{0\}$ with a point in $S^1 \times \{1\}$ corresponding to a critical value or an inverse image of γ_1 respectively. We choose the orientations and the base points of γ_0, γ_1 in such a way that if two critical values are joined by an arc, then they share the same label a_i and if two inverse images are joined by an arc, then they share the same label z_j (see figure 4).

With these choices, it follows that $w(\gamma_0) = w(\gamma_1)$ and hence γ_0 and γ_1 are topologically equivalent by theorem 3.2. □

Remark 4.3. If Γ is a cobordism between γ_0, γ_1 such that $\Delta(\Gamma)$ is diffeomorphic to $\Delta(\gamma_0) \times I$, then it can be shown that Γ is trivial, that is, Γ is $\mathcal{A}$-equivalent to the product cobordism $\gamma_0 \times \mathrm{id} : S^1 \times I \to S^1 \times I$ by diffeomorphisms $\Phi, \Psi : S^1 \times I \to S^1 \times I$ such that $\Phi|_{S^1 \times \{0\}}, \Psi|_{S^1 \times \{0\}} = \mathrm{id}$.

To show this, we first choose a diffeomorphism $\psi : \Delta(\gamma_0) \times I \to \Delta(\Gamma)$ such that $\psi(p, 0) = (p, 0)$, for all $p \in \Delta(\gamma_0)$. We denote by $\phi : \gamma_0^{-1}(\Delta(\gamma_0)) \times I \to \Gamma^{-1}(\Delta(\Gamma))$ the induced diffeomorphism by Γ in such a way that $\phi(s, 0) = (s, 0)$, for all $s \in \gamma_0^{-1}(\Delta(\gamma_0))$ and the following diagram is commutative:

$$\begin{array}{ccc} \Gamma^{-1}(\Delta(\Gamma)) & \xrightarrow{\Gamma} & \Delta(\Gamma) \\ \uparrow \phi & & \uparrow \psi \\ \gamma_0^{-1}(\Delta(\gamma_0)) \times I & \xrightarrow{\gamma_0 \times id} & \Delta(\gamma_0) \times I \end{array}$$

We extend the diffeomorphisms ϕ, ψ to $S^1 \times I$. This can be done by using standard arguments of extensions of vector fields. Details are left to the reader.

5. EXTENDING THE CONE STRUCTURE

Let $f : U \to V$ be a good representative of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. Since $\Delta(f)$ is a 1-dimensional analytic subset, we can also shrink the neighborhoods U, V so that this set is contractible. In this case $\Delta(f) \setminus \{0\}$ has a finite number of connected components, each one of them is an edge joining the origin with the boundary of V. We orient each one of this edges from 0 to ∂V. We denote by $X : \Delta(f) \setminus \{0\} \to \mathbb{R}^2$ the unit normal vector field of $\Delta(f) \setminus \{0\}$ with respect to this orientation (see figure 5).

Definition 5.1. Let $f : U \to V$ be a good representative of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\Delta(f)$ is contractible. We say that $\epsilon > 0$ is a *convenient* radius for f if the following conditions hold:

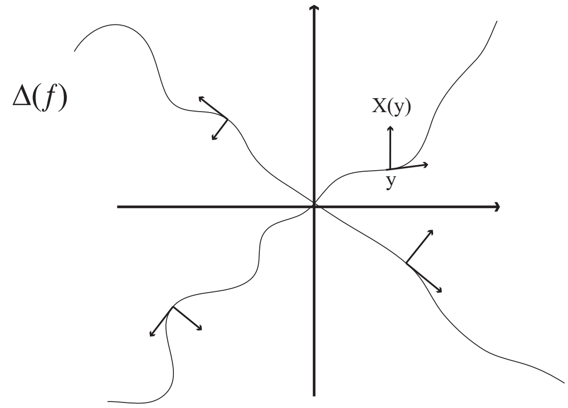

FIGURE 5.

(1) S^1_ϵ is transverse to $\Delta(f)$,
(2) $\widetilde{S}^1_\epsilon$ is diffeomorphic to S^1,
(3) S^1_ϵ intersects $\Delta(f)$ *properly*, that is, S^1_ϵ intersects each connected component of $\Delta(f)\backslash\{0\}$ at exactly one point.

It is easy to see that S^1_ϵ intersects $\Delta(f)$ properly if and only if S^1_ϵ cuts each point of $\Delta(f)$ following the orientation of the outward-pointing normal of S^1_ϵ. In other words, S^1_ϵ cuts $\Delta(f)$ properly if and only if

$$\det(X(y), y) > 0, \quad \forall y \in S^1_\epsilon \cap \Delta(f).$$

If ϵ_0 is a Milnor-Fukuda radius, then S^1_ϵ intersects $\Delta(f)$ properly for any $0 < \epsilon \leq \epsilon_0$, but in general, this may not be true.

Theorem 5.2. *Let $f : U \to V$ be a good representative of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\Delta(f) \subset V$ is contractible and let $\epsilon > 0$ be a convenient radius for f. Then,*

(1) *$f|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon$ is topologically equivalent to the link of f.*
(2) *$f|_{\widetilde{D}^2_\epsilon} : \widetilde{D}^2_\epsilon \to D^2_\epsilon$ is topologically equivalent to the cone of $f|_{\widetilde{S}^1_\epsilon}$.*

Proof. Let $\epsilon_0 > 0$ be a Milnor-Fukuda radius for f. If $\epsilon \leq \epsilon_0$, then the result follows from theorem 2.5. We assume $\epsilon > \epsilon_0$ and take $0 < \delta < \epsilon_0$. We consider the two associated links $\gamma_0 = f|_{\widetilde{S}^1_\delta}$ and $\gamma_1 = f|_{\widetilde{S}^1_\epsilon}$ and we denote by

$$C^2_{\delta,\epsilon} = \{y \in \mathbb{R}^2 : \delta \leq \|y\|^2 \leq \epsilon\}, \quad \widetilde{C}^2_{\delta,\epsilon} = f^{-1}(C^2_{\delta,\epsilon}),$$

and $\Gamma = f|_{\widetilde{C}^2_{\delta,\epsilon}} : \widetilde{C}^2_{\delta,\epsilon} \to C^2_{\delta,\epsilon}$, which defines a cobordism between γ_0 and γ_1. We only need to show that γ_0 and γ_1 are topologically equivalent, since in this case we have that the cone structure of $f|_{\widetilde{D}^2_\delta}$ can be extended to $f|_{\widetilde{D}^2_\epsilon}$.

Let $\Delta_1, \dots, \Delta_r$ be the connected components of $\Delta(f) \setminus \{0\}$. Since $\Delta(f) \subset V$ is closed, contractible and regular outside the origin, we have that each Δ_i is diffeomorphic to an open interval, whose end points are the origin and another point of ∂V. Now, both S^1_δ and S^1_ϵ intersect $\Delta(f)$ properly, so that $S^1_\delta \cap \Delta_i = \{x_i\}$ and $S^1_\epsilon \cap \Delta_i = \{x'_i\}$ for each $i = 1, \dots, r$. It follows that

$$\Delta(\Gamma) = \overline{x_1 x'_1} \cup \cdots \cup \overline{x_r x'_r},$$

where $\overline{x_i x_i'}$ is the closed interval in Δ_i joining the points x_i and x_i' (see figure 6). Therefore, $\Delta(\Gamma)$ is diffeomorphic to $\{x_1, \ldots, x_r\} \times [\delta, \epsilon]$ and γ_0 and γ_1 are topologically equivalent by lemma 4.2. □

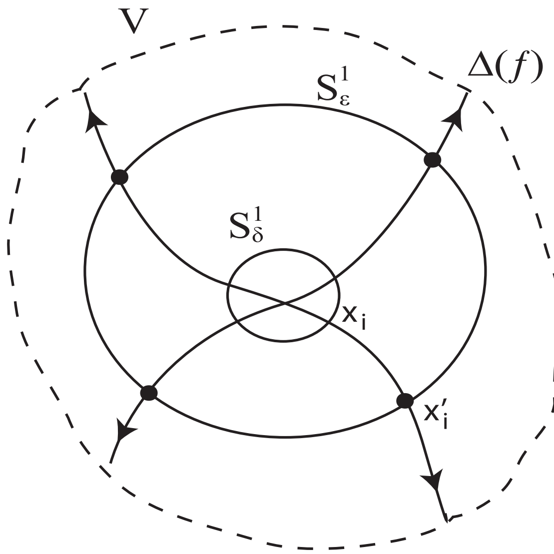

FIGURE 6.

6. TOPOLOGICAL TRIVIALITY OF FAMILIES

Given a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, a 1- parameter unfolding is a map germ $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $F(x,t) = (f_t(x), t)$ and such that $f_0 = f$. Here, we consider that the unfolding is origin preserving, that is, $f_t(0) = 0$ for any t. Hence, we have a 1-parameter family of map germs $f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

Definition 6.1. Let F be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

(1) We say that F is *excellent* if there is a representative $F : U \to V \times I$, where U, V, I are open neighborhoods of the origin in $\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}$ respectively, such that for any $t \in I$, $f_t : U_t \to V$ is a good representative in the sense of definition 2.2.
(2) We say that F has *constant topological type* if for any $t \neq t'$, the map germs f_t and f_t' are topologically equivalent.
(3) We say that F is *topologically trivial* if there are homeomorphism germs $\Psi, \Phi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ such that they are unfoldings of the identity and $F = \Psi \circ (f \times \mathrm{id}) \circ \Phi$.
(4) We say that F is *μ-constant* if the Milnor number $\mu(\Delta(f_t))$ is independent of t.

Example 6.2. Any topologically trivial unfolding F has constant topological type, but the converse is not true in general. Let us consider $h_t : \mathbb{R}^2 \to \mathbb{R}$ the equation of $S(f_t)$ for each t, given by

$$h_t(x, y) = (x + 3y)(5x - 2y)s_t(x, y),$$

with $s_t(x,y) = ((x-2)^2 + (y-3)^2)t - \epsilon^2 t$ (see figure 7). Then, we set:

$$f_t(x,y) = (x, \int h_t(x,y)dy).$$

It is not difficult to check that the Gauss word is constant $w(f_t) = a\overline{b}\overline{a}bc\overline{d}\overline{c}d$. As a consequence, the map germs f_t and $f_{t'}$ are topologically equivalent for any $t \neq t'$. However, it is clear that our family is not topologically trivial.

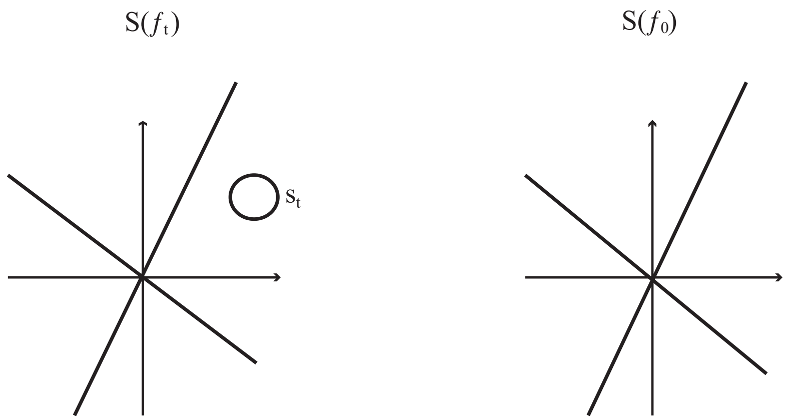

FIGURE 7.

Theorem 6.3. *Let F be an excellent unfolding of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. If $\Delta(F)$ is topologically trivial, then F is topologically trivial.*

Proof. Let $F : U \to V \times I$ be a representative of the unfolding F, where U, V, I are open neighborhoods of the origin in $\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}$ respectively, and such that $f_t : U_t \to V$ is a good representative of the map germ f_t, for any $t \in I$. We can shrink the neighborhoods if necessary and assume that $\Delta(f_0) \subset V$ is contractible.

On the other hand, since $\Delta(F)$ is topologically trivial, by shrinking again the neighbourhoods if necessary, there is a homeomorphism $\Psi : V \times I \to V \times I$ of the form $\Psi = (\psi_t, t)$ such that $\psi_0 = \text{id}$ and $\psi_t(\Delta(f_t)) = \Delta(f)$, for any $t \in I$. In particular, $\Delta(f_t)$ is homeomorphic to $\Delta(f_0)$ and it is also contractible.

We take $X : (V \setminus \{0\}) \times I \to \mathbb{R}^2$ such that $X_t(y) = X(y,t)$ is the unit normal vector at each point $y \in \Delta(f_t) \setminus \{0\}$ as in definition 5.1. We also denote by $g_t : U_t \to \mathbb{R}$ the function $g_t(x) = \|f(x)\|^2$ and $G : U \to \mathbb{R}$, given by $G(x,t) = g_t(x)$.

Let $\epsilon_0 > 0$ be a Milnor-Fukuda radius for f and let $0 < \epsilon \leq \epsilon_0$. We have that ϵ is a regular value of g_0, $\widetilde{S}^1_\epsilon = g_0^{-1}(\epsilon)$ is diffeomorphic to S^1 and that S^1_ϵ intersects properly to $\Delta(f)$, that is,

$$\det(X_0(y), y) > 0, \ \forall y \in S^1_\epsilon \cap \Delta(f).$$

Once ϵ is fixed, we can choose $\delta > 0$ such that for any $t \in (-\delta, \delta)$, ϵ is also a regular value of g_t and

$$\det(X_t(y), y) > 0, \ \forall y \in S^1_\epsilon \cap \Delta(f_t).$$

By the fibration theorem, we have that $\widetilde{S}^1_{\epsilon,t} = g_t^{-1}(\epsilon)$ is diffeomorphic to $\widetilde{S}^1_\epsilon$, and hence to S^1. Moreover, the above condition gives that S^1_ϵ is transverse to $\Delta(f_t)$ and that S^1_ϵ intersects $\Delta(f_t)$ properly. In conclusion, we have shown that ϵ is a convenient radius for f_t, for any $t \in (-\delta, \delta)$.

By theorem 5.2, $\gamma_{\epsilon,t} = f_t|_{\widetilde{S}^1_{\epsilon,t}}$ is the link of f_t and $f_t|_{\widetilde{D}^2_{\epsilon,t}}$ is topologically equivalent to the cone of $\gamma_{\epsilon,t}$.

Since $\gamma_{\epsilon,t} : \widetilde{S}^1_{\epsilon,t} \to S^1_\epsilon$, with $t \in (-\delta, \delta)$, is stable, we have that this family of links is trivial. Hence, each $f_t|_{\widetilde{D}^2_{\epsilon,t}}$ is topologically equivalent to $f|_{\widetilde{D}^2_\epsilon}$. By remark 3.3, there is a unique homeomorphism in the source ϕ_t such that $\psi_t \circ f_t \circ \phi_t^{-1} = f$. Note that the unicity of ϕ_t implies that it depends continuously on t. We consider now $\Phi = (\phi_t, t) : F^{-1}(D^2_\epsilon \times (-\delta, \delta)) \to \widetilde{D}^2_\epsilon \times (-\delta, \delta)$. Then Φ is a homeomorphism, it is an unfolding of the identity and $\Psi \circ F \circ \Phi^{-1} = f \times \mathrm{id}$. □

Corollary 6.4. *Any μ-constant unfolding F of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is topologically trivial.*

Proof. Any μ-constant unfolding F is excellent. This is known to be true in the complex case by the results of Gaffney [4]. Since F is analytic we are able to consider its complexification $\widehat{F}$ and we have that $\mu(\Delta(\widehat{f_t})) = \mu(\Delta(f_t))$ is constant. Then, $\widehat{F}$ is excellent, and as a consequence, F is also excellent. On the other hand, the μ-constant condition in the family of plane curves $\Delta(F)$ implies its topological triviality by the results of [7]. By theorem 6.3, F is topologically trivial. □

It is well known that in the complex case, any family of plane curves is topologically trivial if and only if the Milnor number is constant in the family. Hence, the converse of corollary 6.4 is also true in the complex case. In the real case, this is not true in general, as shown in the following example.

Example 6.5. Consider the family $f_t(x, y) = (x, x^4y + y^5 + t^2y^3)$. We have $f_t^{-1}(0) = \{0\}$, $Jf = x^4 + 5y^4 + 3t^2y^2 = 0$ and $S(f_t) = \{0\}$, for any $t \in \mathbb{R}$. Thus, the unfolding $F = (f_t, t)$ is excellent. Moreover, $\Delta(f_t) = \{0\}$ for any $t \in \mathbb{R}$, and hence F is topologically trivial by theorem 6.3.

On the other hand, the discriminant $\Delta(\hat{f}_t)$ of the complexification $\hat{f}_t$ is given by equation:

$$108t^{10}v^2 + 16t^8u^{12} - 900t^6u^4v^2 - 128t^4u^{16} + 2000t^2u^8v^2 + 256u^{20} + 3125v^4 = 0.$$

We have that $\mu(\Delta(f_t)) = 11$ for $t \neq 0$, but $\mu(\Delta(f_0)) = 57$.

7. The number of cusps of an unfolding

In this last section, we follow the arguments of the proof of theorem 6.3 to give a formula for the parity of the number of cusps of an unfolding $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of a finitely determined map germ f. Here, we do not assume that F is excellent, but we only assume the following condition $(*)$: there is a representative $F : U \to V \times I$, where U, V, I are open neighbourhoods of the origin in $\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2, \mathbb{R}$ respectively, such that $f_t : U_t \to V$ is proper and its restriction to $f_t^{-1}(V \setminus \{0\})$ is stable.

Given an unfolding satisfying this condition $(*)$, we introduce the following notation:

(1) $c(f_t^+)$ (respectively $c(f_t^-)$) is the number of cusps of f_t on $f_t^{-1}(V \setminus \{0\})$ for $t > 0$ (respectively $t < 0$).
(2) $r(f_t^+)$ (respectively $r(f_t^-)$) is the number of points of $f_t^{-1}(0)$ for $t > 0$ (respectively $t < 0$).
(3) $\#S(f_t^+)$ (respectively $S(f_t^-)$) is the number of branches of $S(f_t)$ at $f_t^{-1}(0)$ for $t > 0$ (respectively $t < 0$).
(4) $\#S(f_0)$ is the number of branches of $S(f_0)$ at 0.

If the neighbourhoods U, V, I are small enough, then these numbers are well defined. We also denote the multiplicity of a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ by

$$m(f) = \dim_{\mathbb{R}} \frac{\mathcal{E}_2}{\langle f_1, f_2 \rangle}.$$

We have the following congruences, which can be also deduced from the arguments of [5, Proof of Theorem 1.12]

Proposition 7.1. *Let $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ satisfying condition $(*)$. Then,*

$$c(f_t^{\pm}) \equiv 1 - r(f_t^{\pm}) + \#S(f_0) + \#S(f_t^{\pm}) \mod 2.$$

Moreover, if $m(f_t)$ is constant for each $t \in \mathbb{R}$ we have that

$$c(f_t^{\pm}) \equiv \#S(f_0) + \#S(f_t^{\pm}) \mod 2.$$

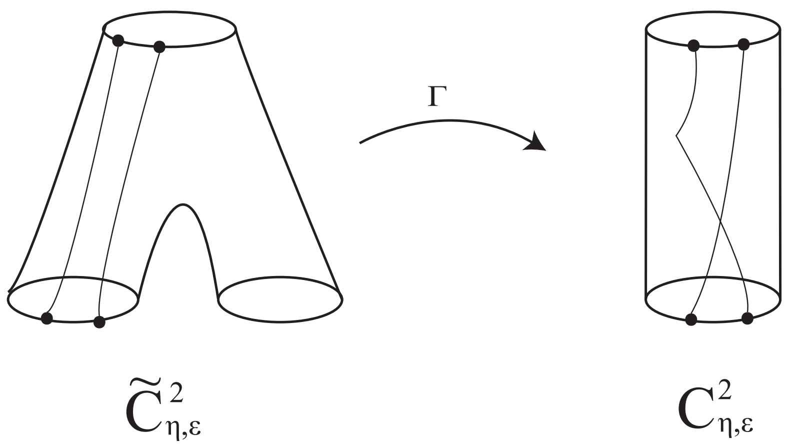

FIGURE 8.

Proof. Let $\epsilon_0 > 0$ be a Milnor-Fukuda radius for f and take $0 < \epsilon \leq \epsilon_0$. There is $\delta > 0$ such that if $t \in (-\delta, \delta)$, then ϵ is a convenient radius for the multigerm $f_t : (\mathbb{R}^2, Z_t) \to (\mathbb{R}^2, 0)$, where $f_t^{-1}(0) = Z_t$.

We fix $0 < t < \delta$, the case $-\delta < t < 0$ being analogous. Take $0 < \eta < \epsilon$, where $\eta \leq \eta_0$, a Milnor Fukuda radius for f_t. We denote:

$$\begin{aligned} \gamma_0 =& f_t|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon, \\ \gamma_1 =& f_t|_{\widetilde{S}^1_\eta} : \widetilde{S}^1_\eta \to S^1_\eta, \\ \Gamma =& f_t|_{\widetilde{C}^2_{\eta,\epsilon}} : \widetilde{C}^2_{\eta,\epsilon} \to C^2_{\eta,\epsilon}. \end{aligned}$$

We have that γ_0 is topologically equivalent to the link of the map germ f, γ_1 is the link of the multigerm f_t and Γ is a cobordism between γ_0, γ_1 (see figure 8). Since Γ is a stable map between compact oriented connected surfaces with boundary, we can apply a result due to Fukuda - Ishikawa [2]:

$$c(\Gamma) \equiv \chi(\widetilde{C}^2_{\eta,\epsilon}) + \deg(\Gamma|_{\partial \widetilde{C}^2_{\eta,\epsilon}})\chi(C^2_{\eta,\epsilon}) + \frac{1}{2}\#(S(\Gamma|_{\partial \widetilde{C}^2_{\eta,\epsilon}})) \mod 2,$$

where $c(\Gamma)$ is the number of cusps of Γ. We have $c(\Gamma) = c(f_t^+)$, $\chi(\widetilde{C}^2_{\eta,\epsilon}) = 1 - r(f_t^+)$, $\chi(C^2_{\eta,\epsilon}) = 0$ and

$$\frac{1}{2}\#(S(\Gamma|_{\partial\widetilde{C}^2_{\eta,\epsilon}})) = \#S(f_0) + \#S(f_t^+).$$

Thus, we arrive to

$$c(f_t^+) \equiv 1 - r(f_t^+) + \#S(f_0) + \#S(f_t^{\pm}) \mod 2.$$

If $m(f_t)$ is constant, we have that $\{f_t^{-1}(0)\} = \{0\}$, $r(f_t^+) = 1$ and hence,

$$c(f_t^+) \equiv \#S(f_0) + \#S(f_t^+) \mod 2.$$

□

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