Journal *of* Singularities
Volume 6 (2012), 158-178
DOI: 10.5427/jsing.2012.6m

Singularities in Geometry and Appl., Będlewo, 2011

SYMPLECTIC W_8 AND W_9 SINGULARITIES

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Abstract. We use the method of algebraic restrictions to classify symplectic W_8 and W_9 singularities. We use discrete symplectic invariants to distinguish symplectic singularities of the curves. We also give the geometric description of symplectic classes.

1. Introduction

In this paper we examine the singularities which are in the list of the simple 1-dimensional isolated complete intersection singularities in the space of dimension greater than 2, obtained by Giusti ([G], [AVG]). Isolated complete intersection singularities (ICIS) were intensively studied by many authors (e. g. see [L]), because of their interesting geometric, topological and algebraic properties. Here using the method of algebraic restrictions we obtain the complete symplectic classification of the singularities of type W_8 and W_9. We calculate discrete symplectic invariants for symplectic orbits of the curves and we give their geometric description. It allows us to explore the specific singular nature of these classical singularities that only appears in the presence of the symplectic structure.

We study the symplectic classification of singular curves under the following equivalence:

Definition 1.1. Let N_1, N_2 be two germs of subsets of the symplectic space $(\mathbb{R}^{2n}, \omega)$. N_1, N_2 are **symplectically equivalent** if there exists a symplectomorphism-germ $\Phi : (\mathbb{R}^{2n}, \omega) \to (\mathbb{R}^{2n}, \omega)$ such that $\Phi(N_1) = N_2$.

We recall that ω is a symplectic form if ω is a smooth nondegenerate closed 2-form, and $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism if Φ is a diffeomorphism and $\Phi^*\omega = \omega$.

Symplectic classification of curves was first studied by V. I. Arnold. In [A1] and [A2] the author studied singular curves in symplectic and contact spaces and introduced the local symplectic and contact algebra. In [A2] V. I. Arnold discovered new symplectic invariants of singular curves. He proved that the A_{2k} singularity of a planar curve (the orbit with respect to standard $\mathcal{A}$-equivalence of parameterized curves) split into exactly $2k+1$ symplectic singularities (orbits with respect to symplectic equivalence of parameterized curves). He distinguished different symplectic singularities by different orders of tangency of the parameterized curve to the *nearest* smooth Lagrangian submanifold. Arnold posed a problem of expressing these invariants in terms of the local algebra's interaction with the symplectic structure and he proposed calling this interaction 'the local symplectic algebra'.

In [IJ1] G. Ishikawa and S. Janeczko classified symplectic singularities of curves in the 2-dimensional symplectic space. All simple curves in this classification are quasi-homogeneous.

We recall that a subset N of $\mathbb{R}^m$ is **quasi-homogeneous** if there exist a coordinate system $(x_1, \cdots, x_m)$ on $\mathbb{R}^m$ and positive numbers $w_1, \cdots, w_m$ (called weights) such that for any point

2000 *Mathematics Subject Classification.* Primary 53D05. Secondary 14H20, 58K50, 58A10.

Key words and phrases. symplectic manifold, curves, local symplectic algebra, algebraic restrictions, relative Darboux theorem, singularities.

$(y_1, \cdots, y_m) \in \mathbb{R}^m$ and any $t > 0$ if $(y_1, \cdots, y_m)$ belongs to N then the point $(t^{w_1}y_1, \cdots, t^{w_m}y_m)$ belongs to N.

The generalization of results in [IJ1] to volume-preserving classification of singular varieties and maps in arbitrary dimensions was obtained in [DR]. A symplectic form on a 2-dimensional manifold is a special case of a volume form on a smooth manifold.

In [K] P. A. Kolgushkin classified the stably simple symplectic singularities of parameterized curves (in the $\mathbb{C}$-analytic category). Symplectic singularity is stably simple if it is simple and if it remains simple if the ambient symplectic space is symplectically embedded (i.e. as a symplectic submanifold) into a larger symplectic space.

In [Z] was developed the local contact algebra. The main results were based on the notion of the algebraic restriction of a contact structure to a subset N of a contact manifold.

In [DJZ2] new symplectic invariants of singular quasi-homogeneous subsets of a symplectic space were explained by the algebraic restrictions of the symplectic form to these subsets.

The algebraic restriction is an equivalence class of the following relation on the space of differential k-forms:

Differential k-forms ω_1 and ω_2 have the same **algebraic restriction** to a subset N if $\omega_1 - \omega_2 = \alpha + d\beta$, where α is a k-form vanishing on N and β is a $(k-1)$-form vanishing on N.

The generalization of the Darboux-Givental theorem ([AG]) to germs of arbitrary subsets of the symplectic space was obtained in [DJZ2]. This result reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets. For non-quasi-homogeneous subsets there is one more cohomological invariant except the algebraic restriction ([DJZ2], [DJZ1]). The dimension of the space of algebraic restrictions of closed 2-forms to a 1-dimensional quasi-homogeneous isolated complete intersection singularity C is equal to the multiplicity of C ([DJZ2]). In [D] it was proved that the space of algebraic restrictions of closed 2-forms to a 1-dimensional (singular) analytic variety is finite-dimensional. In [DJZ2] the method of algebraic restrictions was applied to various classification problems in a symplectic space. In particular the complete symplectic classification of classical A-D-E singularities of planar curves and S_5 singularity were obtained. Most of different symplectic singularity classes were distinguished by new discrete symplectic invariants: the index of isotropy and the symplectic multiplicity.

In [DT1] following ideas from [A1] and [D] new discrete symplectic invariants - the Lagrangian tangency orders were introduced and used to distinguish symplectic singularities of simple planar curves of type A-D-E, symplectic T_7 and T_8 singularities.

In [DT2] was obtained the complete symplectic classification of the isolated complete intersection singularities S_μ for $\mu > 5$.

In this paper we obtain the detailed symplectic classification of W_8 and W_9 singularities. The paper is organized as follows. In Section 2 we recall discrete symplectic invariants (the symplectic multiplicity, the index of isotropy and the Lagrangian tangency orders). Symplectic classification of W_8 and W_9 singularity is presented in Sections 3 and 4 respectively. The symplectic sub-orbits of this singularities are listed in Theorems 3.1 and 4.1. Discrete symplectic invariants for the symplectic classes are calculated in Theorems 3.2 and 4.2. The geometric descriptions of the symplectic orbits is presented in Theorems 3.5 and 4.4. In Section 5 we recall the method of algebraic restrictions and use it to classify symplectic singularities.

2. Discrete symplectic invariants.

We can use discrete symplectic invariants to characterize symplectic singularity classes.

The first invariant is the symplectic multiplicity ([DJZ2]) introduced in [IJ1] as a symplectic defect of a curve.

Let N be a germ of a subset of $(\mathbb{R}^{2n}, \omega)$.

Definition 2.1. The **symplectic multiplicity** $\mu^{sym}(N)$ of N is the codimension of a symplectic orbit of N in an orbit of N with respect to the action of the group of local diffeomorphisms.

The second invariant is the index of isotropy [DJZ2].

Definition 2.2. The **index of isotropy** $ind(N)$ of N is the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all smooth submanifolds M containing N.

This invariant has geometrical interpretation. An equivalent definition is as follows: the index of isotropy of N is the maximal order of tangency between non-singular submanifolds containing N and non-singular isotropic submanifolds of the same dimension. The index of isotropy is equal to 0 if N is not contained in any non-singular submanifold which is tangent to some isotropic submanifold of the same dimension. If N is contained in a non-singular Lagrangian submanifold then the index of isotropy is ∞.

The symplectic multiplicity and the index of isotropy can be expressed in terms of algebraic restrictions (Propositions 5.6 and 5.7 in Section 5).

There is one more discrete symplectic invariant, introduced in [D] (following ideas from [A2]) which is defined specifically for a parameterized curve. This is the maximal tangency order of a curve $f : \mathbb{R} \to M$ to a smooth Lagrangian submanifold. If $H_1 = ... = H_n = 0$ define a smooth submanifold L in the symplectic space then the tangency order of a curve $f : \mathbb{R} \to M$ to L is the minimum of the orders of vanishing at 0 of functions $H_1 \circ f, \cdots, H_n \circ f$. We denote the tangency order of f to L by $t(f, L)$.

Definition 2.3. The **Lagrangian tangency order** $Lt(f)$ **of a curve** f is the maximum of $t(f, L)$ over all smooth Lagrangian submanifolds L of the symplectic space.

The Lagrangian tangency order of a quasi-homogeneous curve in a symplectic space can also be expressed in terms of the algebraic restrictions (Proposition 5.8 in Section 5).

In [DT1] the above invariant was generalized for germs of curves and multi-germs of curves which may be parameterized analytically since the Lagrangian tangency order is the same for every 'good' analytic parametrization of a curve.

Consider a multi-germ $(f_i)_{i\in\{1,\cdots,r\}}$ of analytically parameterized curves f_i. We have r-tuples $(t(f_1, L), \cdots, t(f_r, L))$ for any smooth submanifold L in the symplectic space.

Definition 2.4. For any $I \subseteq \{1, \cdots, r\}$ we define **the tangency order of the multi-germ** $(f_i)_{i\in I}$ to L:

$$t[(f_i)_{i\in\ I}, L] = \min_{i\in\ I} t(f_i, L).$$

Definition 2.5. The **Lagrangian tangency order** $Lt((f_i)_{i\in\ I})$ **of a multi-germ** $(f_i)_{i\in I}$ is the maximum of $t[(f_i)_{i\in\ I}, L]$ over all smooth Lagrangian submanifolds L of the symplectic space.

3. Symplectic W_8-singularities

Denote by (W_8) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$W_8 = \{x \in \mathbb{R}^{2n\geq 4} : x_1^2 + x_3^3 = x_2^2 + x_1x_3 = x_{\geq 4} = 0\}. \tag{3.1}$$

This is the simple 1-dimensional isolated complete intersection singularity W_8 ([G], [AVG][1]). Here N is quasi-homogeneous with weights $w(x_1) = 6,\ w(x_2) = 5,\ w(x_3) = 4$.

We used the method of algebraic restrictions to obtain the complete classification of symplectic singularities of (W_8) presented in the following theorem.

Theorem 3.1. *Any submanifold of the symplectic space* $(\mathbb{R}^{2n}, \sum_{i=1}^n dp_i \wedge dq_i)$ *where* $n \geq 3$ *(respectively,* $n = 2$*) which is diffeomorphic to* W_8 *is symplectically equivalent to one and only one of the normal forms* $W_8^i, i = 0, 1, \cdots, 8$ *(respectively,* $i = 0, 1, 2a, 2b$*) listed below. The parameters* c, c_1, c_2 *of the normal forms are moduli:*

$W_8^0:\ p_1^2 + p_2q_1 = 0,\ \ p_2^2 + q_1^3 = 0,\ \ q_2 = c_1q_1 + c_2p_1,\ \ p_{\geq 3} = q_{\geq 3} = 0;$

$W_8^1:\ q_1^2 + p_1q_2 = 0,\ \ p_1^2 + q_2^3 = 0,\ \ p_2 = c_1p_1 + c_2q_1q_2,\ \ p_{\geq 3} = q_{\geq 3} = 0,\ c_1 \neq 0;$

$W_8^{2a}:\ p_2^2 \pm p_1q_1 = 0,\ \ p_1^2 + q_1^3 = 0,\ \ q_2 = \frac{c_1}{2}q_1^2 + \frac{c_2}{3}q_1^3,\ \ p_{\geq 3} = q_{\geq 3} = 0;$

$W_8^{2b}:\ q_1^2 + p_1q_2 = 0,\ \ p_1^2 + q_2^3 = 0,\ \ p_2 = c_1q_1q_2 + \frac{c_2}{2}q_1^2,\ \ p_{\geq 3} = q_{\geq 3} = 0;$

$W_8^3:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ q_1 = q_2 = 0, q_3 = -p_2p_3 - \frac{c_1}{2}p_2^2 - c_2p_1p_2,\ p_{>3} = q_{>3} = 0;$

$W_8^4:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ q_1 = q_2 = 0, q_3 = \mp\frac{1}{2}p_2^2 - c_1p_1p_2 - c_2p_2p_3^2,\ p_{>3} = q_{>3} = 0;$

$W_8^5:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ q_1 = q_2 = 0, q_3 = -p_1p_2 - cp_2p_3^2,\ p_{>3} = q_{>3} = 0;$

$W_8^6:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ q_1 = q_2 = 0, q_3 = -p_2p_3^2 - \frac{c}{3}p_2^3,\ p_{>3} = q_{>3} = 0;$

$W_8^7:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ q_1 = q_2 = 0, q_3 = -\frac{1}{3}p_2^3,\ p_{>3} = q_{>3} = 0;$

$W_8^8:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ p_{>3} = q_{>0} = 0.$

In Section 3.1 we use the symplectic invariants (in particular the Lagrangian tangency order) to distinguish the symplectic singularity classes. In Section 3.2 we propose a geometric description of these singularities that confirms the classification. Some of the proofs are presented in Section 5.

3.1. **Distinguishing symplectic classes of W_8 by the Lagrangian tangency order and the index of isotropy.** A curve $N \in (W_8)$ can be described as a parametrical curve $C(t)$. Its parametrization is given in the second column of Table 1. To characterize the symplectic classes we use the following invariants:

- $L_N = Lt(N) = \max\limits_L(t(C(t), L))$;
- ind - the index of isotropy of N.

Here L is a smooth Lagrangian submanifold of the symplectic space.

Theorem 3.2. *A stratified submanifold* $N \in (W_8)$ *of a symplectic space* $(\mathbb{R}^{2n}, \omega_0)$ *with the canonical coordinates* $(p_1, q_1, \cdots, p_n, q_n)$ *is symplectically equivalent to one and only one of the curves presented in the second column of Table 1. The parameters* c, c_1, c_2 *are moduli. The index of isotropy and the Lagrangian tangency order of the curve* N *are presented in the third and fourth column of Table 1.*

Remark 3.3. The invariants can be calculated by knowing the algebraic restrictions for the symplectic classes. We use Proposition 5.7 to calculate the index of isotropy. The Lagrangian tangency order we can calculate using Proposition 5.8 or by applying directly the definition of the Lagrangian tangency order and finding a Lagrangian submanifold the nearest to the curve $C(t)$.

[1] There is a mistake in the description of W_8 singularity in [AVG]. We find there $W_8 = \{x \in \mathbb{R}^{2n \geq 4} : x_1^2 + x_2^3 = x_2^2 + x_1x_3 = x_{\geq 4} = 0\}$ which is not an isolated complete intersection singularity.

class		parametrization of N	ind	L_N
$(W_8)^0$	$2n \geq 4$	$(t^5, -t^4, t^6, -c_1t^4 + c_2t^5, 0, \cdots)$	0	5
$(W_8)^1$	$2n \geq 4$	$(t^6, t^5, c_1t^6 - c_2t^9, -t^4, 0, \cdots)$	0	6
$(W_8)^{2a}$	$2n \geq 4$	$(\pm t^6, -t^4, t^5, \frac{c_1}{2}t^8 - \frac{c_2}{3}t^{12}, 0, \cdots)$	0	6
$(W_8)^{2b}$	$2n \geq 4$	$(t^6, t^5, -c_1t^9 + \frac{c_2}{2}t^{10}, -t^4, , 0, \cdots)$	0	6
$(W_8)^3$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, t^9 - \frac{c_1}{2}t^{10} - c_2t^{11}, 0, \cdots)$	1	9
$(W_8)^4$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, \mp t^{10} - c_1t^{11} - c_2t^{13}, 0, \cdots)$	1	10
$(W_8)^5$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, -t^{11} - ct^{13}, 0, \cdots)$	1	11
$(W_8)^6$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, -t^{13} - \frac{c}{3}t^{15}, 0, \cdots)$	2	13
$(W_8)^7$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, -t^{15}, 0, \cdots)$	2	15
$(W_8)^8$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, 0, 0, \cdots)$	∞	∞

TABLE 1. The symplectic invariants for symplectic classes of W_8 singularity.

Remark 3.4. The comparison of invariants presented in Table 1 shows that the Lagrangian tangency order distinguishes more symplectic classes than the index of isotropy. Symplectic classes $(W_8)^{2a}$ and $(W_8)^{2b}$ can not be distinguished by any of the invariants but we can distinguish them by geometric conditions.

3.2. **Geometric conditions for the classes $(W_8)^i$.** We can characterize the symplectic classes $(W_8)^i$ by geometric conditions independent of any local coordinate system. Let $N \in (W_8)$. Denote by W the tangent space at 0 to some (and then any) non-singular 3-manifold containing N. We can define the following subspaces of this space: ℓ − the tangent line at 0 to the curve N, V − the 2-space tangent at 0 to the curve N. For $N = W_8 =$(3.1) it is easy to calculate

$$W = \mathrm{span}(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3), \ \ell = \mathrm{span}(\partial/\partial x_3), \ V = \mathrm{span}(\partial/\partial x_2, \partial/\partial x_3). \tag{3.2}$$

The classes $(W_8)^i$ satisfy special conditions in terms of the restriction $\omega|_W$, where ω is the symplectic form.

Theorem 3.5. *If a stratified submanifold $N \in (W_8)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ belongs to the class $(W_8)^i$ then the couple (N, ω) satisfies the corresponding conditions in the last column of Table 2.*

Sketch of the proof of Theorem 3.5. We have to show that the conditions in the row of $(W_8)^i$ are satisfied for any $N \in (W_8)^i$.

Each of the conditions in the last column of Table 2 is invariant with respect to the action of the group of diffeomorphisms in the space of pairs (N, ω). Because each of these conditions depends only on the algebraic restriction $[\omega]_N$ we can take the simplest 2-forms ω^i representing the normal forms $[W_8]^i$ for algebraic restrictions: ω^0, ω^1, $\omega^{2,a}$, $\omega^{2,b}$, ω^3, ω^4, ω^5, ω^6, ω^7, ω^8 and we can check that the pair $(W_8, \omega = \omega^i)$ satisfies the condition in the last column of Table 2.

We note that in the case $N = W_8 =$ (3.1) one has the description (3.2) of the subspaces W, ℓ and V. By simple calculation and observation of the Lagrangian tangency order we obtain that the conditions corresponding to the classes $(W_8)^i$ are satisfied. $\square$

class	normal form	geometric conditions
$(W_8)^0$	$[W_8]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$	$\omega\|_V \neq 0$
$(W_8)^1$	$[W_8]^1 : [c_1\theta_2 + \theta_3 + c_2\theta_4]_{W_8}$, $c_1 \neq 0$	$\omega\|_V = 0$ and $\ker\omega \neq \ell$
$(W_8)^{2a}$	$[W_8]^{2a} : [\pm\theta_2 + c_1\theta_4 + c_2\theta_7]_{W_8}$	$\omega\|_V = 0$ and $\ker\omega \neq \ell$
$(W_8)^{2b}$	$[W_8]^{2b} : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{W_8}$	$\omega\|_V = 0$ and $\ker\omega = \ell$
		$\omega\|_W = 0$
$(W_8)^3$	$[W_8]^3 : [\theta_4 + c_1\theta_5 + c_2\theta_6]_{W_8}$	$L_N = 9$
$(W_8)^4$	$[W_8]^4 : [\pm\theta_5 + c_1\theta_6 + c_2\theta_7]_{W_8}$	$L_N = 10$
$(W_8)^5$	$[W_8]^5 : [\theta_6 + c\theta_7]_{W_8}$	$L_N = 11$
$(W_8)^6$	$[W_8]^6 : [\theta_7 + c\theta_8]_{W_8}$	$L_N = 13$
$(W_8)^7$	$[W_8]^7 : [\theta_8]_{W_8}$	$L_N = 15$
$(W_8)^8$	$[W_8]^8 : [0]_{W_8}$	N is contained in a smooth Lagrangian submanifold

TABLE 2. Geometric interpretation of singularity classes of W_8. (W is the tangent space to a non-singular 3-dimensional manifold in $(\mathbb{R}^{2n\geq 4}, \omega)$ containing $N \in (W_8)$. The forms $\theta_1, \ldots, \theta_8$ are described in Theorem 5.10 on the page 168.)

4. Symplectic W_9-singularities

Denote by (W_9) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$W_9 = \{x \in \mathbb{R}^{2n\geq 4} : x_1^2 + x_2x_3^2 = x_2^2 + x_1x_3 = x_{\geq 4} = 0\}. \tag{4.1}$$

This is the simple 1-dimensional isolated complete intersection singularity W_9 ([G], [AVG]). Here N is quasi-homogeneous with weights $w(x_1)=5$, $w(x_2)=4$, $w(x_3)=3$.

We present the complete classification of the symplectic singularities of (W_9) which was obtained using the method of algebraic restrictions.

Theorem 4.1. *Any submanifold of the symplectic space $(\mathbb{R}^{2n}, \sum_{i=1}^n dp_i \wedge dq_i)$ where $n \geq 3$ (respectively, $n = 2$) which is diffeomorphic to W_9 is symplectically equivalent to one and only one of the normal forms $W_9^i, i = 0, 1, \cdots, 9$ (respectively, $i = 0, 1, 2$) listed below. The parameters c, c_1, c_2 of the normal forms are moduli:*

W_9^0 : $p_1^2 + p_2q_2^2 = 0$, $p_2^2 + p_1q_2 = 0$, $q_1 = c_1q_2 + c_2p_2$, $p_{\geq 3} = q_{\geq 3} = 0$;

W_9^1 : $p_1^2 + p_2q_1^2 = 0$, $p_2^2 \pm p_1q_1 = 0$, $q_2 = -c_1p_1 + \frac{c_2}{2}q_1^2$, $p_{\geq 3} = q_{\geq 3} = 0$;

W_9^2 : $p_1^2 + q_1p_2^2 = 0$, $q_1^2 + p_1p_2 = 0$, $q_2 = c_1q_1p_2 - c_2p_1p_2$, $p_{\geq 3} = q_{\geq 3} = 0$;

W_9^3 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0, q_3 = \mp p_2p_3 - c_1p_1p_3 - c_2p_1p_2$, $q_1 = q_2 = p_{>3} = q_{>3} = 0$;

W_9^4 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $q_3 = -p_1p_3 - c_1p_1p_2 - c_2p_2p_3^2, q_1 = q_2 = p_{>3} = q_{>3} = 0$;

W_9^5 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $q_3 = \mp p_1p_2 - c_1p_2p_3^2 - c_2p_1p_3^2, q_1 = q_2 = p_{>3} = q_{>3} = 0$;

W_9^6 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $q_1 = q_2 = 0, q_3 = -p_2p_3^2 - cp_1p_3^2$, $p_{>3} = q_{>3} = 0$;

W_9^7 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $q_1 = q_2 = 0, q_3 = \mp p_1p_3^2 - cp_2p_3^3$, $p_{>3} = q_{>3} = 0$;

W_9^8 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $q_1 = q_2 = 0, q_3 = \mp p_2p_3^3$, $p_{>3} = q_{>3} = 0$;

W_9^9 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $p_{>3} = q_{>0} = 0$.

In Section 4.1 we use the Lagrangian tangency orders to distinguish the symplectic classes. In Section 4.2 we propose a geometric description of the symplectic singularities. Some of the proofs are presented in Section 5.

4.1. **Distinguishing symplectic classes of W_9 by Lagrangian tangency orders.** The Lagrangian tangency orders were used to distinguish the symplectic classes of (W_9). A curve $N \in (W_9)$ may be described as a union of two parametrical branches: C_1 and C_2. The curve C_1 is nonsingular and the curve C_2 is singular. Their parametrization in the coordinate system $(p_1, q_1, p_2, q_2, \cdots, p_n, q_n)$ is presented in the second column of Table 3. To characterize the symplectic classes of this singularity we use the following two invariants:

- $L_N = Lt(C_1, C_2) = \max_L(\min\{t(C_1, L), t(C_2, L)\})$,
- $L_2 = Lt(C_2) = \max_L t(C_2, L)$.

Here L is a smooth Lagrangian submanifold of the symplectic space.

Theorem 4.2. *A stratified submanifold $N \in (W_9)$ of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ with the canonical coordinates $(p_1, q_1, \cdots, p_n, q_n)$ is symplectically equivalent to one and only one of the curves presented in the second column of Table 3. The parameters c, c_1, c_2 are moduli. The Lagrangian tangency orders are presented in the third and fourth columns of the table.*

class	parametrization of branches	L_N	L_2
$(W_9)^0$	$C_1 : (0, c_1 t, 0, t, 0, 0, \cdots)$, $C_2 : (t^5, -c_1 t^3 - c_2 t^4, -t^4, -t^3, 0, \cdots)$	4	4
$(W_9)^1$	$C_1 : (0, \pm t, 0, \frac{c_2}{2} t^2, 0, \cdots)$, $C_2 : (t^5, \mp t^3, -t^4, -c_1 t^5 + \frac{c_2}{2} t^6, 0, \cdots)$	5	5
$(W_9)^2$	$C_1 : (0, 0, t, 0, 0, \cdots)$, $C_2 : (t^5, -t^4, -t^3, -c_1 t^7 + c_2 t^8, 0, \cdots)$	5	5
$(W_9)^3$	$C_1 : (0, 0, 0, 0, t, 0, \cdots)$, $C_2 : (t^5, 0, -t^4, 0, t^3, \mp t^7 + c_1 t^8 + c_2 t^9, 0, \cdots)$	7	7
$(W_9)^4$	$C_1 : (0, 0, 0, 0, t, 0, \cdots)$, $C_2 : (t^5, 0, -t^4, 0, t^3, t^8 + c_1 t^9 + c_2 t^{10}, 0, \cdots)$	8	8
$(W_9)^5$	$C_1 : (0, 0, 0, 0, t, 0, \cdots)$, $C_2 : (t^5, 0, -t^4, 0, t^3, \pm t^9 + c_1 t^{10} - c_2 t^{11}, 0, \cdots)$	9	∞
$(W_9)^6$	$C_1 : (0, 0, 0, 0, t, 0, \cdots)$, $C_2 : (t^5, 0, -t^4, 0, t^3, t^{10} - c t^{11}, 0, \cdots)$	10	∞
$(W_9)^7$	$C_1 : (0, 0, 0, 0, t, 0, \cdots)$, $C_2 : (t^5, 0, -t^4, 0, t^3, \mp t^{11} - c t^{13}, 0, \cdots)$	11	∞
$(W_9)^8$	$C_1 : (0, 0, 0, 0, t, 0, \cdots)$, $C_2 : (t^5, 0, -t^4, 0, t^3, \mp t^{13}, 0, \cdots)$	13	∞
$(W_9)^9$	$C_1 : (0, 0, 0, 0, t, 0, \cdots)$, $C_2 : (t^5, 0, -t^4, 0, t^3, 0, 0, \cdots)$	∞	∞

TABLE 3. The Lagrangian tangency orders for symplectic classes of W_9 singularity.

Remark 4.3. The invariants can be calculated by knowing the parametrization of branches C_1 and C_2. We apply directly the definition of the Lagrangian tangency order finding a Lagrangian submanifold the nearest to the branches.

4.2. **Geometric conditions for the classes $(W_9)^i$.**

Let $N \in (W_9)$. Denote by W the tangent space at 0 to some (and then any) non-singular 3-manifold containing N. We can define the following subspaces of this space:

ℓ – the tangent line at 0 to both branches of N,

V – 2-space tangent at 0 to the singular branch of N.

The classes $(W_9)^i$ satisfy special conditions in terms of the restriction $\omega|_W$, where ω is the symplectic form.

Theorem 4.4. *A stratified submanifold $N \in (W_9)$ of the symplectic space $(\mathbb{R}^{2n}, \omega)$ belongs to the class $(W_9)^i$ if and only if the couple (N, ω) satisfies the corresponding conditions in the last column of Table 4.*

class	normal form	geometric conditions
$(W_9)^0$	$[W_9]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$	$\omega\vert_V \neq 0$ (2-space tangent to N is not isotropic)
$(W_9)^1$	$[W_9]^1 : [\pm\theta_2 + c_1\theta_3 + c_2\theta_4]_{W_9}$	$\omega\vert_V = 0$ and $\ker\omega \neq \ell$
$(W_9)^2$	$[W_9]^2 : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{W_9}$	$\omega\vert_V = 0$ and $\ker\omega = \ell$
		$\omega\vert_W = 0$
$(W_9)^3$	$[W_9]^3 : [\pm\theta_4 + c_1\theta_5 + c_2\theta_6]_{W_9}$	$L_N = 7$
$(W_9)^4$	$[W_9]^4 : [\theta_5 + c_1\theta_6 + c_2\theta_7]_{W_9}$	$L_N = 8$
$(W_9)^5$	$[W_9]^5 : [\pm\theta_6 + c_1\theta_7 + c_2\theta_8]_{W_9}$	$L_N = 9$
$(W_9)^6$	$[W_9]^6 : [\theta_7 + c\theta_8]_{W_9}$	$L_N = 10$
$(W_9)^7$	$[W_9]^7 : [\pm\theta_8 + c\theta_9]_{W_9}$	$L_N = 11$
$(W_9)^8$	$[W_9]^8 : [\pm\theta_9]_{W_9}$	$L_N = 13$
$(W_9)^9$	$[W_9]^9 : [0]_{W_9}$	N is contained in a smooth Lagrangian submanifold

TABLE 4. Geometric characterization of symplectic classes of W_9 singularity. (The forms $\theta_1, \ldots, \theta_9$ are described in Theorem 5.23 on the page 173.)

Sketch of the proof of Theorem 4.4. The conditions on the pair (ω, N) in the last column of Table 4 are disjoint. It suffices to prove that these conditions in the row of $(W_9)^i$, are satisfied for any $N \in (W_9)^i$.

We can take the simplest 2-forms ω^i representing the normal forms $[W_9]^i$ for algebraic restrictions and we can check that the pair $(W_9, \omega = \omega^i)$ satisfies the condition in the last column of Table 4.
We note that in the case $N = W_9 =$ (4.1) one has
$\ell = \mathrm{span}(\partial/\partial x_3)$, $V = \mathrm{span}(\partial/\partial x_2, \partial/\partial x_3$, $W = \mathrm{span}(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$.
By simple calculation and observation of the Lagrangian tangency orders we obtain that the conditions corresponding to the classes $(W_9)^i$ are satisfied. □

5. PROOFS

5.1. **The method of algebraic restrictions.** In this section we present basic facts on the method of algebraic restrictions, which is a very powerful tool for the symplectic classification. The details of the method and proofs of all results of this section can be found in [DJZ2].

Given a germ of a non-singular manifold M denote by $\Lambda^p(M)$ the space of all germs at 0 of differential p-forms on M. Given a subset $N \subset M$ introduce the following subspaces of $\Lambda^p(M)$:

$$\Lambda^p_N(M) = \{\omega \in \Lambda^p(M) : \quad \omega(x) = 0 \text{ for any } x \in N\};$$

$$\mathcal{A}^p_0(N, M) = \{\alpha + d\beta : \quad \alpha \in \Lambda^p_N(M),\ \beta \in \Lambda^{p-1}_N(M).\}$$

Definition 5.1. Let N be the germ of a subset of M and let $\omega \in \Lambda^p(M)$. The **algebraic restriction** of ω to N is the equivalence class of ω in $\Lambda^p(M)$, where the equivalence is as follows: ω is equivalent to $\widetilde{\omega}$ if $\omega - \widetilde{\omega} \in \mathcal{A}^p_0(N, M)$.

Notation. The algebraic restriction of the germ of a p-form ω on M to the germ of a subset $N \subset M$ will be denoted by $[\omega]_N$. By writing $[\omega]_N = 0$ (or saying that ω has zero algebraic restriction to N) we mean that $[\omega]_N = [0]_N$, i.e. $\omega \in A_0^p(N, M)$.

Definition 5.2. Two algebraic restrictions $[\omega]_N$ and $[\widetilde{\omega}]_{\widetilde{N}}$ are called **diffeomorphic** if there exists the germ of a diffeomorphism $\Phi : \widetilde{M} \to M$ such that $\Phi(\widetilde{N}) = N$ and $\Phi^*([\omega]_N) = [\widetilde{\omega}]_{\widetilde{N}}$.

The method of algebraic restrictions applied to singular quasi-homogeneous subsets is based on the following theorem.

Theorem 5.3 (Theorem A in [DJZ2]). *Let N be the germ of a quasi-homogeneous subset of $\mathbb{R}^{2n}$. Let ω_0, ω_1 be germs of symplectic forms on $\mathbb{R}^{2n}$ with the same algebraic restriction to N. There exists a local diffeomorphism Φ such that $\Phi(x) = x$ for any $x \in N$ and $\Phi^*\omega_1 = \omega_0$.*

Two germs of quasi-homogeneous subsets N_1, N_2 of a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ are symplectically equivalent if and only if the algebraic restrictions of the symplectic form ω to N_1 and N_2 are diffeomorphic.

Theorem 5.3 reduces the problem of symplectic classification of germs of singular quasi-homogeneous subsets to the problem of diffeomorphic classification of algebraic restrictions of the germ of the symplectic form to the germs of singular quasi-homogeneous subsets.

The geometric meaning of the zero algebraic restriction is explained by the following theorem.

Theorem 5.4 (Theorem **B** in [DJZ2]). *The germ of a quasi-homogeneous set N of a symplectic space $(\mathbb{R}^{2n}, \omega)$ is contained in a non-singular Lagrangian submanifold if and only if the symplectic form ω has zero algebraic restriction to N.*

In the remainder of this paper we use the following notations:

- $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$: the vector space consisting of the algebraic restrictions of germs of all 2-forms on $\mathbb{R}^{2n}$ to the germ of a subset $N \subset \mathbb{R}^{2n}$;
- $\left[Z^2(\mathbb{R}^{2n})\right]_N$: the subspace of $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ consisting of the algebraic restrictions of germs of all closed 2-forms on $\mathbb{R}^{2n}$ to N;
- $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$: the open set in $\left[Z^2(\mathbb{R}^{2n})\right]_N$ consisting of the algebraic restrictions of germs of all symplectic 2-forms on $\mathbb{R}^{2n}$ to N.

To obtain a classification of the algebraic restrictions we use the following proposition.

Proposition 5.5. *Let $a_1, \cdots, a_p$ be a quasi-homogeneous basis of quasi-degrees $\delta_1 \le \cdots \le \delta_s < \delta_{s+1} \le \cdots \le \delta_p$ of the space of algebraic restrictions of closed 2-forms to quasi-homogeneous subset N. Let $a = \sum_{j=s}^{p} c_j a_j$, where $c_j \in \mathbb{R}$ for $j = s, \cdots, p$ and $c_s \neq 0$.*

If there exists a tangent quasi-homogeneous vector field X over N such that $\mathcal{L}_X a_s = r a_k$ for $k > s$ and $r \neq 0$ then a is diffeomorphic to $\sum_{j=s}^{k-1} c_j a_j + \sum_{j=k+1}^{p} b_j a_j$, for some $b_j \in \mathbb{R}$, $j = k+1, \cdots, p$.

Proposition 5.5 is a modification of Theorem 6.13 formulated and proved in [D]. It was formulated for algebraic restrictions to a parameterized curve but we can generalize this theorem for any quasi-homogeneous subset N. The proofs of the cited theorem and Proposition 5.5 are based on the Moser homotopy method.

For calculating discrete invariants we use the following propositions.

Proposition 5.6 ([DJZ2]). *The symplectic multiplicity of the germ of a quasi-homogeneous subset N in a symplectic space is equal to the codimension of the orbit of the algebraic restriction $[\omega]_N$ with respect to the group of local diffeomorphisms preserving N in the space of algebraic restrictions of closed 2-forms to N.*

Proposition 5.7 ([DJZ2]). *The index of isotropy of the germ of a quasi-homogeneous subset N in a symplectic space $(\mathbb{R}^{2n}, \omega)$ is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction $[\omega]_N$.*

Proposition 5.8 ([D]). *Let f be the germ of a quasi-homogeneous curve such that the algebraic restriction of a symplectic form to it can be represented by a closed 2-form vanishing at 0. Then the Lagrangian tangency order of the germ of a quasi-homogeneous curve f is the maximum of the order of vanishing on f over all 1-forms α such that $[\omega]_f = [d\alpha]_f$*

5.2. **Proofs for W_8 singularity.**

5.2.1. *Algebraic restrictions to W_8 and their classification.* One has the following relations for (W_8)-singularities:

$$[d(x_2^2 + x_1x_3)]_{W_8} = [2x_2dx_2 + x_1dx_3 + x_3dx_1]_{W_8} = 0, \tag{5.1}$$

$$[d(x_1^2 + x_3^3)]_{W_8} = [2x_1dx_1 + 3x_3^2dx_3]_{W_8} = 0. \tag{5.2}$$

Multiplying these relations by suitable 1-forms we obtain the relations in Table 5.

δ	relations	proof
14	$[x_2dx_2 \wedge dx_3]_N = -\frac{1}{2}[x_3dx_1 \wedge dx_3]_N$	(5.1)$\wedge dx_3$
15	$[x_1dx_2 \wedge dx_3]_N = [x_3dx_1 \wedge dx_2]_N$	(5.1)$\wedge dx_2$
16	$[x_2dx_1 \wedge dx_2]_N = -\frac{1}{2}[x_1dx_1 \wedge dx_3]_N = 0$	(5.2)$\wedge dx_3$ and (5.1)$\wedge dx_1$
17	$[x_3^2dx_2 \wedge dx_3]_N = \frac{2}{3}[x_1dx_1 \wedge dx_2]_N$	(5.2)$\wedge dx_2$
18	$[x_3^2dx_1 \wedge dx_3]_N = 2[x_2x_3dx_2 \wedge dx_3]_N = 0$	(5.2)$\wedge dx_1$ and (5.1)$\wedge x_3dx_3$
19	$[x_2^2dx_2 \wedge dx_3]_N = -\frac{1}{2}[x_2x_3dx_1 \wedge dx_3]_N$ $[x_2^2dx_2 \wedge dx_3]_N = -[x_1x_3dx_2 \wedge dx_3]_N =$ $= -[x_3^2dx_1 \wedge dx_2]_N$	(5.1)$\wedge x_2dx_3$ (5.1)$\wedge x_3dx_2$ and $[x_2^2 + x_1x_3]_N = 0$
20	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 20	relations for $\delta \in \{14, 15, 16\}$ and $[x_2^2 + x_1x_3]_N = 0$
21	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 21	relations for $\delta \in \{15, 16, 17\}$ and $[x_1^2 + x_3^3]_N = [x_2^2 + x_1x_3]_N = 0$
22	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 22	relations for $\delta \in \{16, 17, 18\}$ and $[x_1^2 + x_3^3]_N = [x_2^2 + x_1x_3]_N = 0$
23	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 23	relations for $\delta \in \{17, 18, 19\}$ and $[x_1^2 + x_3^3]_N = 0$
24	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 24	relations for $\delta \in \{18, 19, 20\}$ and $[x_1^2 + x_3^3]_N = 0$
25	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 25	relations for $\delta \in \{19, 20, 21\}$
$\delta > 25$	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree $\delta > 25$	relations for $\delta > 19$

TABLE 5. Relations towards calculating $[\Lambda^2(\mathbb{R}^{2n})]_N$ for $N = W_8$.

Using the method of algebraic restrictions and Table 5 we obtain the following proposition.

Proposition 5.9. *The space $[\Lambda^2(\mathbb{R}^{2n})]_{W_8}$ is a 9-dimensional vector space spanned by the algebraic restrictions to W_8 of the 2-forms:*

$$\theta_1 = dx_2 \wedge dx_3,\ \ \theta_2 = dx_1 \wedge dx_3,\ \ \theta_3 = dx_1 \wedge dx_2,\quad \theta_4 = x_3 dx_2 \wedge dx_3,\ \ \theta_5 = x_2 dx_2 \wedge dx_3,$$
$$\sigma_1 = x_1 dx_2 \wedge dx_3,\quad \sigma_2 = x_2 dx_1 \wedge dx_3,\quad \theta_7 = x_3^2 dx_2 \wedge dx_3,\quad \theta_8 = x_2^2 dx_2 \wedge dx_3.$$

Proposition 5.9 and results of Section 5.1 imply the following description of the space $[Z^2(\mathbb{R}^{2n})]_{W_8}$ and the manifold $[\mathrm{Symp}(\mathbb{R}^{2n})]_{W_8}$.

Theorem 5.10. *The space $[Z^2(\mathbb{R}^{2n})]_{W_8}$ is an 8-dimensional vector space spanned by the algebraic restrictions to W_8 of the quasi-homogeneous 2-forms θ_i of degree δ_i:*

$\theta_1 = dx_2 \wedge dx_3, \quad \delta_1 = 9,$

$\theta_2 = dx_1 \wedge dx_3, \quad \delta_2 = 10,$

$\theta_3 = dx_1 \wedge dx_2, \quad \delta_3 = 11,$

$\theta_4 = x_3 dx_2 \wedge dx_3, \quad \delta_4 = 13,$

$\theta_5 = x_2 dx_2 \wedge dx_3, \quad \delta_5 = 14,$

$\theta_6 = \sigma_1 + \sigma_2 = x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3, \quad \delta_6 = 15,$

$\theta_7 = x_3^2 dx_2 \wedge dx_3, \quad \delta_7 = 17,$

$\theta_8 = x_2^2 dx_2 \wedge dx_3, \quad \delta_8 = 19.$

If $n \geq 3$ then $[\mathrm{Symp}(\mathbb{R}^{2n})]_{W_8} = [Z^2(\mathbb{R}^{2n})]_{W_8}$. The manifold $[\mathrm{Symp}(\mathbb{R}^4)]_{W_8}$ is an open part of the 8-space $[Z^2(\mathbb{R}^4)]_{W_8}$ consisting of algebraic restrictions of the form $[c_1\theta_1 + \cdots + c_8\theta_8]_{W_8}$ such that $(c_1, c_2, c_3) \neq (0, 0, 0)$.

Theorem 5.11.

(i) Any algebraic restriction in $[Z^2(\mathbb{R}^{2n})]_{W_8}$ can be brought by a symmetry of W_8 to one of the normal forms $[W_8]^i$ given in the second column of Table 6.

(ii) The codimension in $[Z^2(\mathbb{R}^{2n})]_{W_8}$ of the singularity class corresponding to the normal form $[W_8]^i$ is equal to i, the symplectic multiplicity and the index of isotropy are given in the fourth and fifth columns of Table 6.

(iii) The singularity classes corresponding to the normal forms are disjoint.

(iv) The parameters c, c_1, c_2 of the normal forms $[W_8]^i$ are moduli.

In the first column of Table 6 we denote by $(W_8)^i$ a subclass of (W_8) consisting of $N \in (W_8)$ such that the algebraic restriction $[\omega]_N$ is diffeomorphic to some algebraic restriction of the normal form $[W_8]^i$, where i is the codimension of the class. Classes $(W_8)^{2a}$ and $(W_8)^{2b}$ have the same codimension equal to 2 but they can be distinguished geometrically (see Table 2).

The proof of Theorem 5.11 is presented in Section 5.2.3.

5.2.2. *Symplectic normal forms.* Let us transfer the normal forms $[W_8]^i$ to symplectic normal forms. Fix a family ω^i of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[W_8]^i$ of algebraic restrictions. We can fix, for example,

$\omega^0 = \theta_1 + c_1\theta_2 + c_2\theta_3 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n};$

$\omega^1 = c_1\theta_2 + \theta_3 + c_2\theta_4 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}, \quad c_1 \neq 0;$

$\omega^{2,a} = \pm\theta_2 + c_1\theta_4 + c_2\theta_7 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n};$

$\omega^{2,b} = \theta_3 + c_1\theta_4 + c_2\theta_5 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n};$

symplectic class	normal forms for algebraic restrictions	cod	$\mu^{\rm sym}$	ind
$(W_8)^0 \quad (2n \geq 4)$	$[W_8]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$,	0	2	0
$(W_8)^1 \quad (2n \geq 4)$	$[W_8]^1 : [c_1\theta_2 + \theta_3 + c_2\theta_4]_{W_8},\ c_1 \neq 0$	1	3	0
$(W_8)^{2,a} \quad (2n \geq 4)$	$[W_8]^{2,a} : [\pm\theta_2 + c_1\theta_4 + c_2\theta_7]_{W_8}$,	2	4	0
$(W_8)^{2,b} \quad (2n \geq 4)$	$[W_8]^{2,b} : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{W_8}$,	2	4	0
$(W_8)^3 \quad (2n \geq 6)$	$[W_8]^3 : [\theta_4 + c_1\theta_5 + c_2\theta_6]_{W_8}$	3	5	1
$(W_8)^4 \quad (2n \geq 6)$	$[W_8]^4 : [\pm\theta_5 + c_1\theta_6 + c_2\theta_7]_{W_8}$	4	6	1
$(W_8)^5 \quad (2n \geq 6)$	$[W_8]^5 : [\theta_6 + c\theta_7]_{W_8}$	5	6	1
$(W_8)^6 \quad (2n \geq 6)$	$[W_8]^6 : [\theta_7 + c\theta_8]_{W_8}$	6	7	2
$(W_8)^7 \quad (2n \geq 6)$	$[W_8]^7 : [\theta_8]_{W_8}$	7	7	2
$(W_8)^8 \quad (2n \geq 6)$	$[W_8]^8 : [0]_{W_8}$	8	8	∞

TABLE 6. Classification of symplectic W_8 singularities.
cod – codimension of the classes; μ^{sym}– symplectic multiplicity;
ind – the index of isotropy.

$$\omega^3 = \theta_4 + c_1\theta_5 + c_2\theta_6 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$$
$$\omega^4 = \pm\theta_5 + c_1\theta_6 + c_2\theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$$
$$\omega^5 = \theta_6 + c\theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$$
$$\omega^6 = \theta_7 + c\theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$$
$$\omega^7 = \theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$$
$$\omega^8 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

Let $\omega_0 = \sum_{i=1}^{m} dp_i \wedge dq_i$, where $(p_1, q_1, \cdots, p_n, q_n)$ is the coordinate system on $\mathbb{R}^{2n}, n \geq 3$ (resp. $n = 2$). Fix, for $i = 0, 1, \cdots, 8$ (resp. for $i = 0, 1, 2$) a family Φ^i of local diffeomorphisms which bring the family of symplectic forms ω^i to the symplectic form ω_0: $(\Phi^i)^*\omega^i = \omega_0$. Consider the families $W_8^i = (\Phi^i)^{-1}(W_8)$. Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ which is diffeomorphic to W_8 is symplectically equivalent to one and only one of the normal forms $W_8^i, i = 0, 1, \cdots, 8$ (resp. $i = 0, 1, 2$) presented in Theorem 3.1. By Theorem 5.11 we obtain that parameters c, c_1, c_2 of the normal forms are moduli.

5.2.3. *Proof of Theorem 5.11.* In our proof we use vector fields tangent to $N \in W_8$. Any vector fields tangent to $N \in W_8$ can be described as $V = g_1E + g_2\mathcal{H}$ where E is the Euler vector field and $\mathcal{H}$ is a Hamiltonian vector field and g_1, g_2 are functions. It was shown in [DT1] (Prop. 6.13) that the action of a Hamiltonian vector field on the algebraic restriction of a closed 2-form to any 1-dimensional complete intersection is trivial.

The germ of a vector field tangent to W_8 of non trivial action on algebraic restrictions of closed 2-forms to W_8 may be described as a linear combination of germs of vector fields: $X_0 = E,\ X_1 = x_3E,\ X_2 = x_2E,\ X_3 = x_1E,\ X_4 = x_3^2E,\ X_5 = x_2x_3E,\ X_6 = x_2^2E,\ X_7 = x_1x_3E$, where E is the Euler vector field

$$E = 6x_1\frac{\partial}{\partial x_1} + 5x_2\frac{\partial}{\partial x_2} + 4x_3\frac{\partial}{\partial x_3}. \tag{5.3}$$

Proposition 5.12. *The infinitesimal action of germs of quasi-homogeneous vector fields tangent to $N \in (W_8)$ on the basis of the vector space of algebraic restrictions of closed 2-forms to N is presented in Table 7.*

$\mathcal{L}_{X_i}[\theta_j]$	$[\theta_1]$	$[\theta_2]$	$[\theta_3]$	$[\theta_4]$	$[\theta_5]$	$[\theta_6]$	$[\theta_7]$	$[\theta_8]$
$X_0=E$	$9[\theta_1]$	$10[\theta_2]$	$11[\theta_3]$	$13[\theta_4]$	$14[\theta_5]$	$15[\theta_6]$	$17[\theta_7]$	$19[\theta_8]$
$X_1=x_3E$	$13[\theta_4]$	$-28[\theta_5]$	$5[\theta_6]$	$17[\theta_7]$	$[0]$	$-57[\theta_8]$	$[0]$	$[0]$
$X_2=x_2E$	$14[\theta_5]$	$10[\theta_6]$	$[0]$	$[0]$	$19[\theta_8]$	$[0]$	$[0]$	$[0]$
$X_3=x_1E$	$5[\theta_6]$	$[0]$	$\frac{51}{2}[\theta_7]$	$-19[\theta_8]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_4=x_3^2E$	$17[\theta_7]$	$[0]$	$-19[\theta_8]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_5=x_2x_3E$	$[0]$	$-38[\theta_8]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_6=x_2^2E$	$19[\theta_8]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_7=x_1x_3E$	$-19[\theta_8]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$

TABLE 7. Infinitesimal actions on algebraic restrictions of closed 2-forms to W_8. (E is defined as in (5.3.))

Let $\mathcal{A} = [c_1\theta_1 + c_2\theta_2 + c_3\theta_3 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7 + c_8\theta_8]_{W_8}$ be the algebraic restriction of a symplectic form ω.

The first statement of Theorem 5.11 follows from the following lemmas.

Lemma 5.13. *If $c_1 \neq 0$ then the algebraic restriction $\mathcal{A} = [\sum_{k=1}^{8} c_k\theta_k]_{W_8}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{W_8}$.*

Proof. Using the data of Table 7, we can see that for any algebraic restriction $[\theta_k]_{W_8}$, where $k \in \{4, 5, \dots, 8\}$ we can find a vector field V_k tangent to W_8 such that $\mathcal{L}_{V_k}[\theta_1]_{W_8} = [\theta_k]_{W_8}$. We deduce from Proposition 5.5 that the algebraic restriction $\mathcal{A}$ is diffeomorphic to $[c_1\theta_1 + c_2\theta_2 + c_3\theta_3]_{W_8}$.

By the condition $c_1 \neq 0$, we have a diffeomorphism $\Psi \in Symm(W_8)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (c_1^{-\frac{6}{9}}x_1, c_1^{-\frac{5}{9}}x_2, c_1^{-\frac{4}{9}}x_3) \tag{5.4}$$

and finally we obtain

$$\Psi^*([c_1\theta_1 + c_2\theta_2 + c_3\theta_3]_{W_8}) = [\theta_1 + c_2c_1^{-\frac{10}{9}}\theta_2 + c_3c_1^{-\frac{11}{9}}\theta_3]_{W_8} = [\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{W_8}.$$

□

Lemma 5.14. *If $c_1 = 0$ and $c_2 \cdot c_3 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\widetilde{c}_2\theta_2 + \theta_3 + \widetilde{c}_4\theta_4]_{W_8}$.*

Proof of Lemma 5.14. We use the homotopy method to prove that $\mathcal{A}$ is diffeomorphic to $[\widetilde{c}_2\theta_2 + \theta_3 + \widetilde{c}_4\theta_4]_{W_8}$.

Let $\mathcal{B}_t = [c_2\theta_2 + c_4\theta_3 + c_4\theta_4 + (1-t)c_5\theta_5 + (1-t)c_6\theta_6 + (1-t)c_7\theta_7 + (1-t)c_8\theta_8]_{W_8}$ for $t \in [0;1]$. Then $\mathcal{B}_0 = \mathcal{A}$ and $\mathcal{B}_1 = [c_2\theta_2 + c_3\theta_3 + c_4\theta_4]_{W_8}$. We prove that there exists a family $\Phi_t \in Symm(W_8)$, $t \in [0;1]$ such that

$$\Phi_t^*\mathcal{B}_t = \mathcal{B}_0,\ \Phi_0 = id. \tag{5.5}$$

Let V_t be a vector field defined by $\frac{d\Phi_t}{dt} = V_t(\Phi_t)$. Then, by differentiating (5.5) we obtain

$$\mathcal{L}_{V_t}\mathcal{B}_t = [c_5\theta_5 + c_6\theta_6 + c_7\theta_7 + c_8\theta_8]_{W_8}. \tag{5.6}$$

We are looking for V_t in the form $V_t = \sum_{k=1}^{5} b_k(t)X_k$ where the $b_k(t)$ for $k = 1, \ldots, 5$ are smooth functions $b_k : [0;1] \to \mathbb{R}$. Then, by Proposition 5.12, equation (5.6) has the form

$$\begin{bmatrix} -28c_2 & 0 & 0 & 0 & 0 \\ 5c_3 & 10c_2 & 0 & 0 & 0 \\ 17c_4 & 0 & \frac{51}{2}c_3 & 0 & 0 \\ -57c_6(1-t) & 19c_5(1-t) & -19c_4 & -19c_3 & -38c_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix}. \tag{5.7}$$

If $c_2 \cdot c_3 \neq 0$ we can solve (5.7) and Φ_t may be obtained as a flow of the vector field V_t. The family Φ_t preserves W_8, because V_t is tangent to W_8 and $\Phi_t^*\mathcal{B}_t = \mathcal{A}$. Using the homotopy arguments, we have that $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1 = [c_2\theta_2 + c_3\theta_3 + c_4\theta_4]_{W_8}$. By the condition $c_3 \neq 0$, we have a diffeomorphism $\Psi \in Symm(W_8)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (c_3^{-\frac{6}{11}}x_1, c_3^{-\frac{5}{11}}x_2, c_3^{-\frac{4}{11}}x_3), \tag{5.8}$$

and we obtain

$$\Psi^*(\mathcal{B}_1) = [c_2c_3^{-\frac{10}{11}}\theta_2 + \theta_3 + c_4c_3^{-\frac{13}{11}}\theta_3]_{W_8} = [\widetilde{c}_2\theta_2 + \theta_3 + \widetilde{c}_4\theta_4]_{W_8}.$$

□

Lemma 5.15. *If $c_1 = c_3 = 0$ and $c_2 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\pm\theta_2 + \widetilde{c}_4\theta_4 + \widetilde{c}_7\theta_7]_{W_8}$.*

Proof. We can see from Table 7 that for any algebraic restriction $[\theta_k]_{W_8}$, where $k \in \{5, 6, 8\}$ there exists a vector field V_k tangent to W_8 such that $\mathcal{L}_{V_k}[\theta_2]_{W_8} = [\theta_k]_{W_8}$. Using Proposition 5.5 we obtain that $\mathcal{A}$ is diffeomorphic to $[c_2\theta_2 + c_4\theta_4 + \widehat{c}_7\theta_7]_{W_8}$ for some $\widehat{c}_7 \in \mathbb{R}$.

By the condition $c_2 \neq 0$ we can use a diffeomorphism $\Psi \in Symm(W_8)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (|c_2|^{-\frac{6}{10}}x_1, |c_2|^{-\frac{5}{10}}x_2, |c_2|^{-\frac{4}{10}}x_3) \tag{5.9}$$

and we obtain

$$\Psi^*([c_2\theta_2 + c_4\theta_4 + \widehat{c}_7\theta_7]_{W_8}) = [\frac{c_2}{|c_2|}\theta_2 + c_4|c_2|^{-\frac{13}{10}}\theta_4 + \widehat{c}_7|c_2|^{-\frac{17}{10}}\theta_7]_{W_8} = [\pm\theta_2 + \widetilde{c}_4\theta_4 + \widetilde{c}_7\theta_7]_{W_8}.$$

The algebraic restrictions $[\theta_2 + \widetilde{c}_4\theta_4 + \widetilde{c}_7\theta_7]_{W_8}$ and $[-\theta_2 + \widetilde{b}_4\theta_4 + \widetilde{b}_7\theta_7]_{W_8}$ are not diffeomorphic. Any diffeomorphism $\Phi = (\Phi_1, \ldots, \Phi_{2n})$ of $(\mathbb{R}^{2n}, 0)$ preserving W_8 has to preserve a curve $C(t) = (t^6, t^5, -t^4, 0, \ldots, 0)$ which means that

$\Phi_1(t^6, t^5, -t^4, 0, \ldots, 0) = (\psi(t))^6$,
$\Phi_2(t^6, t^5, -t^4, 0, \ldots, 0) = (\psi(t))^5$,
$\Phi_3(t^6, t^5, -t^4, 0, \ldots, 0) = -(\psi(t))^4$,
$\Phi_k(t^6, t^5, -t^4, 0, \ldots, 0) = 0$ for $k > 3$,

where $\psi(t) = a_1t + a_2t^2 + a_3t^3 + \ldots$ is a diffeomorphism of $(\mathbb{R}, 0)$.

Hence Φ has a linear part

$$(5.10)\qquad \begin{array}{lcccccccccc}
\Phi_1: & A^6x_1 & + & & & & & A_{14}x_4 & + \cdots + & A_{1,2n}x_{2n} \\
\Phi_2: & A_{2,1}x_1 & + & A^5x_2 & & & + & A_{24}x_4 & + \cdots + & A_{2,2n}x_{2n} \\
\Phi_3: & A_{3,1}x_1 & + & A_{3,2}x_2 & + & A^4x_3 & + & A_{34}x_4 & + \cdots + & A_{3,2n}x_{2n} \\
\Phi_4: & & & & & & & A_{44}x_4 & + \cdots + & A_{4,2n}x_{2n} \\
\vdots & & & & & & & \vdots & \vdots \quad \vdots \quad \vdots & \\
\Phi_{2n}: & & & & & & & A_{2n,4}x_4 & + \cdots + & A_{2n,2n}x_{2n},
\end{array}$$

where $A, A_{i,j} \in \mathbb{R}$.
If we assume that $\Phi^*([\theta_2 + \widetilde{c}_4\theta_4 + \widetilde{c}_7\theta_7]_{W_8}) = [-\theta_2 + \widetilde{b}_4\theta_4 + \widetilde{b}_7\theta_7]_{W_8}$, then $A^{10}dx_1 \wedge dx_3|_0 = -dx_1 \wedge dx_3|_0$, which is a contradiction. $\square$

Lemma 5.16. *If $c_1 = c_2 = 0$ and $c_3 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_3 + \widetilde{c}_4\theta_4 + \widetilde{c}_5\theta_5]_{W_8}$.*

Lemma 5.17. *If $c_1 = c_2 = c_3 = 0$ and $c_4 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_4 + \widetilde{c}_5\theta_5 + \widetilde{c}_6\theta_6]_{W_8}$.*

Lemma 5.18. *If $c_1 = 0, \ldots, c_4 = 0$ and $c_5 \neq 0$, then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\pm\theta_5 + \widetilde{c}_6\theta_6 + \widetilde{c}_7\theta_7]_{W_8}$.*

Lemma 5.19. *If $c_1 = 0, \ldots, c_5 = 0$ and $c_6 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_6 + \widetilde{c}_7\theta_7]_{W_8}$.*

Lemma 5.20. *If $c_1 = 0, \ldots, c_6 = 0$ and $c_7 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_7 + \widetilde{c}_8\theta_8]_{W_8}$.*

Lemma 5.21. *If $c_1 = 0, \ldots, c_7 = 0$ and $c_8 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_8]_{W_8}$.*

The proofs of Lemmas 5.16 – 5.21 are similar to the proofs of Lemmas 5.13 – 5.15 and are based on Table 7.

Statement (ii) of Theorem 5.11 follows from the conditions in the proof of part (i) (the codimension) and from Theorem 5.4 and Proposition 5.6 (the symplectic multiplicity) and Proposition 5.7 (the index of isotropy).

To prove statement (iii) of Theorem 5.11 we have to show that singularity classes corresponding to normal forms are disjoint. The singularity classes that can be distinguished by geometric conditions obviously are disjoint. From Theorem 3.5 we see that only classes $(W_8)^1$ and $(W_8)^{2,a}$ can not be distinguished by the geometric conditions but their symplectic multiplicities are distinct , hence the classes are disjoint.

To prove statement (iv) of Theorem 5.11 we have to show that the parameters c, c_1, c_2 are moduli in the normal forms. The proofs are very similar in all cases. We consider as an example the normal form with two parameters $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$. From Table 7 we see that the tangent space to the orbit of $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$ at $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$ is spanned by the linearly independent algebraic restrictions $[9\theta_1 + 10c_1\theta_2 + 11c_2\theta_3]_{W_8}$, $[\theta_4]_{W_8}, [\theta_5]_{W_8}, [\theta_6]_{W_8}, [\theta_7]_{W_8}$ and $[\theta_8]_{W_8}$. Hence, the algebraic restrictions $[\theta_2]_{W_8}$ and $[\theta_3]_{W_8}$ do not belong to it. Therefore, the parameters c_1 and c_2 are independent moduli in the normal form $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$.

5.3. **Proofs for W_9 singularity.**

5.3.1. *Algebraic restrictions to W_9 and their classification.*
One has the following relations for (W_9)-singularities

$$[d(x_1^2 + x_2x_3^2)]_{W_9} = [2x_1dx_1 + 2x_2x_3dx_3 + x_3^2dx_2]_{W_9} = 0, \tag{5.11}$$

$$[d(x_2^2 + x_1x_3)]_{W_9} = [2x_2dx_2 + x_3dx_1 + x_1dx_3]_{W_9} = 0. \tag{5.12}$$

Multiplying these relations by suitable 1-forms we obtain the relations in Table 8.

δ	relations	proof
11	$[x_3dx_1 \wedge dx_3]_N = -2[x_2dx_2 \wedge dx_3]_N$	(5.12)$\wedge dx_3$
12	$[x_1dx_2 \wedge dx_3]_N = [x_3dx_1 \wedge dx_2]_N$	(5.12)$\wedge dx_2$
13	$[x_3^2dx_2 \wedge dx_3]_N = -2[x_1dx_1 \wedge dx_3]_N = 4[x_2dx_1 \wedge dx_1]_N$	(5.11)$\wedge dx_3$ and (5.12)$\wedge dx_1$
14	$[x_3^2dx_1 \wedge dx_3]_N = -2[x_1dx_1 \wedge dx_2]_N = -2[x_2x_3dx_2 \wedge dx_3]_N$	(5.11)$\wedge dx_2$, (5.12)$\wedge x_3dx_3$
15	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 15	relations for $\delta \in \{11, 12\}$ and (5.11)$\wedge dx_1$ and $[x_2^2 + x_1x_3]_N = 0$
16	$[x_3^3dx_2 \wedge dx_3]_N = -2[x_1x_3dx_1 \wedge dx_3]_N = 4[x_2x_3dx_1 \wedge dx_2]_N$ $[x_1x_3dx_1 \wedge dx_3]_N = -2[x_1x_2dx_2 \wedge dx_3]_N = -[x_2^2dx_1 \wedge dx_3]_N$	relations for $\delta = 13$ relations for $\delta \in \{11, 12\}$ and $[x_2^2 + x_1x_3]_N = 0$
17	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 17	relations for $\delta \in \{12, 13, 14\}$ and $[x_1^2 + x_2x_3^2]_N = 0$ i $[x_2^2 + x_1x_3]_N = 0$
18	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 18	relations for $\delta \in \{13, 14, 15\}$ and $[x_1^2 + x_2x_3^2]_N = 0$
19	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 19	relations for $\delta \in \{14, 15, 16\}$ and $[x_1^2 + x_2x_3^2]_N = 0$
20	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 20	relations for $\delta \in \{15, 16, 17\}$
21	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 21	relations for $\delta \in \{16, 17, 18\}$
>21	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree $\delta > 21$	relations for $\delta > 16$

TABLE 8. Relations towards calculating $[\Lambda^2(\mathbb{R}^{2n})]_N$ for $N = W_9$.

Using the method of algebraic restrictions and Table 8 we obtain the following proposition:

Proposition 5.22. *The space $[\Lambda^2(\mathbb{R}^{2n})]_{W_9}$ is a 10-dimensional vector space spanned by the algebraic restrictions to W_9 of the 2-forms*
$\theta_1 = dx_2 \wedge dx_3$, $\theta_2 = dx_1 \wedge dx_3$, $\theta_3 = dx_1 \wedge dx_2$,
$\theta_4 = x_3dx_2 \wedge dx_3$, $\theta_5 = x_3dx_1 \wedge dx_3$, $\sigma_1 = x_1dx_2 \wedge dx_3$, $\sigma_2 = x_2dx_1 \wedge dx_3$,
$\theta_7 = x_3^2dx_2 \wedge dx_3$, $\theta_8 = x_3^2dx_1 \wedge dx_3$, $\theta_9 = x_3^3dx_2 \wedge dx_3$.

Proposition 5.22 and results of Section 5.1 imply the following description of the space $[Z^2(\mathbb{R}^{2n})]_{W_9}$ and the manifold $[\mathrm{Symp}(\mathbb{R}^{2n})]_{W_9}$.

Theorem 5.23. *The space $[Z^2(\mathbb{R}^{2n})]_{W_9}$ is a 9-dimensional vector space spanned by the algebraic restrictions to W_9 of the quasi-homogeneous 2-forms θ_i of degree δ_i*

$\theta_1 = dx_2 \wedge dx_3$, $\delta_1 = 7$,

$\theta_2 = dx_1 \wedge dx_3$, $\delta_2 = 8$,

$\theta_3 = dx_1 \wedge dx_2$, $\delta_3 = 9$,

$\theta_4 = x_3 dx_2 \wedge dx_3, \quad \delta_4 = 10,$

$\theta_5 = x_3 dx_1 \wedge dx_3, \quad \delta_5 = 11,$

$\theta_6 = \sigma_1 + \sigma_2 = x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3, \quad \delta_6 = 12,$

$\theta_7 = x_3^2 dx_2 \wedge dx_3, \quad \delta_7 = 13,$

$\theta_8 = x_3^2 dx_1 \wedge dx_3, \quad \delta_8 = 14,$

$\theta_9 = x_3^3 dx_2 \wedge dx_3, \quad \delta_8 = 16,$

If $n \geq 3$ then $[\mathrm{Symp}(\mathbb{R}^{2n})]_{W_9} = [Z^2(\mathbb{R}^{2n})]_{W_9}$. The manifold $[\mathrm{Symp}(\mathbb{R}^4)]_{W_9}$ is an open part of the 9-space $[Z^2(\mathbb{R}^4)]_{W_9}$ consisting of algebraic restrictions of the form $[c_1\theta_1 + \cdots + c_9\theta_9]_{W_9}$ such that $(c_1, c_2, c_3) \neq (0,0,0)$.

Theorem 5.24.

(i) Any algebraic restriction in $[Z^2(\mathbb{R}^{2n})]_{W_9}$ can be brought by a symmetry of W_9 to one of the normal forms $[W_9]^i$ given in the second column of Table 9.

(ii) The codimension in $[Z^2(\mathbb{R}^{2n})]_{W_9}$ of the singularity class corresponding to the normal form $[W_9]^i$ is equal to i, the symplectic multiplicity and the index of isotropy are given in the fourth and fifth columns of Table 9.

(iii) The singularity classes corresponding to the normal forms are disjoint.

(iv) The parameters c, c_1, c_2 of the normal forms $[W_9]^i$ are moduli.

symplectic class	normal forms for algebraic restrictions	cod	μ^{sym}	ind
$(W_9)^0 \quad (2n \geq 4)$	$[W_9]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$,	0	2	0
$(W_9)^1 \quad (2n \geq 4)$	$[W_9]^1 : [\pm\theta_2 + c_1\theta_3 + c_2\theta_4]_{W_9}$	1	3	0
$(W_9)^2 \quad (2n \geq 4)$	$[W_9]^2 : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{W_9}$,	2	4	0
$(W_9)^3 \quad (2n \geq 6)$	$[W_9]^3 : [\pm\theta_4 + c_1\theta_5 + c_2\theta_6]_{W_9}$,	3	5	1
$(W_9)^4 \quad (2n \geq 6)$	$[W_9]^4 : [\theta_5 + c_1\theta_6 + c_2\theta_7]_{W_9}$	4	6	1
$(W_9)^5 \quad (2n \geq 6)$	$[W_9]^5 : [\pm\theta_6 + c_1\theta_7 + c_2\theta_8]_{W_9}$	5	7	1
$(W_9)^6 \quad (2n \geq 6)$	$[W_9]^6 : [\theta_7 + c\theta_8]_{W_9}$	6	7	2
$(W_9)^7 \quad (2n \geq 6)$	$[W_9]^7 : [\pm\theta_8 + c\theta_9]_{W_9}$	7	8	2
$(W_9)^8 \quad (2n \geq 6)$	$[W_9]^8 : [\pm\theta_9]_{W_9}$	8	8	3
$(W_9)^9 \quad (2n \geq 6)$	$[W_9]^9 : [0]_{W_9}$	9	9	∞

TABLE 9. Classification of symplectic W_9 singularities.
(*cod* – codimension of the classes; μ^{sym}– the symplectic multiplicity;
ind – the index of isotropy.)

In the first column of Table 9 by $(W_9)^i$ we denote a subclass of (W_9) consisting of $N \in (W_9)$ such that the algebraic restriction $[\omega]_N$ is diffeomorphic to some algebraic restriction of the normal form $[W_9]^i$.

The proof of Theorem 5.24 is presented in Section 5.3.3.

5.3.2. *Symplectic normal forms.*

Let us transfer the normal forms $[W_9]^i$ to symplectic normal forms. Fix a family ω^i of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[W_9]^i$ of algebraic restrictions. We can fix, for example

$\omega^0 = \theta_1 + c_1\theta_2 + c_2\theta_3 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^1 = \pm\theta_2 + c_1\theta_3 + c_2\theta_4 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^2 = \theta_3 + c_1\theta_4 + c_2\theta_5 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^3 = \pm\theta_4 + c_1\theta_5 + c_2\theta_6 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^4 = \theta_5 + c_1\theta_6 + c_2\theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^5 = \pm\theta_6 + c_1\theta_7 + c_2\theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^6 = \theta_7 + c\theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^7 = \pm\theta_8 + c\theta_9 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^8 = \pm\theta_9 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^9 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$.

Let $\omega_0 = \sum_{i=1}^m dp_i \wedge dq_i$, where $(p_1, q_1, \cdots, p_n, q_n)$ is the coordinate system on $\mathbb{R}^{2n}, n \geq 3$ (resp. $n = 2$). Fix, for $i = 0, 1, \cdots, 9$ (resp. for $i = 0, 1, 2$) a family Φ^i of local diffeomorphisms which bring the family of symplectic forms ω^i to the symplectic form ω_0: $(\Phi^i)^*\omega^i = \omega_0$. Consider the families $W_9^i = (\Phi^i)^{-1}(W_8)$. Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ which is diffeomorphic to W_9 is symplectically equivalent to one and only one of the normal forms $W_9^i, i = 0, 1, \cdots, 9$ (resp. $i = 0, 1, 2$) presented in Theorem 4.1. By Theorem 5.24 we obtain that parameters c, c_1, c_2 of the normal forms are moduli.

5.3.3. *Proof of Theorem 5.24.*

In our proof we use vector fields tangent to $N \in W_9$.
The germ of a vector field tangent to W_8 of non trivial action on algebraic restrictions of closed 2-forms to W_9 may be described as a linear combination of germs of the following vector fields: $X_0 = E,\ X_1 = x_3E,\ X_2 = x_2E,\ X_3 = x_1E,\ X_4 = x_3^2E, X_5 = x_2x_3E,\ X_6 = x_2^2E,\ X_7 = x_1x_3E,$ $X_8 = x_1x_2E,\ X_9 = x_3^3E,$
where E is the Euler vector field

$$E = 5x_1\frac{\partial}{\partial x_1} + 4x_2\frac{\partial}{\partial x_2} + 3x_3\frac{\partial}{\partial x_3}. \tag{5.13}$$

Proposition 5.25. *The infinitesimal action of germs of quasi-homogeneous vector fields tangent to $N \in (W_9)$ on the basis of the vector space of algebraic restrictions of closed 2-forms to N is presented in Table 10.*

$\mathcal{L}_{X_i}[\theta_j]$	$[\theta_1]$	$[\theta_2]$	$[\theta_3]$	$[\theta_4]$	$[\theta_5]$	$[\theta_6]$	$[\theta_7]$	$[\theta_8]$	$[\theta_9]$
$X_0 = E$	$7[\theta_1]$	$8[\theta_2]$	$9[\theta_3]$	$10[\theta_4]$	$11[\theta_5]$	$12[\theta_6]$	$13[\theta_7]$	$14[\theta_8]$	$16[\theta_9]$
$X_1 = x_3E$	$10[\theta_4]$	$11[\theta_5]$	$4[\theta_6]$	$13[\theta_7]$	$14[\theta_8]$	$[0]$	$16[\theta_9]$	$[0]$	$[0]$
$X_2 = x_2E$	$-\frac{11}{2}[\theta_5]$	$8[\theta_6]$	$\frac{13}{4}[\theta_7]$	$-7[\theta_8]$	$[0]$	$12[\theta_9]$	$[0]$	$[0]$	$[0]$
$X_3 = x_1E$	$4[\theta_6]$	$-\frac{13}{2}[\theta_7]$	$-7[\theta_8]$	$[0]$	$-8[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_4 = x_3^2E$	$13[\theta_7]$	$14[\theta_8]$	$[0]$	$16[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_5 = x_2x_3E$	$-7[\theta_8]$	$[0]$	$4[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_6 = x_2^2E$	$[0]$	$8[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_7 = x_1x_3E$	$[0]$	$-8[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_8 = x_1x_2E$	$4[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_9 = x_3^3E$	$16[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$

TABLE 10. Infinitesimal actions on algebraic restrictions of closed 2-forms to W_9. (E is defined as in (5.13).)

Let $\mathcal{A} = [\sum_{k=1}^{9} \theta_k]_{W_9}$ be the algebraic restriction of a symplectic form ω.

The first statement of Theorem 5.24 follows from the following lemmas.

Lemma 5.26. *If $c_1 \neq 0$ then the algebraic restriction $\mathcal{A} = [\sum_{k=1}^{9} c_k\theta_k]_{W_9}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{W_9}$.*

Lemma 5.27. *If $c_1 = 0$ and $c_2 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\pm\theta_2 + \widetilde{c}_3\theta_3 + \widetilde{c}_4\theta_4]_{W_9}$.*

Lemma 5.28. *If $c_1 = c_2 = 0$ and $c_3 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\theta_3 + \widetilde{c}_4\theta_4 + \widetilde{c}_5\theta_5]_{W_9}$.*

Lemma 5.29. *If $c_1 = c_2 = c_3 = 0$ and $c_4 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\pm\theta_4 + \widetilde{c}_5\theta_5 + \widetilde{c}_6\theta_6]_{W_9}$.*

Lemma 5.30. *If $c_1 = \ldots = c_4 = 0$ and $c_5 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\theta_5 + \widetilde{c}_6\theta_6 + \widetilde{c}_7\theta_7]_{W_9}$.*

Lemma 5.31. *If $c_1 = \ldots = c_5 = 0$ and $c_6 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\pm\theta_6 + \widetilde{c}_7\theta_7 + \widetilde{c}_8\theta_8]_{W_9}$.*

Lemma 5.32. *If $c_1 = \ldots = c_6 = 0$ and $c_7 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\theta_7 + \widetilde{c}_8\theta_8]_{W_9}$.*

Lemma 5.33. *If $c_1 = \ldots = c_7 = 0$ and $c_8 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\pm\theta_8 + \widetilde{c}_9\theta_9]_{W_9}$.*

Lemma 5.34. *If $c_1 = \ldots = c_8 = 0$ and $c_9 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\pm\theta_9]_{W_9}$.*

The proofs of Lemmas 5.26 – 5.34 are similar to the proofs of the lemmas for the W_8 singularity. As an example we give the proof of Lemma 5.27.

Proof of Lemma 5.27. We see from Table 10 that for any algebraic restriction $[\theta_k]_{W_9}$, where $k \in \{5,6,7,8,9\}$, there exists a vector field V_k tangent to W_9 such that $\mathcal{L}_{V_k}[\theta_2]_{W_9} = [\theta_k]_{W_9}$. Using Proposition 5.5, we obtain that $\mathcal{A}$ is diffeomorphic to $[c_2\theta_2 + c_3\theta_3 + c_4\theta_4]_{W_9}$.

By the condition $c_2 \neq 0$, we have a diffeomorphism $\Psi \in Symm(W_9)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (|c_2|^{-\frac{5}{8}} x_1, |c_2|^{-\frac{4}{8}} x_2, |c_2|^{-\frac{3}{8}} x_3) \tag{5.14}$$

and we obtain

$$\Psi^*([c_2\theta_2 + c_3\theta_3 + c_4\theta_4]_{W_9}) = [\frac{c_2}{|c_2|}\theta_2 + c_3|c_2|^{-\frac{9}{8}}\theta_3 + c_4|c_2|^{-\frac{10}{8}}\theta_4]_{W_9} = [\pm\theta_2 + \widetilde{c}_3\theta_3 + \widetilde{c}_4\theta_4]_{W_9}.$$

The algebraic restrictions $[\theta_2 + \widetilde{c}_3\theta_3 + \widetilde{c}_4\theta_4]_{W_9}$ and $[-\theta_2 + \widetilde{b}_3\theta_3 + \widetilde{b}_4\theta_4]_{W_9}$ are not diffeomorphic. Any diffeomorphism $\Phi = (\Phi_1, \ldots, \Phi_{2n})$ of $(\mathbb{R}^{2n}, 0)$ preserving W_9 has to preserve a curve $C_2(t) = (t^5, -t^4, -t^3, 0, \ldots, 0)$. Hence, Φ has a linear part

$$\begin{array}{lccccccccccc}
\Phi_1 : & A^5 x_1 & + & & & & & A_{14}x_4 & + & \cdots & + & A_{1,2n}x_{2n} \\
\Phi_2 : & A_{2,1}x_1 & + & A^4 x_2 & + & & & A_{24}x_4 & + & \cdots & + & A_{2,2n}x_{2n} \\
\Phi_3 : & A_{3,1}x_1 & + & A_{3,2}x_2 & + & A^3 x_3 & + & A_{34}x_4 & + & \cdots & + & A_{3,2n}x_{2n} \\
\Phi_4 : & & & & & & & A_{44}x_4 & + & \cdots & + & A_{4,2n}x_{2n} \\
\vdots & & & & & & & \vdots & \vdots & \vdots & \vdots & \\
\Phi_{2n} : & & & & & & & A_{2n,4}x_4 & + & \cdots & + & A_{2n,2n}x_{2n}
\end{array} \tag{5.15}$$

where $A, A_{i,j} \in \mathbb{R}$.
If we assume that $\Phi^*([\theta_2 + \widetilde{c}_3\theta_3 + \widetilde{c}_4\theta_4]_{W_9}) = [-\theta_2 + \widetilde{b}_3\theta_3 + \widetilde{b}_4\theta_4]_{W_9}$, then
$A^8 dx_1 \wedge dx_3|_0 = -dx_1 \wedge dx_3|_0$, which is a contradiction. □

Statement (ii) of Theorem 5.24 follows from the conditions in the proof of part (i) (the codimension) and from Theorem 5.4 and Proposition 5.6 (the symplectic multiplicity) and Proposition 5.7 (the index of isotropy).

Statement (iii) of Theorem 5.24 follows from Theorem 4.4. The singularity classes corresponding to normal forms are disjoint because they can be distinguished by the geometric conditions.

To prove statement (iv) of Theorem 5.24 we have to show that the parameters c, c_1, c_2 are moduli in the normal forms. The proofs are very similar in all cases. We consider as an example the normal form with two parameters $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$. From Table 10 we see that the tangent space to the orbit of $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$ at $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$ is spanned by the linearly independent algebraic restrictions $[7\theta_1 + 8c_1\theta_2 + 9c_2\theta_3]_{W_9}$, $[\theta_4]_{W_9}, [\theta_5]_{W_9}, [\theta_6]_{W_9}, [\theta_7]_{W_9}, [\theta_8]_{W_9}$ and $[\theta_9]_{W_9}$. Hence, the algebraic restrictions $[\theta_2]_{W_9}$ and $[\theta_3]_{W_9}$ do not belong to it. Therefore, the parameters c_1 and c_2 are independent moduli in the normal form $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$.

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