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BRIANÇON-SPEDER EXAMPLES AND THE FAILURE OF WEAK WHITNEY REGULARITY

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1. Introduction

In [3, 5] we introduced a weakened form of Whitney's condition (b), motivated by the work of M. Ferrarotti on metric properties of Whitney stratified sets [11, 12]. The resulting weakly Whitney stratified sets retain many properties of Whitney stratified sets, in particular they are locally topologically trivial along strata [26, 16], because they are Bekka (c)-regular (see section 5) and so they have the structure of abstract stratified sets [3, 4], and thus are triangulable [14]. Weakly Whitney stratified sets also have many metric properties known to hold for Whitney stratified sets [7]. Orro and Trotman [20], Parusiński [23], Pflaum [24], and Schürmann [25] have described and developed further useful properties of weakly Whitney stratified sets.

There are real algebraic varieties with weakly Whitney regular stratifications which are not Whitney regular, and we give such an example in section 3 below. No examples are known among complex analytic varieties however, so that the natural question arises : do Whitney regularity and weak Whitney regularity coincide in the complex case? As a test, in this paper we study the weak Whitney regularity of the well-known Briançon-Speder examples, consisting of Milnor number constant families of complex surface singularities in $\mathbb{C}^3$ which are not Whitney regular [9], although they are (c)-regular because they are weighted homogeneous with constant weights.

We investigate systematically all of these (infinitely many) Briançon-Speder examples, and establish in particular that none of the examples are weakly Whitney regular. We determine **all** the complex curves along which Whitney (b)-regularity fails and **all** the complex curves along which weak Whitney regularity fails. It turns out that for each example there are a finite number of curves γ_i with the property that weak Whitney regularity fails along every curve tangent to one of the γ_i at the origin, while weak Whitney regularity holds along all other curves. For example, the classical Briançon-Speder example

$$f_t(x,y,z) = x^5 + txy^6 + y^7z + z^{15}$$

for which $\mu(f_t) = 364$, has 16 such curves $\gamma_1, \dots, \gamma_{16}$, where each $\gamma_i(s)$ is of the form

$$(s^8, as^5, 4a^{-7}s^5, -5a^{-6}s^2) \in \mathbb{C}^4,$$

with $a^{16} = -8$ (hence the 16 distinct complex solutions).

It should be of interest to interpret these curves in the light of other studies of the metric geometry of singular complex surfaces, for example the recent work of Birbrair, Neumann and Pichon characterising their inner bilipschitz geometry [8], and the work of Neumann and Pichon characterising outer bilipschitz triviality [19], or the work of Garcia Barroso and Teissier on the local concentrations of curvature [13].

Further evidence that weak Whitney regularity and Whitney regularity might be equivalent for complex analytic stratifications, at least for complex analytic hypersurfaces, comes from a recent result of the second author with Duco van Straten [29] that equimultiplicity of a family of complex analytic hypersurfaces follows from weak Whitney regularity.

The second author acknowledges the support of the University of Rennes 1 during several visits to Rennes, when much of the work in this paper was done.

2. Definitions.

We start by recalling the Whitney conditions.

Let X, Y be two submanifolds of a riemannian manifold M and take $y \in X \cap Y$.

Condition (a): The triple (X, Y, y) satisfies Whitney's condition (a) if for each sequence of points $\{x_i\}$ of X converging to $y \in Y$ such that $T_{x_i}X$ converges to τ (in the corresponding grassmannian in TM), then $T_yY \subset \tau$.

Condition (b): The triple (X, Y, y) satisfies Whitney's condition (b) if for each local diffeomorphism $h : \mathbb{R}^n \to M$ onto a neighbourhood U of y in M and for each sequence of points $\{(x_i, y_i)\}$ of $h^{-1}(X) \times h^{-1}(Y)$ converging to $(h^{-1}(y), h^{-1}(y))$, such that the sequence $\{T_{x_i}h^{-1}(X)\}$ converges to τ in the corresponding grassmannian and the sequence $\{\overline{x_i y_i}\}$ converges to ℓ in $\mathbb{P}^{n-1}(\mathbb{R})$, then $\ell \subset \tau$.

Condition (b^π): The triple (X, Y, y) satisfies Whitney's condition (b^π) if for each local diffeomorphism $h : \mathbb{R}^n \to M$ onto a neighbourhood U of y in M and for each sequence of points $\{x_i\}$ of $h^{-1}(X)$ converging to $h^{-1}(y)$, such that the sequence $\{T_{x_i}h^{-1}(X)\}$ converges to τ in the corresponding grassmannian and the sequence $\{\overline{x_i\pi(x_i)}\}$ converges to ℓ in $\mathbb{P}^{n-1}(\mathbb{R})$, then $\ell \subset \tau$.

One says that (X, Y) satisfies condition (a) (resp.(b), (b^π)) if (X, Y, y) satisfies (a) (resp. (b), (b^π)) at each $y \in X \cap Y$.

Remark 2.1. It is an easy exercise to check that condition (b) implies condition (a) [16]. Also (b) is equivalent to both (a) and (b^π) holding [18].

We now introduce a regularity condition (δ), obtained by weakening condition (b).

Given a euclidean vector space V, and two vectors $v_1, v_2 \in V^* = V - \{0\}$, define the sine of the angle $\theta(v_1, v_2)$ between them by :

$$\sin\theta(v_1, v_2) = \frac{||v_1 \wedge v_2||}{||v_1||.||v_2||}$$

where $v_1 \wedge v_2$ is the usual vector product and $||.||$ is the norm on V induced by the euclidean structure. Given two vector subspaces S and T of V we define the sine of the angle between S and T by :

$$\sin\theta(S, T) = \sup\{\sin\theta(s, T) : s \in S^*\}$$

where

$$\sin\theta(s, T) = \inf\{\sin\theta(s, t) : t \in T^*\}.$$

If $\pi_T : V \longrightarrow T^\perp$ is the orthogonal projection onto the orthogonal complement of T, then

$$\sin\theta(s, T) = \frac{||\pi_T(s)||}{||s||}.$$

The definition for lines is similar to that for vectors - take unit vectors on the lines.

One verifies easily that :

$$\sin\theta(v_1, v_3) \leq \sin\theta(v_1, v_2) + \sin\theta(v_2, v_3)$$

for all $v_1, v_2, v_3 \in V^*$, and

$$\sin\theta(S_1 + S_2, T) \leq \sin\theta(S_1, T) + \sin\theta(S_2, T),$$

for subspaces S_1, S_2, T of V such that S_1 is orthogonal to S_2.

Condition (δ): We say that the triple (X, Y, y) satisfies condition (δ) if there exists a local diffeomorphism $h : \mathbb{R}^n \longrightarrow M$ to a neighbourhood U of y in M, and there exists a real number δ_y, $0 \leq \delta_y < 1$, such that for every sequence $\{x_i, y_i\}$ of $h^{-1}(X) \times h^{-1}(Y)$ which converges to $(h^{-1}(y), h^{-1}(y))$ such that the sequence $\overline{x_i y_i}$ converges to l in $\mathbb{P}^{n-1}(\mathbb{R})$, and the sequence $T_{x_i} h^{-1}(X)$ converges to τ, then $\sin\theta(l, \tau) \leq \delta_y$.

Remark 2.2. Clearly condition (b) implies (δ) : just take $\delta_y = 0$.

Definition 2.3. *A weakly Whitney stratification of a subspace A of a manifold M is a locally finite partition of A into connected submanifolds, called the strata, such that :*

(1). Frontier Condition : If X and Y are distinct strata such that $\overline{X} \cap Y \neq \emptyset$, that is X and Y are adjacent, then $Y \subset \overline{X}$.

(2). Each pair of adjacent strata satisfies condition (a).

(3). Each pair of adjacent strata satisfies condition (δ).

Examples. (1). Every Whitney stratification is weakly Whitney regular.

(2). Let X be the open logarithmic spiral with polar equation,

$$\{(r, \theta) \in \mathbb{R}^2 \,|\, r = e^{\frac{t}{\tan(\beta)}}, \theta = t (\operatorname{mod} 2\pi)\} \quad \text{where } 0 < \beta < \frac{\pi}{2}\}$$

and let $Y = \{0\} \subset \mathbb{R}^2$. Condition (a) is trivially satisfied for $(X, Y, \{0\})$, and condition (δ) is also satisfied, but condition (b) fails because the angle $\theta(\overline{x0}, T_x X) = \beta$ is constant and nonzero for all x in X. So this is a weakly Whitney regular stratification which is not Whitney regular.

(3). If X is the open spiral with polar equation

$$\{(r, t) \in \mathbb{R}^2 \,|\, r = e^{-\sqrt{t}}, t \geq 0\}$$

and $Y = \{0\} \subset \mathbb{R}^2$, then the stratified space $X \cup Y$ is not weakly Whitney.

Remark 2.4. In the definition of weakly Whitney stratification, we could further weaken condition (δ) as follows : If π is a local C^1 retraction associated to a C^1 tubular neighbourhood of Y near y, a condition (δ^π) is obtained from the definition of (δ) by replacing the sequence $\{y_i\}$ by the sequence $\{\pi(x_i)\}$. Clearly $(b^\pi$ implies (δ^π). Recall that $(b) \Longleftrightarrow (b^\pi) + (a)$ [18], as noted above in Remark 2.1.

Lemma 2.5. $(\delta) + (a) \Longleftrightarrow (\delta^\pi) + (a)$.

Proof. Clearly $(\delta) \Longrightarrow (\delta^\pi)$, so it suffices to show that $(\delta^\pi) + (a) \Longrightarrow (\delta)$. In the definition of (δ) decompose the limiting vector l as the sum of a vector l_1 tangent to Y at y, and a vector l_2 tangent to $\pi^{-1}(y)$ at y. Then

$$\sin\theta(l, \tau) = \sin\theta(l_1 + l_2, \tau) \leq \sin\theta(l_1, \tau) + \sin\theta(l_2, \tau).$$

By condition (a), $\sin\theta(l_1, \tau) = 0$, hence $\sin\theta(l, \tau) \leq \sin\theta(l_2, \tau)$, which is less than or equal to δ_y by hypothesis, implying (δ). □

Using Lemma 2.5 will make checking weak Whitney regularity easier.

3. Real algebraic examples.

Because many of the important applications of Whitney stratifications arise in real algebraic geometry and real singularity theory, it is necessary to know how weak Whitney regularity compares with Whitney regularity for semi-algebraic or real algebraic stratifications, as well as for complex algebraic/analytic stratifications. The following simple example illustrates that weak Whitney regularity is strictly weaker than Whitney regularity for real algebraic stratifications. No such example is currently known in the case of complex algebraic stratifications, and this will be the motivation for the calculations in sections 7, 8 and 9 of this paper.

Example 3.1. *Let* $V = \{(x, y, t) \in \mathbb{R}^3 \,|\, y^6 = t^6x^2 + x^6\}$*, let* Y *denote the* t*-axis, and let* $X = V \setminus Y$*. One can check that the triple* $(X, Y, (0, 0, 0))$ *satisfies conditions* (a) *and* (δ)*, but not condition* (b)*. See* [6] *for details.*

The following example illustrates the independence of the conditions (a) and (δ) in the case of real algebraic stratifications.

Example 3.2. *Let* $V = \{(x, y, t) \in \mathbb{R}^3 \,|\, y^{20} = t^4x^6 + x^{10}\}$*, let* Y *denote the* t*-axis and let* $X = V \setminus Y$*. Then the triple* $(X, Y, (0, 0, 0))$ *satisfies condition* (δ)*, but not condition* (a)*. For details see* [6]*.*

4. Some properties of weakly Whitney stratified spaces.

Like Whitney stratified spaces, weakly Whitney stratified spaces are filtered by dimension.

Proposition 4.1. *Suppose that a triple* (X, Y, y)*,* $y \in Y \cap \overline{X}$*, satisfies conditions* (a) *and* (δ)*. Then* $\dim Y < \dim X$*.*

Definition 4.2. *If* (A, Σ)*,* (B, Σ') *are weakly Whitney stratified spaces in* M*, then* (A, Σ) *and* (B, Σ') *are said to be in general position if for each pair of strata* $X \in \Sigma$ *and* $X' \in \Sigma'$*,* X *and* X' *are in general position in* M*, i.e. the natural map :*

$$T_xM \longrightarrow T_xM/T_xX \oplus T_xM/T_xX'$$

is surjective for all $x \in X \cap X'$*.*

Proposition 4.3. *Let* V *be a submanifold of* M *in general position with respect to* (A, Σ)*. Then* $(A \cap V, \Sigma \cap V)$ *is weakly Whitney regular, if* (A, Σ) *is weakly Whitney regular.*

A proof is given in [6]. A stronger statement, in the case of two stratified sets transverse to each other, is given in [22].

If A is locally closed and (A, Σ) is weakly Whitney (without assuming the frontier condition) then the stratified space (A, Σ_c), whose strata are the connected components of the strata of Σ, automatically satisfies the frontier condition. See [3, 4] for the (c)-regular case, which includes the case of weakly Whitney stratifications, as remarked below.

Proposition 4.4. *Let* $f : M \to M'$ *be a* C^1 *map, and let* (A, Σ) *be a weakly Whitney stratified space in* M'*. If* f *is transverse to each stratum* $X \in \Sigma$*, then the pull-back* $(f^{-1}(A), f^{-1}(\Sigma))$ *is weakly Whitney stratified.*

See [6] for proofs.

5. (c)-regularity of weakly Whitney stratifications.

In this section we recall the fact that weakly Whitney stratified spaces are (c)-regular. It follows [3, 4] that they can be given the structure of abstract stratified sets in the sense of Thom-Mather [16], implying in particular local topological triviality along strata and triangulability [14].

Let (U, ϕ) be a C^1 chart at y for a submanifold $Y \subseteq M$ where $\dim Y = d$,

$$\phi : (U, U \cap Y, y) \longrightarrow (\mathbb{R}^n, \mathbb{R}^d \times \{0\}^{n-d}, 0).$$

Then ϕ defines a tubular neighbourhood T_ϕ of $U \cap Y$ in U, induced by the standard tubular neighbourhood of $\mathbb{R}^d \times \{0\}^{n-d}$ in $\mathbb{R}^n$:

- with retraction $\pi_\phi = \phi^{-1} \circ \pi_d \circ \phi$ where $\pi_d : \mathbb{R}^n \to \mathbb{R}^d$ is the canonical projection,
- and distance function $\rho_\phi = \rho \circ \phi : U \to \mathbb{R}^+$ where $\rho : \mathbb{R}^n \to \mathbb{R}^+$ is the function defined by $\rho(x_1, \cdots, x_n) = \Sigma_{i=d+1}^n x_i^2$.

It is well-known (see [16, 27, 28]) that if a pair (X, Y) of submanifolds of M satisfies Whitney's condition (b) then for *any* sufficiently small tubular neighbourhood T_Y of Y in M, the map

$$(\pi_Y, \rho_Y)|_{X\cap T_Y} : X \cap T_Y \longrightarrow Y \times \mathbb{R}$$

is a submersion. In fact this property characterises (b)-regularity [27]. For comparison, when the pair (X, Y) is weakly Whitney, there exists *some* tubular neighbourhood T_Y such that the map

$$(\pi_Y, \rho_Y)|_{X\cap T_Y} : X \cap T_Y \longrightarrow Y \times \mathbb{R}$$

is a submersion.

Proposition 5.1. *Let X, Y be two submanifolds of M, such that $Y \subset \overline{X}$ and let $y \in Y$. If the triple (X, Y, y) satisfies the weak Whitney conditions, then there exists a C^1 chart (U, ϕ) at y for Y in M and a neighbourhood U' of y, $U' \subset U$, such that $(\pi_\phi, \rho_\phi)|_{U'\cap X}$ is a submersion.*

Corollary 5.2. *Let X, Y be two submanifolds of M such that $Y \subset \overline{X}$ and the pair (X, Y) satisfies the conditions (a) and (δ). Then there exists a tubular neighbourhood T_Y of Y in M such that $(\pi_Y, \rho_Y)|_X : X \cap T_Y \longrightarrow Y \times \mathbb{R}$ is a submersion.*

Proposition 5.3. *Every weakly Whitney stratified space is (c)-regular, and hence is locally topologically trivial along strata.*

For the proofs see [6]. We note that, when weak Whitney regularity holds, the control function in the definition of (c)-regularity can be chosen to be a standard distance function arising from a tubular neighbourhood. This means that weak Whitney regularity is a much stronger condition than mere (c)-regularity, for which the control function may be weighted homogeneous or even infinitely flat along Y.

6. Complex stratifications.

In Example 3.1 we saw an example of a weakly Whitney regular real algebraic stratification in $\mathbb{R}^3$ which is not Whitney (b)-regular. We are now interested in comparing weak Whitney regularity and Whitney regularity of *complex* analytic or *complex* algebraic stratifications, the main question being whether the extra 'rigidity' of complex analytic varieties prevents the existence of weakly Whitney complex analytic stratifications which are not Whitney regular.

Let F be an analytic function germ from $\mathbb{C}^n \times \mathbb{C}$ to $\mathbb{C}$, defined in a neighbourhood of 0,

$$\begin{array}{rccc} F: & \mathbb{C}^n \times \mathbb{C}, 0 & \longrightarrow & \mathbb{C}, 0 \\ & (x,t) & \longmapsto & F(x,t) \end{array}$$

where $F(0, t) = 0$. We denote by π the projection on the second factor, and let $V = F^{-1}(0)$, $Y = \{0\}^n \times \mathbb{C}$ and $V_t = \{x \in \mathbb{C}^n \mid F(x, t) = 0\}$. We assume that each V_t has an isolated singularity at $(0, t)$, the critical set of the restriction of π to V is Y, and $X = V \setminus Y$ is an analytic complex manifold of dimension n.

For each point $(x, t) \in X$ we have

$$T_{(x,t)}X = \left\{ (u, v) \in \mathbb{C}^n \times \mathbb{C} \,\middle|\, \sum_{i=1}^n u_i \frac{\partial F}{\partial x_i}(x,t) + v\frac{\partial F}{\partial t}(x,t) = 0 \right\} = \left(\mathbb{C}\overline{\mathrm{grad}}F\right)^\perp.$$

Let $\mathrm{grad}F = (\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial t})$, $\mathrm{grad}_x F = (\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n})$ and

$$\|\mathrm{grad}_x F\|^2 = \sum_{i=1}^n \|\frac{\partial F}{\partial x_i}\|^2.$$

The following characterisations of conditions (a), (b^π) and (δ^π) are straightforward.

Whitney's condition (a)

The pair (X,Y) satisfies Whitney's condition (a) at 0 if and only if

$$\lim_{\substack{(x,t)\to 0\\(x,t)\in X}}\left(\frac{\frac{\partial F}{\partial t}(x,t)}{\|\mathrm{grad}_x F(x,t)\|}\right)=0.$$

Whitney's condition (b^π)

The couple (X,Y) satisfies Whitney's condition (b^π) at 0 if and only if

$$\lim_{\substack{(x,t)\to 0\\(x,t)\in X}}\left(\frac{\sum_{i=1}^n x_i\frac{\partial F}{\partial x_i}(x,t)}{\|x\|\|\mathrm{grad}_x F(x,t)\|}\right)=0.$$

Condition (δ^π)

The pair (X,Y) satisfies the (δ^π) condition at 0 if and only if there exists a real number $0\le\delta<1$ such that

$$\lim_{\substack{(x,t)\to 0\\(x,t)\in X}}\left(\frac{\sum_{i=1}^n x_i\frac{\partial F}{\partial x_i}(x,t)}{\|x\|\|\mathrm{grad}_x F(x,t)\|}\right)\le\delta.$$

Recall that Whitney's condition (b) implies $(a)+(\delta^\pi)$.

Question. Is the converse true in the complex hypersurface case, i.e. does $(a)+(\delta^\pi)$ imply (b) or, equivalently, does $(a)+(\delta^\pi)$ imply (b^π) ?

Remark 6.1. Because weak Whitney regularity implies local topological triviality along strata, if we wish to decide whether weak Whitney regularity and Whitney regularity are equivalent for complex hypersurfaces, we can restrict to studying families of isolated singularities of complex hypersurfaces with constant Milnor number (Milnor number is a topological invariants). But we know by the fundamental result of Lê Dung Tràng and K. Saito [15] that a family of complex hypersurfaces with isolated singularities has constant Milnor number if and only if

$$\lim_{(x,t)\to 0}\left(\frac{\frac{\partial F}{\partial t}(x,t)}{\|\mathrm{grad}_x F(x,t)\|}\right)=0,$$

which implies condition (a).

The following lemma due to Briançon and Speder [10] gives an equivalent condition to (b^π) when (a) is satisfied.

Let $\gamma:([0,1],0)\to(\mathbb{C}^n\times\mathbb{C},0)$, be a germ of an analytic arc and ν the valuation along γ in the local ring $\mathcal{O}_{n+1,0}$.

Notation. Let $\nu(x):=\inf\{\nu(x_i)|1\le i\le n\}$ and $\nu(J_x(F)):=\inf\{\nu(\frac{\partial F}{\partial x_i}|1\le i\le n\}$.

Lemma 6.2. *The following statements are equivalent:*

(i) the pair (X,Y) satisfies (b^π) at 0,

(ii)

$$\lim_{\substack{(x,t)\to 0\\(x,t)\in X}}\left(\frac{t\frac{\partial F}{\partial t}(x,t)}{\|x\|\|\mathrm{grad}_x F(x,t)\|}\right)=0.$$

In other words, the following statements are equivalent:

(i) $\nu(\sum_{i=1}^n x_i\frac{\partial F}{\partial x_i})>\nu(x)+\nu(J_x(F))$

(ii) $\nu(t)+\nu(\frac{\partial F}{\partial t})>\nu(x)+\nu(J_x(F))$

where ν is the valuation along germs of analytic arcs $\gamma:[0,1]\to X$.

Proof. For $s \in [0,1]$, $\gamma(s) = (x_1(s), \dots, x_n(s), t(s))$.

Since $F \circ \gamma \equiv 0$, we have

$$\sum_{i=1}^{n} x_i'(s) \frac{\partial F \circ \gamma}{\partial x_i}(s) = -t'(s) \frac{\partial F \circ \gamma}{\partial t}(s). \qquad (*)$$

If $a = \nu(x)$ and $b = \nu(J_x(F))$, there exist two non zero vectors of $\mathbb{C}^n$, A and B, such that

$$(x_1(s), \dots, x_n(s)) = As^a + \dots$$

and

$$\left(\frac{\partial F \circ \gamma}{\partial x_1}(s), \dots, \frac{\partial F \circ \gamma}{\partial x_n}(s) \right) = Bs^b + \dots.$$

We suppose (i) holds. Then since

$$\sum_{i=1}^{n} x_i(s) \frac{\partial F \circ \gamma}{\partial x_i}(s) = \langle A, B \rangle s^{a+b} + \dots,$$

we must have $\langle A, B \rangle = 0$.

From (*) we have

$$t'(s) \frac{\partial F \circ \gamma}{\partial t}(s) = -\sum_{i=1}^{n} x_i'(s) \frac{\partial F \circ \gamma}{\partial x_i}(s) = -a \langle A, B \rangle s^{a+b-1} + \dots.$$

Then

$$\nu(t) + \nu(\frac{\partial F}{\partial t}) = \nu(t'(s) \frac{\partial F \circ \gamma}{\partial t}) + 1 > (a + b - 1) + 1 = a + b.$$

We suppose now that (ii) holds. Then since

$$t'(s) \frac{\partial F \circ \gamma}{\partial t} = -a \langle A, B \rangle s^{a+b-1} + \dots,$$

we must have again $\langle A, B \rangle = 0$, which is exactly condition (i). □

7. The Briançon and Speder example with $\mu = 364$.

In this section we study the original example, due to Briançon and Speder [9], of a topologically trivial family of isolated complex hypersurface singularities which is not Whitney regular. The examples of Briançon and Speder given in [9] were the *only* such examples known, until very recently.

Initially we shall carry out explicit calculations for the most well-known example of Briançon and Speder, analysed in their celebrated note of January 1975 :

$$F(x, y, z, t) = F_t(x, y, z) = x^5 + txy^6 + y^7 z + z^{15}$$

for which $\mu(F_t) = 364$ for all t near 0.

Theorem 7.1. *The Briançon and Speder example $F(x, y, z, t) = x^5 + txy^6 + y^7 z + z^{15}$ is not weakly Whitney regular.*

Proof. Let $F(x, y, z, t) = x^5 + txy^6 + y^7 z + z^{15}$. Then F is a quasihomogenous μ-constant family of type $(3, 2, 1; 15)$. Thus the stratification $(F^{-1}(0) \setminus (0t), (0t))$ is (a)-regular by Remark 6.1

We shall construct an explicit analytic path $\gamma(s) = (x(s), y(s), z(s), t(s))$ contained in $F^{-1}(0)$ such that

$$\Delta(x, y, z, t) = \left(\frac{\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}(x, y, z, t)}{\|x\| \|\mathrm{grad}_x F(x, y, z, t)\|} \right)$$

tends to 1 when (x,y,z,t) tends to 0 along $\gamma(s)$. This means that condition (δ^π) is not satisfied at 0, by the characterisation given in section 6. By Lemma 2.4 it then follows using (a)-regularity that (δ) is not satisfied at 0, so that weak Whitney regularity fails.

Following [9] and [27] we take

$$\begin{aligned} x(s) &= s^8 \\ y(s) &= as^5 \\ z(s) &= \frac{4}{a^7}\lambda s^5 \\ t(s) &= -\frac{5}{a^6}s^2 \end{aligned}$$

with $a \neq 0$.

For $\gamma(s)$ to lie on $F^{-1}(0)$ we must have that

$$F(\gamma(s)) = (1 - \frac{5}{a^6}a^6 + 4\lambda + (\frac{4}{a^7})^{15}\lambda^{15}s^{35})s^{40} \equiv 0,$$

so that

$$G(\lambda, s) = -4 + 4\lambda + (\frac{4}{a^7})^{15}\lambda^{15}s^{35} \equiv 0.$$

ince $\frac{\partial G}{\partial \lambda}(\lambda, 0) = 4 \neq 0$, it follows by the implicit function theorem that λ is a function of s for s near 0.

Note that $\lambda(0) = 1$.

Then we have along $\gamma(s)$ near $s = 0$,

$$\begin{aligned} \frac{\partial F}{\partial x} &= 5x^4 + ty^6 = 5s^{32} - \frac{5}{a^6}a^6 s^{32} = 0, \\ \frac{\partial F}{\partial y} &= 6txy^5 + 7y^6 z = \left(\frac{-30}{a} + \frac{28}{a}\lambda\right)s^{35}, \\ \frac{\partial F}{\partial z} &= y^7 + 15z^{14} = a^7 s^{35} + 15(\frac{4}{a^7})^{14}\lambda^{14}s^{70} \sim a^7 s^{35}. \end{aligned}$$

Because $\lambda(0) = 1$, the limit of the orthogonal secant vectors

$$\frac{(x,y,z)}{\|(x,y,z)\|}$$

is

$$(0 : a : \frac{4}{a^7}) = (0 : a^8 : 4),$$

and the limit of the normal vectors

$$\frac{\operatorname{grad}_x F(x,y,z,t)}{\|\operatorname{grad}_x F(x,y,z,t)\|}$$

is

$$(0 : \frac{-2}{a} : a^7) = (0 : -2 : a^8).$$

Then $\Delta(\gamma(s))$ tends to 1 if and only if $(0 : a^8 : 4) = (0 : -2 : a^8)$, i.e.

$$\frac{a^8}{4} = \frac{-2}{a^8} \iff a^{16} = -8.$$

It follows that (δ^π) is not satisfied along γ if and only if $a^{16} = -8$. Choosing a to be one of these 16 complex numbers, we have the desired conclusion, i.e. that (δ^π) fails. It follows as above that weak Whitney regularity fails, proving the theorem. $\square$

Note that in the proof above we cannot exclude the possibility that there are other curves on which (δ^π) fails. To clarify the situation, in the next section we make a systematic study of all curves $\gamma(s)$ on $F^{-1}(0)$ and passing through the origin.

A similar calculation as in the theorem above for the simpler μ-constant family

$$F(x,y,z,t) = x^3 + txy^3 + y^4z + z^9,$$

for which $\mu = 56$ (also due to Briançon and Speder [9]), shows that (δ^π) fails for this example too. Our systematic study to determine all curves on which (δ^π) fails for this simpler example will be extended to a more general study, given in section 9 below, of an infinite family of examples, of which $x^3 + txy^3 + y^4z + z^9$ is the first, again defined by Briançon and Speder in their celebrated 1975 note [9].

In what follows we determine the initial terms of *all* curves along which condition (δ) fails, or equivalently along which (δ^π) fails, by Lemma 2.5.

8. Failure of weak Whitney regularity: a complete analysis.

Take again

$$F(x,y,z,t) = x^5 + txy^6 + y^7z + z^{15}.$$

Let $\gamma : ([0,1],0) \to (F^{-1}(0),0) \subset (\mathbb{C}^n \times \mathbb{C}, 0)$ be a germ of an analytic arc and let ν be the valuation along γ.

Let $X = (x,y,z)$ and

$$J_X F = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}).$$

We will use the notations

$$\nu(X) := \inf\{\nu(x), \nu(y), \nu(z)\},$$

and

$$\nu(J_X(F)) := \inf\{\nu(\frac{\partial F}{\partial x}), \nu(\frac{\partial F}{\partial y}), \nu(\frac{\partial F}{\partial z})\}.$$

We begin by determining the curves along which condition (b^π) holds (because (a)-regularity holds, by Remark 6.1, we know that (b^π) is equivalent to (b), by Remark 2.1).

By Lemma 6.2, the μ-constant property and the Lê-Saito theorem (see Remark 6.1), if $\nu(t) \geq \nu(X)$ then

$$\begin{aligned} \nu(t) + \nu(\frac{\partial F}{\partial t}) &\geq \nu(X) + \nu(\frac{\partial F}{\partial t}) \\ &> \nu(X) + \nu(J_X(F))), \end{aligned}$$

so that (b^π) holds by Lemma 6.2, and hence Whitney's condition (b) holds also.

We can therefore suppose from now on that $\nu(t) < \nu(X)$.

If

$$\nu(\frac{\partial F}{\partial x}) = \inf\{4\nu(x), \nu(t) + 6\nu(y)\} \tag{1}$$

we have:

(1). when $4\nu(x) \geq \nu(t) + 6\nu(y)$, so that $\nu(x) > \nu(y)$, then

$$\begin{aligned}
\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + 6\nu(y) \\
&> \nu(y) + \nu(t) + 6\nu(y) \\
&= \nu(y) + \nu(\frac{\partial F}{\partial x}) && \text{(by (1))} \\
&\geq \nu(X) + \nu(J_X(F)),
\end{aligned}$$

When however

$$\nu(\frac{\partial F}{\partial x}) > \inf\{4\nu(x), \nu(t) + 6\nu(y)\} \tag{2}$$

then we must have

$$4\nu(x) = \nu(t) + 6\nu(y). \tag{3}$$

Because $F \circ \gamma \equiv 0$, we have that

$$x^5 + txy^6 = -y^7 z - z^{15}. \tag{4}$$

On the other hand

$$x^5 + ty^6 x = -4x^5 + x\frac{\partial F}{\partial x}$$

and (2) imply that

$$\nu(x^5 + ty^6 x) = 5\nu(x).$$

Hence, by (4),

$$5\nu(x) \geq \inf\{7\nu(y) + \nu(z), 15\nu(z)\}$$

and, unless $\nu(y) = 2\nu(z)$, it follows that

$$5\nu(x) = \inf\{7\nu(y) + \nu(z), 15\nu(z)\}. \tag{5}$$

(i) If $\nu(y) > 2\nu(z)$, it follows that $\nu(x) = 3\nu(z)$. Then, by (1) and using that

$$\frac{\partial F}{\partial z} = y^7 + 15z^{14} \tag{6}$$

it follows that

$$\begin{aligned}
\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + 6\nu(y) \\
&> \nu(t) + 15\nu(z) \\
&> 15\nu(z) \\
&= \nu(z) + \nu(\frac{\partial F}{\partial z}) \\
&\geq \nu(X) + \nu(J_X(F)),
\end{aligned}$$

and hence (b^π) holds.

(ii) If $\nu(y) = 2\nu(z)$,

(a) and $\nu(x) = 3\nu(z)$, then from (3) we obtain

$$12\nu(z) = \nu(t) + 12\nu(z),$$

i.e. $\nu(t) = 0$, which is impossible;

(b) and $\nu(x) > 3\nu(z)$, then

$$\nu(y^7 + z^{14}) > 14\nu(z)$$

and from (6) it follows that

$$\nu(\frac{\partial F}{\partial z}) = 14\nu(z)$$

so that

$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + 6\nu(y) \\ &> \nu(t) + 15\nu(z) \\ &> 15\nu(z) \\ &= \nu(z) + \nu(\frac{\partial F}{\partial z}) \\ &\geq \nu(X) + \nu(J_X(F)),\end{aligned}$$

so that (b^π) holds, using Lemma 6.2 again.

(iii) If $\nu(y) < 2\nu(z)$, we have from Equation (5) that

$$5\nu(x) = 7\nu(y) + \nu(z). \tag{7}$$

Subtracting (2) from (7) gives

$$\nu(x) + \nu(t) = \nu(y) + \nu(z). \tag{8}$$

We can suppose now that

$$\nu(x) + \nu(t) = \nu(y) + \nu(z)$$

and

$$\nu(y) < 2\nu(z).$$

We carry on with the last cases:

(I) If $\nu(z) > \nu(y)$ we have

$$\nu(z^{14}) > \nu(y^7),$$

so that

$$\nu(\frac{\partial F}{\partial z}) = 7\nu(y). \tag{9}$$

Then (1), (8) and (9) give

$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + 6\nu(y) \\ &= \nu(z) + 7\nu(y) \\ &> \nu(y) + 7\nu(y) \\ &= \nu(y) + \nu(\frac{\partial F}{\partial z}) \\ &\geq \nu(X) + \nu(J_X(F)),\end{aligned}$$

and we have that (b^π) holds, by Lemma 6.2.

(II) If $2\nu(z) > \nu(y) > \nu(z)$, then

$$\nu(\frac{\partial F}{\partial z}) = 7\nu(y) > 6\nu(y) + \nu(z),$$

so that

$$\frac{\partial F}{\partial y} = 6txy^6 + 7y^7z = 6txy^6 - 7(x^5 + txy^6 + z^{15})$$

or

$$\frac{\partial F}{\partial y} = -7x^5 - txy^6 - 7z^{15}. \tag{10}$$

Now because $2\nu(z) > \nu(y)$ we have that

$$\begin{aligned} 15\nu(z) &> \nu(z) + 7\nu(y), \\ &= \nu(x) + \nu(t) + 6\nu(y) && \text{(by (8))} \\ &= \nu(txy^6). && (11) \end{aligned}$$

Also by (2)

$$\nu(\frac{\partial F}{\partial x}) > 4\nu(x).$$

This means that

$$\nu(5x^4 + ty^6) > 4\nu(x)$$

which implies in turn that

$$\nu(-7x^5 - txy^6) = \nu(txy^6). \tag{12}$$

It follows from (10), (11) and (12) that

$$\nu(y\frac{\partial F}{\partial y}) = \nu(txy^6),$$

i.e.

$$\nu(\frac{\partial F}{\partial y}) = \nu(txy^5). \tag{13}$$

Then

$$\begin{aligned} \nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + 6\nu(y) \\ &= \nu(txy^5) + \nu(y) \\ &> \nu(txy^5) + \nu(z) \\ &= \nu(\frac{\partial F}{\partial y}) + \nu(z) && \text{(by (13))} \\ &\geq \nu(X) + \nu(J_X(F)), \end{aligned}$$

and again (b^π) holds, by Lemma 6.2

Résumé: a germ of arc $(x(s), y(s), z(s), t(s))$ along which Whitney condition (b) is not satisfied must fulfil the following conditions:

- $\nu(x) > \nu(y) = \nu(z) > \nu(t)$
- $\nu(x) + \nu(t) = \nu(z) + \nu(y)$
- $4\nu(x) = \nu(t) + 6\nu(y)$.

Resolving these equations we find that

$$5\nu(x) = 8\nu(y) \text{ and } 5\nu(t) = 2\nu(y),$$

so that the set of germs of analytic arcs along which Whitney condition (b) is not satisfied is contained in the set

$$\begin{aligned}\mathcal{A} := \{\ \gamma(s) = (x(s), y(s), z(s), t(s)) : [0,1] \to \mathbb{C}^n \times \mathbb{C}\ |\\ x(s) &= a_1 s^{\alpha_1} + \cdots \\ y(s) &= a_2 s^{\alpha_2} + \cdots \\ z(s) &= a_3 s^{\alpha_3} + \cdots \\ t(s) &= a_4 s^{\alpha_4} + \cdots\end{aligned}$$

with $5\alpha_1 = 8\alpha, \alpha_2 = \alpha_3 = \alpha, 5\alpha_4 = 2\alpha, \alpha \equiv 0[5], \text{and } a_i \in \mathbb{C}^*\text{satisfying some conditions}\}$

It remains to characterize the subset of arcs along which the (δ) condition is not satisfied, or equivalently along which the (δ^π) condition is not satisfied, using (a)-regularity and Lemma 2.5. Let $\gamma \in \mathcal{A}$. We may suppose $a_1 = 1$, and write $a_2 = a, a_3 = b$ and $a_4 = c$.Then

$$F \circ \gamma(s) = (s^{8\alpha} + ...) + (c.a^6 s^{8\alpha} + \dots) + (a^7 b s^{8\alpha} + \dots) + (b^{15} s^{15\alpha} + \dots) \equiv 0$$

so that

$$s^{8\alpha}(1 + c.a^6 + a^7.b + s(\dots + b^{15} s^{7\alpha - 1} + \dots)) \equiv 0,$$

and we must have

$$1 + c.a^6 + a^7.b = 0.$$

Thus along $\gamma(s)$ near $s = 0$ we have,

$$\begin{aligned}\frac{\partial F}{\partial x} &= 5x^4 + ty^6 = s^{\frac{32}{5}\alpha}(5 + ca^6) + \dots \\ \frac{\partial F}{\partial y} &= 6txy^5 + 7y^6 z = a^5(6c + 7ab)s^{7\alpha} + \dots \\ \frac{\partial F}{\partial z} &= y^7 + 15z^{14} = a^7 s^{7\alpha} + \dots + 14b^{14} s^{14\alpha} + \dots.\end{aligned}$$

But now, using (2), $\nu(\frac{\partial F}{\partial x}) > \frac{32}{5}\alpha$ imposes the condition $5 + ca^6 = 0$. It follows that

$$c = -\frac{5}{a^6}, b = \frac{4}{a^7}$$

and

$$a^5(6c + 7ab) = -\frac{2}{a}.$$

The limit of orthogonal secant vectors

$$\frac{(x, y, z)}{\|(x, y, z)\|}$$

is

$$(0 : a : b) = (0 : a^8 : 4),$$

and the limit of normal vectors

$$\frac{\mathrm{grad}_x F(x, y, z, t)}{\|\mathrm{grad}_x F(x, y, z, t)\|}$$

is

$$(0 : a^5(6c + 7ab) : a^7) = (0 : -\frac{2}{a} : a^7) = (0 : -2 : a^8).$$

As at the end of section 7 we deduce that (δ^π) is not satisfied along γ if and only if

$$(0 : a^8 : 4) = (0 : -2 : a^8),$$

or equivalently when $a^{16} = -8$. Choosing a to be one of these 16 complex numbers, we have the desired conclusion, namely that (δ^π) fails precisely on those curves

$$\gamma(s) = (x(s), y(s), z(s), t(s))$$

whose initial terms are

$$(s^8, as^5, 4a^{-7}s^5, -5a^{-6}s^2).$$

By Lemma 2.5 and (a)-regularity, these are precisely the curves on which (δ) fails, that is to say we have identified all of the curves on which weak Whitney regularity fails to hold.

9. Other Briançon and Speder examples

We perform similar calculations for the infinite family of examples, also due to Briançon and Speder [9] :

$$F(x, y, z, t) = x^3 + txy^\alpha + y^\beta z + z^{3\alpha},$$

where $\alpha \geq 3$ and $3\alpha = 2\beta + 1$.
The functions $f_t(x, y, z) = F_t(x, y, z)$ are quasihomogenous of type $(\alpha, 2, 1; 3\alpha)$ with isolated singularity at the origin, for each t, and so each

$$\mu_t = (3\alpha - 1)(3\alpha - 2) = 2\beta(2\beta - 1),$$

by the Milnor-Orlik formula [17]. Thus f_t is a μ-constant family.

We are again hunting for analytic arc germs where condition (δ) fails.

Clearly

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 3x^2 + ty^\alpha \\
\frac{\partial F}{\partial y} &= \alpha txy^{\alpha-1} + \beta y^{\beta-1}z \\
\frac{\partial F}{\partial z} &= y^\beta + 3\alpha z^{3\alpha-1} = y^\beta + 3\alpha z^{2\beta}.
\end{aligned}$$

Let $\gamma : ([0, 1], 0) \to (F^{-1}(0), 0) \subset (\mathbb{C}^n \times \mathbb{C}, 0)$, be a germ of an analytic arc and ν the valuation along γ.

As above we let $X = (x, y, z)$ and $J_X F = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})$, then write

$$\nu(X) := \inf\{\nu(x), \nu(y), \nu(z)\}$$

and

$$\nu(J_X(F)) := \inf\{\nu(\frac{\partial F}{\partial x}), \nu(\frac{\partial F}{\partial y}), \nu(\frac{\partial F}{\partial z})\}.$$

We begin by determining along which curves condition (b^π) holds. Note that again Remark 2.1 implies that for the examples studied here (b^π) is equivalent to Whitney's condition (b), because Whitney's condition (a) holds by the Lê -Saito theorem (Remark 6.1).

Suppose that $\nu(t) \geq \nu(X)$. Again by the μ-constant condition and Remark 6.1,

$$\nu(t) + \nu(\frac{\partial F}{\partial t}) > \nu(t) + \nu(\frac{\partial F}{\partial x})$$

so that, since $\nu(t) \geq \nu(X)$,

$$\nu(t) + \nu(\frac{\partial F}{\partial t}) > \nu(X) + \nu(\frac{\partial F}{\partial x}),$$

so that

$$\nu(t) + \nu(\frac{\partial F}{\partial t}) > \nu(X) + \nu(J_X(F))$$

and by Lemma 6.2 (b^π) holds, and Whitney's condition (b) holds using (a) and Remark 2.1.

We shall assume from now on that $\nu(t) < \nu(X)$.

$$\nu(\frac{\partial F}{\partial x}) = \inf\{2\nu(x), \nu(t) + \alpha\nu(y)\} \tag{14}$$

we have:

(1) either $2\nu(x) \geq \nu(t) + \alpha\nu(y)$, and then we must have $\nu(x) > \nu(y)$ and

$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + \alpha\nu(y)\\ &> \nu(y) + \nu(t) + \alpha\nu(y)\\ &\geq \nu(X) + \nu(\frac{\partial F}{\partial x}) && \text{(by (14))}\\ &\geq \nu(X) + \nu(J_X(F)),\end{aligned}$$

so that as in section 8 we obtain that (b^π) holds, using Lemma 6.2;

(2) or we have $2\nu(x) < \nu(t) + \alpha\nu(y)$, and then

$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + \alpha\nu(y)\\ &> \nu(x) + 2\nu(x)\\ &\geq \nu(x) + \nu(\frac{\partial F}{\partial x}) && \text{(by (14))}\\ &\geq \nu(X) + \nu(J_X(F)),\end{aligned}$$

and again (b^π) holds by Lemma 6.2.

It follows that from now on we are reduced to studying the case when

$$\nu(\frac{\partial F}{\partial x}) > \inf\{2\nu(x), \nu(t) + \alpha\nu(y)\}, \tag{15}$$

and hence that

$$2\nu(x) = \nu(t) + \alpha\nu(y). \tag{16}$$

Now we are assuming that $F \circ \gamma \equiv 0$, i.e.

$$x^3 + txy^\alpha = -y^\beta z - z^{3\alpha}. \tag{17}$$

Write

$$x^3 + txy^\alpha = -2x^3 + x\frac{\partial F}{\partial x}.$$

Then (15) implies that

$$\nu(x^3 + txy^\alpha) = 3\nu(x).$$

Using (17) we see that

$$3\nu(x) \geq \inf\{\beta\nu(y) + \nu(z), 3\alpha\nu(z)\}$$

and that

$$3\nu(x) = \inf\{\beta\nu(y) + \nu(z), 3\alpha\nu(z)\} \quad \text{if} \quad \nu(y) \neq 2\nu(z), \tag{18}$$

using that $3\alpha - 1 = 2\beta$.

(i) If $\nu(y) > 2\nu(z)$ then, by (18), $\nu(x) = \alpha\nu(z)$. Also

$$\nu(\frac{\partial F}{\partial z}) = (3\alpha - 1)\nu(z). \tag{19}$$

Then

$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + \alpha\nu(y)\\ &> \nu(t) + 3\alpha\nu(z)\\ &> \nu(z) + \nu(\frac{\partial F}{\partial z}) && \text{(by (19))}\\ &\geq \nu(X) + \nu(J_X(F))\end{aligned}$$

so that (b^π) holds by Lemma 6.2.

(ii) If $\nu(y) = 2\nu(z)$, then from (16) it follows immediately that $\nu(x) > \alpha\nu(z)$.
Now

$$\begin{aligned}\nu(\frac{\partial F}{\partial y}) &= \inf\{\nu(t) + \nu(x) + (\alpha - 1)\nu(y), (\beta - 1)\nu(y) + \nu(z)\}\\ &= \inf\{3\nu(x) - \nu(y), (2\beta - 1)\nu(z)\} && \text{(by (16))}\\ &= \inf\{3\nu(x) - 2\nu(z), (3\alpha - 2)\nu(z)\} && \text{(since } \nu(y) = 2\nu(z))\\ &= (3\alpha - 2)\nu(z) && \text{(since } \nu(x) > \alpha\nu(z).)\end{aligned}$$

Then

$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + \alpha\nu(y)\\ &= 3\nu(x) && \text{(by (16))}\\ &> 3\alpha\nu(z)\\ &= \nu(z) + (3\alpha - 1)\nu(z)\\ &> \nu(z) + \nu(\frac{\partial F}{\partial y})\\ &\geq \nu(X) + \nu(J_X(F)),\end{aligned}$$

and (b^π) holds by Lemma 6.2.

(iii) If $\nu(y) < 2\nu(z)$, we have

$$3\nu(x) = \beta\nu(y) + \nu(z),$$

and (16) gives

$$\nu(x) + \nu(t) = (\beta - \alpha)\nu(y) + \nu(z).$$

Thus we can suppose from now on that

$$\nu(x) + \nu(t) = (\beta - \alpha)\nu(y) + \nu(z) \tag{20}$$

and

$$\nu(y) < 2\nu(z).$$

We carry on with the last cases:

(I) If $\nu(z) > \nu(y)$ we have

$$\nu(z^{3\alpha-1}) = \nu(z^{2\beta}) > \nu(y^{\beta})$$

so that

$$\nu(\frac{\partial F}{\partial z}) = \beta\nu(y).$$

Then

$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + \alpha\nu(y)\\ &= \nu(z) + \beta\nu(y)\\ &> \nu(y) + \nu(\frac{\partial F}{\partial z})\\ &\geq \nu(X) + \nu(J_X(F)),\end{aligned}$$

and (b^{π}) holds by Lemma 6.2

(II) If $2\nu(z) > \nu(y) > \nu(z)$, then

$$\nu(z^{3\alpha-1}) = \nu(z^{2\beta}) > \nu(y^{\beta})$$

so that

$$\nu(\frac{\partial F}{\partial z}) = \beta\nu(y) > (\beta-1)\nu(y) + \nu(z).$$

Now

$$\begin{aligned}y\frac{\partial F}{\partial y} &= \alpha txy^{\alpha} + \beta y^{\beta} z\\ &= \alpha txy^{\alpha} - \beta(x^3 + txy^{\alpha} + z^{3\alpha}) && \text{(on } F^{-1}(0))\\ &= -\beta x^3 - (\beta-\alpha)txy^{\alpha} - \beta z^{3\alpha},\\ &= x(ty^{\alpha} - \frac{\beta}{3}(\frac{\partial F}{\partial x})) - \beta z^{3\alpha} && (21)\end{aligned}$$

since $3\alpha = 2\beta + 1$.

Also

$$\begin{aligned}\nu(z^{3\alpha}) &= 3\alpha\nu(z)\\ &= \nu(z) + 2\beta\nu(z)\\ &> \nu(z) + \beta(y)\\ &= \nu(txy^{\alpha}) && \text{(by (20))}\end{aligned}$$

so we have that

$$\nu(z^{3\alpha}) > \nu(txy^{\alpha}). \tag{22}$$

From (15) and (16),

$$\nu(\frac{\partial F}{\partial x}) > \nu(t) + \alpha\nu(y). \tag{23}$$

Using (21), (22) and (23) we find that

$$\nu(y\frac{\partial F}{\partial y}) = \nu(txy^{\alpha}),$$

and thus

$$\nu(\frac{\partial F}{\partial y}) = \nu(txy^{\alpha-1}). \tag{24}$$

Then

$$\begin{aligned}
\nu(t)+\nu(\frac{\partial F}{\partial t}) &= \nu(t)+\nu(x)+\alpha\nu(y) \\
&= \nu(txy^{\alpha-1})+\nu(y) \\
&> \nu(txy^{\alpha-1})+\nu(z) \\
&= \nu(\frac{\partial F}{\partial y})+\nu(z) && \text{(by (24))} \\
&\geq \nu(X)+\nu(J_X(F)),
\end{aligned}$$

and again (b^π) holds by Lemma 6.2.

Résumé: a germ of arc along which Whitney condition (b) is not satisfied must fulfil the following conditions:

- $\nu(x) > \nu(y) = \nu(z) > \nu(t)$
- $\nu(x)+\nu(t) = (\beta-\alpha)\nu(y)+\nu(z)$
- $2\nu(x) = \nu(t)+\alpha\nu(y)$

Finally the set of germs of analytic arcs along which Whitney condition (b) is not satisfied is contained in the set

$$\begin{aligned}
\mathcal{A} := \{\ \gamma(s) = (x(s),y(s),z(s),t(s)) : [0,1] \to \mathbb{C}^n\times\mathbb{C}\ | \\
x(s) &= a_1 s^{\alpha_1}+\cdots \\
y(s) &= a_2 s^{\alpha_2}+\cdots \\
z(s) &= a_3 s^{\alpha_3}+\cdots \\
t(s) &= a_4 s^{\alpha_4}+\cdots
\end{aligned}$$

$3\alpha_1 = (\beta+1)m, \alpha_2 = \alpha_3 = m, 3\alpha_4 = m, \alpha \equiv 0[3], a_i \in \mathbb{C}^*$ satisfying some conditions $\}$.

It remains to characterize the subset of arcs along which the (δ) condition is not satisfied.

Let $\gamma \in \mathcal{A}$. We may suppose $a_1 = 1$, and write $a_2 = a, a_3 = b$ and $a_4 = c$.

Now

$$\begin{aligned}
F\circ\gamma(s) &= (s^{(\beta+1)m}+...)+(c.a^\alpha s^{(\beta+1)m}+\ldots)+(a^\beta b s^{(\beta+1)m}+\ldots)+(b^{3\alpha}s^{3\alpha m}+\ldots) \\
&\equiv 0,
\end{aligned}$$

so then

$$s^{(\beta+1)m}(1+a^\alpha.c+a^\beta.b+s(\ldots+b^{3\alpha}s^{\beta m}+\ldots)) \equiv 0,$$

and we must have

$$1+c.a^\alpha+b.a^\beta = 0.$$

Hence along $\gamma(s)$ near $s=0$ we have,

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 3x^2+ty^\alpha = s^{\frac{2(\beta+1)m}{3}\alpha}(3+ca^\alpha)+\ldots \\
\frac{\partial F}{\partial y} &= \alpha txy^{\alpha-1}+\beta y^{\beta-1}z = (\alpha ca^{\alpha-1}+\beta b.a^{\beta-1})s^{\beta m}+\ldots \\
\frac{\partial F}{\partial z} &= y^\beta+3\alpha z^{3\alpha-1} = y^\beta+3\alpha z^{2\beta} = a^\beta s^{\beta m}+\ldots+(2\beta)b^{2\beta)}s^{(2\beta)m}+\ldots.
\end{aligned}$$

However, the condition

$$\nu(\frac{\partial F}{\partial x}) > \beta m$$

implies that

$$3+ca^\alpha = 0$$

and it follows that
$$c = -\frac{3}{a^\alpha},\ b = \frac{2}{a^\beta}$$
and so
$$\alpha c.a^{\alpha-1} + \beta b.a^{\beta-1} = -\frac{1}{a}.$$
The limit of orthogonal secant vectors
$$\frac{(x,y,z)}{\|(x,y,z)\|}$$
is thus
$$(0:a:b) = (0:a:\frac{2}{a^\beta}),$$
and the limit of normal vectors
$$\frac{\mathrm{grad}_x F(x,y,z,t)}{\|\mathrm{grad}_x F(x,y,z,t)\|}$$
is
$$(0:\alpha c.a^{\alpha-1} + \beta b.a^{\beta-1} : a^\beta) = (0:-\frac{1}{a}:a^\beta).$$
It follows that (δ) is not satisfied along γ if and only if
$$a^{2\beta+2} = -2.$$
Choosing α to be one of these $2\beta+2 = 3\alpha+1$ complex numbers, we have the desired conclusion, i.e. that (δ) fails.

10. Other examples.

A Milnor number constant family,
$$F_t(x,y,z) = z^{12} + zy^3x + ty^2x^3 + x^6 + y^5,$$
with $\mu = 166$, which is also not Whitney regular over the t-axis, was studied by E. Artal Bartolo, J. Fernandez de Bobadilla, I. Luengo and A. Melle-Hernandez in a recent paper [2]. Also a series of Milnor number constant but non Whitney regular families, depending on a parameter ℓ, was given by Abderrahmane [1] as follows:
$$F_t^\ell(x,y,z) = x^{13} + y^{20} + zx^6y^5 + tx^6y^8 + t^2x^{10}y^3 + z^\ell,$$
for integers $\ell \geq 7$. Here $\mu = 153\ell + 32$, while $\mu^2(F_0) = 260$ and $\mu^2(F_t) = 189$, according to Abderrahmane. We do not yet know whether weak Whitney regularity holds or fails for these examples.

References

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