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# Hodge-theoretic splitting mechanisms for projective maps

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**Abstract**

According to the decomposition and relative hard Lefschetz theorems, given a projective map of complex quasi projective algebraic varieties and a relatively ample line bundle, the rational intersection cohomology groups of the domain of the map split into various direct summands. While the summands are canonical, the splitting is certainly not, as the choice of the line bundle yields at least three different splittings by means of three mechanisms in a triangulated category introduced by Deligne. It is known that these three choices yield splittings of mixed Hodge structures. In this paper, we use the relative hard Lefschetz theorem and elementary linear algebra to construct five distinct splittings, two of which seem to be new, and to prove that they are splittings of mixed Hodge structures.

## Contents



# 1 Introduction and main theorem

Let $f : X \to Y$ be a projective map of complex quasi projective varieties, let

$$H := \oplus_{d\geq 0} IH^d(X, \mathbb{Q})$$

be the total intersection cohomology rational vector space of $X$ and let $\eta \in H^2(X, \mathbb{Q})$ be the first Chern class of an $f$-ample line bundle on $X$. We refer to the survey [6] for the background concerning perverse sheaves and the decomposition and relative hard Lefscehtz theorems etc. that we use.

The map $f$ endows $H$ with the perverse Leray filtration $P$. The graded objects

$$H_p := P_p H / P_{p-1} H$$

are non-trivial only in a certain interval $[-r, r]$, with $r = r(f) \in \mathbb{Z}_{\geq 0}$. We thus obtain the two objects $\mathbb{H} := (H, P)$ and $\mathbb{H}_* = \oplus_p (H_p, T[-p])$ ($T[-p]$ = the trivial filtration translated to position $p$) in $\mathscr{V}_{\mathbb{Q}}\mathscr{F}$, the filtered category of finite dimensional rational vector spaces. Obviously, every filtration on a vector space by vector subspaces splits and we have a good (= inducing the identity on the graded pieces) isomorphisms $\varphi : \mathbb{H}_* \cong \mathbb{H}$ in $\mathscr{V}_{\mathbb{Q}}\mathscr{F}$.

The vector space $H$ underlies a natural mixed Hodge structure (MHS). The subspaces $P_p H$ are mixed Hodge substructures (MHSS) so that the graded objects $H_p$ are endowed with a natural MHS. Let $\mathscr{M}HS$ be the Abelian category of rational mixed Hodge structures. It is natural to ask whether there are good-isomorphisms $\varphi : \mathbb{H}_* \cong \mathbb{H}$ in $\mathscr{M}HS\mathscr{F}$, the filtered category of mixed Hodge structures. In English, do we have splittings $\varphi : \oplus_p H_p \cong H$ such that the component $H_p \to H$ is a map of MHS so that, in particular, the image is a MHSS?

In this paper, we list five distinct such mixed-Hodge theoretic good splittings. They are built by using the $f$-ample $\eta$ and they depend on it (see Theorem 1.1.1):

$$\omega_{\mathrm{I}}(\eta),\ \omega_{\mathrm{II}}(\eta),\ \phi_{\mathrm{I}}(\eta),\ \phi_{\mathrm{II}}(\eta),\ \phi_{\mathrm{III}}(\eta)\ :\quad \mathbb{H}_* \xrightarrow{\ \cong\ } \mathbb{H} \qquad \text{in } \mathscr{M}HS\mathscr{F} \tag{1}$$

The key to our approach is the relative hard Lefschetz theorem (RHL). The cup product with $\eta$ induces an arrow $\eta : \mathbb{H} \to \mathbb{H}[2](1)$ in $\mathscr{M}HS\mathscr{F}$. This means that $\eta : H \to H(1)$ (Tate shift (1)) is such that $\eta : P_p H \to P_{p+2} H(1)$ (translation of filtration [2] and Tate shift (1)). RHL yields isomorphisms in $\mathscr{M}HS$:

$$\eta^k\ : H_{-k} \xrightarrow{\ \cong\ } H_k(k), \qquad \forall\, k \geq 0. \tag{2}$$

Our main technical result, which in fact is proved in an elementary way, is as follows: let $\mathscr{A}$ be an Abelian category with shift functors $(n)$ and let $(\mathbb{V}, e)$ be a pair where $e : \mathbb{V} \to \mathbb{V}[2](1)$ is an arrow in the filtered category $\mathscr{A}\mathscr{F}$ inducing isomorphisms $e^k : V_{-k} \cong V_k(k)$, for every $k \geq 0$; then there is a natural isomorphism $\omega_{\mathrm{I}}(e) : \mathbb{V}_* \cong \mathbb{V}$ in $\mathscr{A}\mathscr{F}$.

With this result in hand, we easily verify that we can construct the remaining four splittings within $\mathscr{A}\mathscr{F}$. We then set $\mathscr{A} = \mathscr{M}HS$ and deduce (1). Let us stress again, that we use RHL in an essential way and that the point made in this paper is that once you have this deep result, the splittings (1) stem from elementary linear algebra considerations.

The construction of the three splittings of type $\phi$ is borrowed from [8]. However, it seems that [8] only yields $\phi$-type splittings in $\mathscr{V}_{\mathbb{Q}}\mathscr{F}$, i.e., not necessarily in its refinement $\mathscr{M}HS\mathscr{F}$. By coupling [8] with the theory of mixed Hodge modules, one can indeed prove that the splittings of type $\phi$ take place in $\mathscr{M}HS\mathscr{F}$. By way of contrast, as pointed out above, the constructions of this paper are based on the elementary construction of $\omega_{\mathrm{I}}(e)$ in $\mathscr{A}\mathscr{F}$.

The fact that $\phi_{\mathrm{I}}(\eta)$ is mixed-Hodge theoretic had been proved in [2, 4] (projective and quasi projective case, respectively) by using the properties of the cup product, of Poincaré duality and geometric descriptions of the perverse Leray filtration $P$ on $H$ associated with the map $f$. The proof that $\phi_{\mathrm{I}}(\eta)$ is an isomorphism in $\mathscr{M}HS\mathscr{F}$ that we give here is different since it does not use the aforementioned special features of the geometric situation.

The simple Examples 2.6.4 and 2.6.5 show that the five splittings (1) are, in general, pairwise distinct.

There is a natural condition, the existence of an $e$-good splitting, under which the five splittings coincide; see Definition 2.4.5 and Proposition 2.6.3.

In the paper [7], we proved the following result (auxiliary to the main result of [7]): the Hitchin fibration $f : X \to Y$ for the groups $GL_2(\mathbb{C})$, $SL_2(\mathbb{C})$ and $PGL_2(\mathbb{C})$ associated with any compact Riemann surface of genus $g \geq 2$ and with Higgs bundles of odd degree, presents a natural $f$-ample line bundle $\alpha$ and, in the terminology of the present paper, the splitting $\phi_{\mathrm{I}}(\alpha)$ is $\alpha$-good ([7] shows that (55) holds for $\phi_{\mathrm{I}}(\alpha)$). In particular, in this case, the five splittings coincide.

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## 1.1 The main theorem

A splitting $\mathbb{H}_* \cong \mathbb{H}$ in $\mathscr{V}_{\mathbb{Q}}\mathscr{F}$ acquires significance only if we can describe $H_*$. This is the content of the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber (see the survey [6]) which implies the highly non-trivial fact that, up to a simple renumbering of cohomological degrees, we have that $H_p = H(Y, {}^p\mathcal{H}^p(Rf_*IC_X))$, where ${}^p\mathcal{H}^p(Rf_*IC_X)$ is the $p$-th perverse cohomology sheaf of the push-forward $Rf_*IC_X$ of the intersection cohomology complex $IC_X$ of $X$ ($= \mathbb{Q}_X[\dim_{\mathbb{C}} X]$, if $X$ is nonsingular).

Let us briefly discuss how the cohomology groups of these perverse sheaves split according to the decomposition by supports and to the primitive Lefschetz decomposition coming from RHL. Let us start with the one by supports, i.e., the $Z$'s appearing in what follows: each perverse sheaf $\mathcal{H}^p := {}^p\mathcal{H}^p(Rf_*IC_X)$ is a semi-simple perverse sheaf and decomposes canonically by taking supports: $\mathcal{H}^p = \oplus_Z \mathcal{H}^p_Z$, where the sum is finite and the summands are the intersection cohomology complexes with suitable semisimple local coefficients of suitable closed integral subvarieties $Z$ of $Y$.

The RHL induces the primitive Lefschetz decompositions (PLD) of $\mathcal{H}^p$: recall that $p \in [-r, r]$; let $i \geq 0$; set $\mathcal{P}^{\eta}_{-i,Z} = \mathrm{Ker}\,\{\eta^{i+1} : \mathcal{H}^{-i}_Z \to \mathcal{H}^{i+2}_Z\}$ and, for $0 \leq j \leq i$, set $\mathcal{P}^{\eta}_{ij,Z} := \mathrm{Im}\,\{\eta^j : \mathcal{P}^{\eta}_{-i,Z} \to \mathcal{H}^{-i+2j}\}$; then the PLD reads as $\mathcal{H}^p = \oplus_{-i+2j=p} \oplus_Z \mathcal{P}^{\eta}_{ij,Z}$ (sum subject to $0 \leq j \leq i$). Set $P^{\eta}_{-i}(-j)_Z := H(Y, \mathcal{P}^{\eta}_{ij,Z})$; it is a MHSS of $H_{-i+2j}$. Note that $\eta$ induces isomorphisms $\eta^j : (P^{\eta}_{-i}(0)_Z)(-j) \cong P^{\eta}_{-i}(-j)_Z$ of MHS. By combining the decomposition by supports with the primitive one, we obtain the splitting $\oplus_{i,j,Z} P^{\eta}_{-i}(-j)_Z \cong H_*$ which, in view of [1, 2, 4, 5], is a splitting of MHS. We set $\mathbb{P}^{\eta}_p := (\oplus_{-i+2j=p,Z} P^{\eta}_{-i}(-j)_Z, T[-p])$ (sum subject to $0 \leq j \leq i$; trivial filtration translated to position $p$) which is an object in $\mathscr{MHSF}$.

By taking into account these refined splittings, we have the corresponding refinements of (1). We note that everything holds just as well, with the same proofs, for intersection cohomology with compact supports.

The main result of this paper is the following

**Theorem 1.1.1** *Let $f : X \to Y$, $\eta$ and $\mathbb{H}$ be as above. There are five distinguished good splittings* (1) *in $\mathscr{MHSF}$:*

$$\bigoplus_{0\leq j\leq i,\,Z} \mathbb{P}^{\eta}_{-i}(-j)_Z \xrightarrow{\cong} \mathbb{H}_*. \tag{3}$$

*Similarly, for intersection cohomology with compact supports.*

*Proof.* We first work in the abstract setting outlined in §2.1 of the filtered category $\mathscr{AF}$ of an Abelian category $\mathscr{A}$ endowed with a shift functor. In this context, we work with an object $\mathbb{V}$ and an arrow $e : \mathbb{V} \to \mathbb{V}[2](1)$ subject to the HL condition (32).

The splitting $\omega_{\mathrm{II}}(e)$ is obtained by the "dual" procedure as follows. Let $\mathscr{A}^o$ be the category opposite to $\mathscr{A}$; it is Abelian and can be endowed with a shift functor coming from the given one in $\mathscr{A}$. We then have that $e^o : \mathbb{V}^o \to \mathbb{V}^o[2]^o(1)^o$ in $\mathscr{A}^o\mathscr{F}$ satisifes the corresponding HL condition (32). We thus obtain $\omega_{\mathrm{I}}(e^o) : \mathbb{V}^o_* \cong \mathbb{V}^o$. We set $\omega_{\mathrm{II}} := ((\omega_{\mathrm{I}}(e^o))^o)^{-1}$.

The splitting $\phi_{\mathrm{I}}(e)$ is constructed in §2.4. The proof is parallel to [8], §2 with the following two changes: (i) instead of using the existence of a splitting arising from [8], §1, which is proved using some basic features of $t$-categories, features that $\mathscr{AF}$ does not present, we use the existence of either of the splittings $\omega(e)$ established above; (ii) we adapt the proof of [8], Lemma 2.1, which again takes place in the context of a $t$-category, to the context of $\mathscr{AF}$.

The splitting $\phi_{\mathrm{II}}(e)$ is obtained in §2.5 by following the procedure "dual" to the one followed to produce $\phi_{\mathrm{I}}(e)$.

Finally, $\phi_{\mathrm{III}}(e)$, which necessitates that we work with a $\mathbb{Q}$-category (= the Hom-sets are rational vector spaces), is constructed in §2.6 by adapting the corresponding construction in [8], §3. This construction is self-dual.

We now specialize to $\mathscr{A} := \mathscr{MHS}$, with shift functor given by the Tate shift and to $(\mathbb{V}, e) := (\mathbb{H}, \eta)$ and conclude, due to the fact that the HL condition (32) is met by the RHL (2). □

# 2 The five splittings

## 2.1 Filtered category associated with an Abelian category

Let $\mathscr{A}$ be an Abelian category whose elements we denote $V, W$, etc. For ease of exposition only, in this paper we make heavy use of the language of sets and elements.

A filtration $F$ on $V$ is a finite increasing filtration, i.e., an increasing sequence of subobjects

$$\ldots \subseteq F_pV \subseteq F_{p+1}V \subseteq \ldots \subseteq$$

of $V$ such that $F_pV = 0$ for $p \ll 0$ and $F_pV = V$ for $p \gg 0$. We set $\mathrm{Gr}^{\mathrm{F}}_{\mathrm{p}}\mathrm{V} := F_pV/F_{i-1}V$. We denote by $T$ the trivial filtration on $V$: $T_{-1}V = 0 \subseteq T_0V = V$. Given $n \in \mathbb{Z}$, we denote by $F[n]$ the $n$-th translate of $F$: $F[n]_pV := F_{n+p}V$, so $\mathrm{Gr}^{F[n]}_p V = \mathrm{Gr}^F_{n+p}V$; for example $T[-p]$ is the trivial filtration in position $p$.

Given a pair $\mathbb{V} := (V, F)$ and a subquotient $U$ of $V$, the filtration $F$ on $V$ induces a filtration on $U$, which we still denote by $F$; for example $(\mathrm{Gr}^F_p V, F) = (\mathrm{Gr}^F_p V, T[-p])$. In particular, for every $p \leq q \in \mathbb{Z}$, we have the following pairs associated with $\mathbb{V}$:

$$\mathbb{V}_{\leq p} := \qquad (F_p V, F), \qquad\qquad \mathbb{V}_{\geq p} := \qquad (V/F_{p-1}V, F), \tag{4}$$

$$\mathbb{V}_p := \qquad (\mathrm{Gr}^F_p V, F), \qquad\qquad \mathbb{V}_{[p,q]} := \qquad (F_q V/F_{p-1}V, F).$$

We say that $\mathbb{V}$ has type $[a, b]$, with $a \leq b$, if $\mathrm{Gr}^F_p V = 0$ for every $p \notin [a, b]$.

Given $(V, F)$ and $(W, F)$, an arrow $\mathfrak{l} : V \to W$ in $\mathscr{A}$ is said to be a filtered arrow if it respects the given filtrations, i.e., $\mathfrak{l}$ maps $F_p V$ to $F_p W$, for every $p$.

The filtered category $\mathscr{A}F$ associated with $\mathscr{A}$ is the category with objects the pairs $\mathbb{V} = (V, F)$ and arrows the filtered arrows. In particular, an arrow $\mathfrak{l} : \mathbb{V} \to \mathbb{W}$ induces arrows on the objects listed in (4), e.g., $\mathfrak{l}_i : \mathbb{V}_i \to \mathbb{W}_i$. An arrow $\mathfrak{l}$ in $\mathscr{A}\mathscr{F}$ is an isomorphism if and only if $\mathfrak{l}_i$ is an isomorphism for every $i \in \mathbb{Z}$.

We have functors $[n] : \mathscr{A}\mathscr{F} \to \mathscr{A}\mathscr{F}$, $(V, F) = \mathbb{V} \mapsto \mathbb{V}[n] := (V, F[n])$, etc.

We have the following graded-type objects, in $\mathscr{A}$, $\mathscr{A}\mathscr{F}$ and $\mathscr{A}\mathscr{F}$, respectively:

$$V_* := \oplus_p V_p, \qquad \mathbb{V}_* := \bigoplus_p \mathbb{V}_p, \qquad \widetilde{\mathbb{V}}_* := \bigoplus_p (V_p, T[-p]). \tag{5}$$

We say that $\mathbb{V}$ splits in $\mathscr{A}\mathscr{F}$ it there is an isomorphism in $\mathscr{A}\mathscr{F}$:

$$\varphi : \mathbb{V}_* \xrightarrow{\cong} \mathbb{V}. \tag{6}$$

We say that a splitting $\varphi$ is good if, for every $p$, the induced map $\varphi_p : V_p \cong V_p$ is the identity.

**Remark 2.1.1** If $\mathbb{V}$ splits, then there is a good splitting: let $\varphi_p : \mathbb{V}_p \cong \mathbb{V}_p$ be the induced isomorphisms and replace $\varphi$ with $\varphi \circ (\sum_p \varphi_p^{-1})$.

The category $\mathscr{A}\mathscr{F}$ is pre-Abelian (additive with kernels and cokernels), hence pseudo-Abelian (every idempotent has a kernel). In particular, given an idempotent $\pi : \mathbb{V} \to \mathbb{V}$, $\pi^2 = \pi$, we have canonical splittings in $\mathscr{A}\mathscr{F}$:

$$\mathbb{V} = \mathrm{Ker}\,(\mathrm{id} - \pi) \oplus \mathrm{Ker}\,\pi = \mathrm{Im}\,\pi \oplus \mathrm{Ker}\,\pi. \tag{7}$$

The arrow $\iota : (V, T) \to (V, T)[1]$ induced by the identity is such that the induced arrow $\mathrm{Coim}\,\iota \to \mathrm{Im}\,\iota$ is not an isomorphism so that $\mathscr{A}\mathscr{F}$ is not Abelian.

**Example 2.1.2** The example we have in mind is the one where $\mathscr{A}$ is the Abelian category $\mathscr{M}HS$ of integral (or rational) mixed Hodge structures (MHS) where the arrows are the maps that respect the weight and Hodge filtrations. The Tate shift functor, denoted by (1) is such that if $\mathbb{Z}$ is the pure Hodge structure with weight zero and type $(0, 0)$, then $\mathbb{Z}(1)$ is the pure Hodge structure with weight $-2$ and type $(-1, -1)$. Note that, for example, the cup product with the first Chern class $L$ of a line bundle on a complex algebraic variety $X$ induces, for every $k \geq 0$, a map $L : IH^k(X, \mathbb{Z}) \to IH^{k+2}(X, \mathbb{Z})(1)$. An element $\mathbb{M} = (M, F)$ of $\mathscr{M}HS\mathscr{F}$ is a MHS $M$ (with its weight and Hodge filtrations) equipped with an *additional* filtration $F$ for which $F_p M$ is a mixed Hodge substructure (MHSS) of $M$ for every $p$.

Let $(1) : \mathscr{A} \to \mathscr{A}$, $V \mapsto V(1)$, $\mathfrak{l} \mapsto \mathfrak{l}(1)$, be an additive and exact autoequivalence and, for $m \in \mathbb{Z}$, denote by $(m)$ its $m$-th iterate, called the $m$-shift functor.

By exactness, the shift functors lift to functors (also called shift functors):

$$(m) \,:\, \mathscr{A}F \longrightarrow \mathscr{A}F, \quad \mathbb{V} = (V, F) \longmapsto (V(m), F) =: \mathbb{V}(m), \quad \mathfrak{l} \longmapsto \mathfrak{l}(m), \tag{8}$$

and we have

$$(\mathbb{V}(m))_p = \mathbb{V}_p(m), \qquad \mathrm{Gr}^F_p(\mathfrak{l}(m)) = (\mathrm{Gr}^F_p \mathfrak{l})(m). \tag{9}$$

The shift functors commute with the tanslation functors, so that we can write $\mathbb{V}[n](m)$ unambiguously. We have $(\mathbb{V}[n](m))_p = \mathbb{V}_{n+p}(m)$, etc.

For every $p \in \mathbb{Z}$, an arrow $\mathfrak{l} : \mathbb{V} \to \mathbb{W}[n](m)$ induces arrows in $\mathscr{A}$:

$$\mathfrak{l}_p \,:\, V_p \longrightarrow W_{n+p}(m), \tag{10}$$

where it is understood that:

$$W_{n+p}(m) \,=\, \mathrm{Gr}^F_{n+p}(m) \,=\, \mathrm{Gr}^{F[n]}_p W(m). \tag{11}$$

If $\mathbb{V}$ has type $[a, b]$, then we have canonical arrows:

$$\mathbb{V}_a \xrightarrow{i_a} \mathbb{V} \xrightarrow{p_b} \mathbb{V}_b, \tag{12}$$

first inclusion and last quotient, with compositum $\delta_{ab}\mathrm{Id}$.

An arrow $\mathfrak{l}[n](m)$ obtained from an arrow $\mathfrak{l} : \mathbb{V} \to \mathbb{W}$ by shift/translation is simply denoted by $\mathfrak{l} : \mathbb{V}[n](m) \to \mathbb{W}[n](m)$ (e.g., the arrow $\mathfrak{l}^{-1}_a$ in (21) is really $\mathfrak{l}^{-1}_a(-m)$).

Let $\mathbb{V}$ and $\mathbb{W}$ be in $\mathscr{A}\mathscr{F}$, let $m, n \in \mathbb{Z}$, let $\mathfrak{l} : \mathbb{V}_* \to \mathbb{W}_*[n](m)$ be an arrow in $\mathscr{A}\mathscr{F}$ and let $\mathfrak{l}_{pq} : \mathbb{V}_p \to \mathbb{W}_q[n](m)$ be the $(p, q)$-th component of $\mathfrak{l}$. We define the degree $d \in \mathbb{Z}$ homogeneous part $\mathfrak{l}^{\{d\}}$ of $\mathfrak{l}$ by setting:

$$\mathfrak{l}^{\{d\}} := \sum_{q-p=d} \mathfrak{l}_{pq}, \qquad\qquad \left(\mathfrak{l} = \sum_d \mathfrak{l}^{\{d\}}\right). \tag{13}$$

Since $\mathbb{V}_p = (V_p, T[-p])$ and $\mathbb{W}_q[n](m) = (W_q(m), T[-q+n])$, we must have

$$\mathfrak{l}^{\{d\}} = 0, \quad \forall d \geq n + 1. \tag{14}$$

Let $\mathscr{A}^o$ denote the Abelian category opposite to $\mathscr{A}$.

**Remark 2.1.3** Let $\mathscr{A} = \mathscr{V}_{\mathbb{Q}}$ be the category of finite dimensional rational vector spaces and linear maps. The natural contravariant functor $\mathscr{V}_{\mathbb{Q}} \to \mathscr{V}^o_{\mathbb{Q}}$ can be identified with taking dual vector spaces and transposition of linear maps. Similarly, if we take $\mathscr{A} = \mathscr{M}HS$. This observation may make what follows more down-to-earth and the computations of explicit examples easier.

We have the exact anti-equivalence $(-)^o : \mathscr{A} \to \mathscr{A}^o$, $(V \xrightarrow{f} W) \longmapsto (V^o \xleftarrow{f^o} W^o)$ whose second iterate is the identity functor. We endow $\mathscr{A}^o$ with the additive and exact shift functors $(m)^o$: $V^o \mapsto V^o(m^o) := (V(-m))^o$.

A filtered object $\mathbb{V} = (V, F)$ in $\mathscr{A}\mathscr{F}$ gives rise to a filtered object $\mathbb{V}^o = (V^o, F^o)$ in $\mathscr{A}^o\mathscr{F}$ by setting:

$$F^o_i V^o = (V^o)_{\leq i} := (V_{\geq -i})^o. \tag{15}$$

Contemplation of the following diagram may be useful:

$$\begin{tikzcd}
V_{\leq i-1} \arrow[r, "\text{mono}"] \arrow[dr] & V_{\leq i} \arrow[d] & & (V_{\leq i-1})^o & (V_{\leq i})^o \arrow[l, "\text{epi}"'] & \\
& V \arrow[d] & & & (V)^o \arrow[u] \arrow[ul] & \\
& V_{\geq i+1} & V_{\geq i} \arrow[l, "\text{epi}"'] & & (V_{\geq i+1})^o \arrow[u] \arrow[r, "\text{mono}"] & (V_{\geq i})^o.
\end{tikzcd} \tag{16}$$

Clearly, $(V^o)_i = (V_{-i})^o$ and we set $F^o[n] := (F[-n])^o$.

We obtain an anti-equivalence $(-)^o : \mathscr{A}\mathscr{F} \to \mathscr{A}^o\mathscr{F}$ whose second iterate is the identity functor. The anti-equivalence $(-)^o$ is anti-compatible with translations, shifts and taking graded pieces, etc., for example:

$$\mathbb{V}^o[n]^o(m)^o \;=\; (\mathbb{V}[-n](-m))^o. \tag{17}$$

An arrow $\mathfrak{l} : \mathbb{V} \to \mathbb{W}[n](m)$ in $\mathscr{A}\mathscr{F}$ yields the arrow $\mathfrak{l}^o : \mathbb{W}^o \to \mathbb{V}^o[n]^o(m)^o$ in $\mathscr{A}^o\mathscr{F}$. This arrow is really $\mathfrak{l}^o[n]^o(m)^o$, but we omit those decorations for arrows.

We record the following fact for use in the next section.

**Lemma 2.1.4** *Let $\mathscr{B}$ be an additive category and let $\rho : B \to B'$ be an arrow in $\mathscr{B}$. Assume that the kernel $\iota_\rho : \operatorname{Ker}\rho \to B$ of $\rho$ exists and that there is a splitting $r : B' \to B$ of $\rho$, i.e., $\rho \circ r = \mathrm{id}_{B'}$. Then the natural arrow*

$$B' \oplus \operatorname{Ker}\rho \xrightarrow{\;r+\iota_\rho\;} B \tag{18}$$

*is an isomorphism in $\mathscr{B}$.*

*Proof.* Note that $\rho \circ (1 - r\,\rho) = 0$, so that there is an unique arrow $s : B \to \operatorname{Ker}\rho$ such that $1 - r \circ \rho = \iota_\rho \circ s$. It is easy to verify that the arrow:

$$B \xrightarrow{\;(\rho,s)\;} B' \oplus \operatorname{Ker}\rho \tag{19}$$

yields the desired inverse to $r + \iota_\rho$. □

**Remark 2.1.5** Assume, in addition, that $\mathscr{B}$ is pseudo-Abelian, e.g., $\mathscr{B} = \mathscr{A}\mathscr{F}$, and consider the idempotent arrow $\pi := r \circ \rho$. Then we have a canonical isomorphism $B = \operatorname{Im}\pi \oplus \operatorname{Ker}\pi$. This isomorphism can be canonically identified with the one in (19), for $\operatorname{Ker}\pi = \operatorname{Ker}\rho$, and $r$ identifies $B'$ with $\operatorname{Im}\pi$.

## 2.2 A splitting mechanism in $\mathscr{A}\mathscr{F}$

Let $\mathbb{V} = (V, F)$ in $\mathscr{A}\mathscr{F}$ be of type $[a, b]$, let $m \in \mathbb{Z}$ and let:

$$\mathfrak{l} : \mathbb{V} \longrightarrow \mathbb{V}[b-a](m) \tag{20}$$

be an arrow such that the resulting arrow (10) is an isomorphism in $\mathscr{A}$:

$$\mathfrak{l}_a : V_a \xrightarrow{\cong} V_b(m). \tag{21}$$

There is the commutative diagram in $\mathscr{A}\mathscr{F}$ (see (12)):

$$\begin{tikzcd}
\mathbb{V}_a \oplus \mathbb{V}_b \arrow[r, "r", "i_a+\mathfrak{l} i_a \mathfrak{l}_a^{-1}"'] \arrow[drr, "1_{\mathbb{V}_a \oplus \mathbb{V}_b}"'] & \mathbb{V} \arrow[r, "\rho'", "(\mathfrak{l}_a^{-1} p_b \mathfrak{l}, p_b)"'] \arrow[dr, "\rho"] & \mathbb{V}_a \oplus \mathbb{V}_b \arrow[d, "u"', "{\begin{pmatrix} 1_{\mathbb{V}_a} & -\mathfrak{l}_a^{-1} p_b \mathfrak{l}^2 i_a \mathfrak{l}_a^{-1} \\ 0 & 1_{\mathbb{V}_b} \end{pmatrix}}"] \\
& & \mathbb{V}_a \oplus \mathbb{V}_b
\end{tikzcd} \tag{22}$$

so that:

$$\rho \ : \ \mathbb{V} \xrightarrow{\left(\mathfrak{l}_a^{-1} p_b \mathfrak{l} - \mathfrak{l}_a^{-1} p_b \mathfrak{l}^2 i_a \mathfrak{l}_a^{-1} p_b \ , \ p_b\right)} \mathbb{V}_a \oplus \mathbb{V}_b. \tag{23}$$

The kernel $\mathbb{K}\mathrm{er}\,\rho$ of $\rho$ in $\mathscr{A}\mathscr{F}$ is the kernel $\mathrm{Ker}\,\rho$ of the underlying map in $\mathscr{A}$ with the filtration induced by $(V, F)$. The natural inclusion induces a map in $\mathscr{A}F$:

$$\iota_\rho : \mathbb{K}\mathrm{er}\,\rho \longrightarrow \mathbb{V}. \tag{24}$$

**Remark 2.2.1** Since the arrows $u$ and $\mathfrak{l}_a$ in (22) are isomorphisms, we have that:

$$\mathrm{Ker}\,\rho = \mathrm{Ker}\,\rho' = \mathrm{Ker}\,(p_b \circ \mathfrak{l}) \cap \mathrm{Ker}\,p_b, \tag{25}$$

and similarly, if we take into account the induced filtrations.

**Lemma 2.2.2** *The following arrow is an isomorphism in $\mathscr{A}\mathscr{F}$:*

$$w : \mathbb{V}_a \oplus \mathbb{K}\mathrm{er}\,\rho \oplus \mathbb{V}_b \xrightarrow[\cong]{i_a + \iota_\rho + \mathfrak{l} i_a \mathfrak{l}_a^{-1}} \mathbb{V}. \tag{26}$$

*Proof.* Apply Lemma 2.1.4. □

**Remark 2.2.3** The map $w$ (26) is uniquely determined by, and depends on, $\mathfrak{l}$. However, the component $i_a$, being the inclusion of the first subspace of the filtration, is independent of $\mathfrak{l}$.

The object $\mathbb{K}\mathrm{er}\,\rho$ is of type $[a+1,b-1]$, the inclusion $\iota_\rho$ induces natural isomorphisms:

$$\iota_{\rho_p} : (\mathbb{K}\mathrm{er}\,\rho)_p \xrightarrow{\cong} \mathbb{V}_p\,, \qquad \forall\, p \in [a+1,b-1] \tag{27}$$

and, by taking subquotients, a natural isomorphism:

$$\iota_{\rho_{[a+1,b-1]}} : \mathbb{K}\mathrm{er}\,\rho \xrightarrow{\cong} \mathbb{V}_{[a+1,b-1]}. \tag{28}$$

By combining (26) with (28), we obtain an isomorphism:

$$w_{[a,b]} \;:\; \mathbb{V}_a \oplus \mathbb{V}_{[a+1,b-1]} \oplus \mathbb{V}_b \xrightarrow{\cong} \mathbb{V}, \tag{29}$$

as well as its component:

$$w_{[a+1,b-1]} \;:\; \mathbb{V}_{[a+1,b-1]} \longrightarrow \mathbb{V}. \tag{30}$$

Both isomorphisms induce the identity on the $p$-th graded pieces, for every $p$ in (29), for $p \in [a+1,b-1]$ in (30).

One may picture the content of Lemma 2.2.2 as an unwrapping of the outmost layer $\mathbb{V}_a \oplus \mathbb{V}_b$ of $\mathbb{V}$ via $\mathfrak{l}$. Note that in general, there is no natural non-trivial arrow from a subquotient of an object to the object itself. The arrow (30) is made possible by the HL condition.

## 2.3 The splittings $\omega_{\mathrm{I}}(e)$ and $\omega_{\mathrm{II}}(e)$

Let $\mathbb{V} = (V,F)$ in $\mathscr{A}\mathscr{F}$ be of type $[-r.r]$ for some $r \geq 0$. Up to a translation, this condition can always be met and leads to simplified notation in what follows.

Let

$$e : \mathbb{V} \longrightarrow \mathbb{V}[2](1) \tag{31}$$

be an arrow in $\mathscr{A}\mathscr{F}$. In particular, for every $k \geq 0$, we have the iterations $e^k$ and their graded counterparts: (we drop the shift when denoting a shifted map and, in what follows, we drop subscripts for the maps induced on graded objects):

$$e^k \;:\; \mathbb{V} \longrightarrow \mathbb{V}[2k](k), \qquad e^k = V_j \longrightarrow V_{j+2k}(k). \tag{32}$$

**Assumption 2.3.1** (**Condition HL**) We assume that $(\mathbb{V}, e)$ satisfies the hard Lefschetz-type condition (HL), i.e., that the arrows:

$$e^k : V_{-k} \xrightarrow{\cong} V_k(k), \qquad \forall k \geq 0, \tag{33}$$

are isomorphisms in $\mathscr{A}$.

The following proposition ensures that if the HL condition is met by a given $(\mathbb{V}, e)$, then $\mathbb{V}$ splits, i.e., we have an isomorphism $\mathbb{V} \cong \mathbb{V}$ (6). By keeping with the analogy of Remark 2.2.3, we may say that HL allows to completely unwrap $\mathbb{V}$. Recall the notion of good splitting (on inducing the identity on the graded pieces).

**Proposition 2.3.2** *Let $(\mathbb{V}, e)$ satisfy the HL condition. There is a good splitting:*

$$\omega_{\mathrm{I}}(e) = \omega_{\mathrm{I}} \, : \, \mathbb{V}_* = \bigoplus_{p \in [-r.r]} \mathbb{V}_p \xrightarrow{\ \cong\ } \mathbb{V}. \tag{34}$$

*Proof.* By applying Lemma 2.2.2 and to $\mathfrak{l} := e^r$, so that $[a, b] = [-r, r]$ and $m = r$, we obtain

$$w : \mathbb{V}_{-r} \oplus \mathbb{K}\mathrm{er}\, \rho \oplus \mathbb{V}_r \xrightarrow{\ \cong\ } \mathbb{V}. \tag{35}$$

The arrow $\mathfrak{l}$ yields the arrow $\tilde{\mathfrak{l}} := \omega_{\mathrm{I}}^{-1} \circ \mathfrak{l} \circ \omega_{\mathrm{I}}$ on the lhs of (35). Keeping in mind that (26) means that the filtration $F$ on $V$ splits, we obtain the $\mathbb{K}\mathrm{er}\, \rho$-component $\mathfrak{l}'$ of $\tilde{\mathfrak{l}}$:

$$\mathfrak{l}' : \mathbb{K}\mathrm{er}\, \rho \longrightarrow \mathbb{K}\mathrm{er}\, \rho[2](1). \tag{36}$$

In view of (27), we have that $\mathfrak{l}'$ satisfies the HL condition.
By using (28) and (29), we replace $\mathbb{K}\mathrm{er}\, \rho$ with $\mathbb{V}_{[-r+1, r-1]}$ and we obtain the desired splitting $\omega_{\mathrm{I}}(e)$ by descending induction on $r$.
By construction (i.e., $\tilde{\mathfrak{l}} = \omega_{\mathrm{I}}^{-1} \circ \mathfrak{l} \circ \omega_{\mathrm{I}}$, (27) and (33)), the isomorphism $\omega_{\mathrm{I}}(e)$ induces the identity on the graded pieces and is thus good. □

The splitting $\omega_{\mathrm{II}}(e)$ is obtained by the dual construction. This is explained in the proof of Theorem 1.1.1.

**Remark 2.3.3** In general, $\omega_{\mathrm{I}}(e) \neq \omega_{\mathrm{II}}(e)$; see Examples 2.6.4 and 2.6.5. In particular, neither of the two constructions $\omega_{\mathrm{I}}(e)$ and $\omega_{\mathrm{II}}(e)$ is self-dual.

The purpose of the next three sections is to show that if $\mathbb{V}$ splits in $\mathscr{A}\mathscr{F}$, then one can use the HL property (33) to construct three additional natural splittings taking place in $\mathscr{A}\mathscr{F}$. Note that these constructions are based solely on the existence of a splitting.

## 2.4 The first Deligne splitting $\phi_{\mathrm{I}}(e)$

Let $(\mathbb{V}, e)$ be as in (31) and assume that it satisfies the HL condition (33). In particular, in view of (34), $\mathbb{V}$ splits in $\mathscr{A}\mathscr{F}$.

Let $i \geq 0$ and define the primitive objects in $\mathscr{A}$:

$$P_{-i} := \mathrm{Ker}\, \left\{ e^{i+1} : V_{-i} \longrightarrow V_{i+2}(i+1) \right\}. \tag{37}$$

The subquotient $P_{-i}$ of $V$ inherits the filtration induced by $F$ on $V$, i.e., the trivial filtration translated in position $-i$, and we denote the resulting object in $\mathscr{A}\mathscr{F}$ by:

$$\mathbb{P}_{-i} = (P_{-i}, F) = (P_{-i}, T[i]). \tag{38}$$

We have the natural monomorphisms in $\mathscr{A}$:

$$P_{-i}(-j) \longrightarrow V_{-i}(-j) \xrightarrow{e^j} V_{-i+2j} \tag{39}$$

which, taken together, yield the canonical primitive Lefschetz decompositions (PLD) in $\mathscr{A}$ and $\mathscr{A}\mathscr{F}$, respectively:

$$\bigoplus_{0\leq j\leq i} P_{-i}(-j) \xrightarrow[\cong]{\epsilon} V_*, \tag{40}$$

$$\bigoplus_{0\leq j\leq i} (P_{-i}(-j), T[i-2j]) = \bigoplus_{0\leq j\leq i} \mathbb{P}_{-i}(-j)[-2j] \xrightarrow[\cong]{\epsilon} \mathbb{V}_*.$$

We have the commutative diagram of epimorphisms and monomorphisms in $\mathscr{A}\mathscr{F}$:

$$\begin{tikzcd}
 & & & \mathbb{P}_{-i} \arrow[r, "\text{mono}"] \arrow[d, "\text{mono}"'] & \mathbb{V}_{-i} \arrow[dl, "\text{mono}"] \\
\mathbb{V} \arrow[r, "\text{epi}"] & \ldots \arrow[r, "\text{epi}"] & \mathbb{V}_{\geq -i-1} \arrow[r, "\text{epi}"] & \mathbb{V}_{\geq -i}. &
\end{tikzcd} \tag{41}$$

Note that (38) implies that if $\mathfrak{l} : \mathbb{P}_{-i} \to \mathbb{W}$ is an arrow in $\mathscr{A}\mathscr{F}$, then it factors through $\mathbb{W}_{\leq -i}$, i.e., the underlying arrow $\mathfrak{l} : P_{-i} \to W$, factors through $F_{-i}W$.

**Lemma 2.4.1** *Let $(\mathbb{V}, e)$ be as above. There is a unique arrow $f_i : \mathbb{P}_{-i} \to \mathbb{V}$ in $\mathscr{A}\mathscr{F}$ with the following properties:*

1. *it lifts the natural arrow $\mathbb{P}_{-i} \to \mathbb{V}_{\geq -i}$ in (41);*
2. *for every $s > i \geq 0$, the composition of the arrows below is zero:*

$$\mathbb{P}_{-i} \xrightarrow{e^s \circ f_i} \mathbb{V}(s) \longrightarrow \mathbb{V}_{\geq s}(s). \tag{42}$$

*Proof.* The proof is essentially identical to the one of [8], Lemme 2.1 (see also [8], 2.3). We include it, with the necessary changes, for the reader's convenience. Recall that we use the language of sets.

Let $\Phi : \mathscr{A}\mathscr{F} \to \mathscr{B}$ be an additive functor into an Abelian category $\mathscr{B}$. We denote $\Phi(e)$ simply by $e$. Let $i \geq 0$ and $x \in \Phi(V_{\geq -i})$ be such that $0 = e^{i+1}(x) \in \Phi(V_{\geq i+2})$.

*CLAIM 1:* there is a unique lift $y \in \Phi(V_{\geq -i-1})$ of $x$ such that $0 = e^{i+1}(y) \in \Phi(V_{\geq i+1})$.

*Proof.* For every $a \in \mathbb{Z}$, we have the natural maps:

$$\mathbb{V}_a \longrightarrow \mathbb{V}_{\geq a+1} \longrightarrow \mathbb{V}_{\geq a+1}. \tag{43}$$

Since $\Phi$ is additive and, in view of Lemma 2.3.2, $\mathbb{V}$ splits in $\mathscr{A}\mathscr{F}$, we have the short exact sequences in $\mathscr{B}$ stemming from (43):

$$0 \longrightarrow \Phi(\mathbb{V}_a) \longrightarrow \Phi(\mathbb{V}_{\geq a}) \longrightarrow \Phi(\mathbb{V}_{\geq a+1}) \longrightarrow 0.$$

By naturality, we have the following commutative diagram of short exact sequences in $\mathscr{B}$:

$$\begin{tikzcd}
0 \arrow[r] & \Phi(\mathbb{V}_{-i-1}) \arrow[r] \arrow[d, "e^{i+1}", "\cong"'] & \Phi(\mathbb{V}_{\geq -i-1}) \arrow[r] \arrow[d, "e^{i+1}"] & \Phi(\mathbb{V}_{\geq -i}) \arrow[r] \arrow[d, "e^{i+1}"] & 0 \\
0 \arrow[r] & \Phi(\mathbb{V}_{i+1}(i+1)) \arrow[r] & \Phi(\mathbb{V}_{\geq i+1}(i+1)) \arrow[r] & \Phi(\mathbb{V}_{\geq i}(i+1)) \arrow[r] & 0.
\end{tikzcd} \tag{44}$$

The claim follows from a simple diagram-chase starting at $x \in \Phi(\mathbb{V}_{\geq -i})$.

*CLAIM 2:* under the same hypotheses as the ones of CLAIM 1, there is a unique lift $y \in \Phi(\mathbb{V})$ of $x \in \Phi(\mathbb{V}_{\geq i})$ such that $\forall s > i$, we have that $0 = e^s(y) \in \Phi(\mathbb{V}_{\geq s})$.
*Proof.* The element $y$ found in CLAIM 1 satisfies the hypotheses of CLAIM 1 for $i+1$. Since the filtration is finite, we conclude by a repeated use of CLAIM 1.

Let us apply CLAIM 2 to the functor $\Phi(-) := \mathrm{Hom}_{\mathscr{A}\mathscr{F}}(\mathbb{P}_{-i}, -) : \mathscr{A}\mathscr{F} \to \mathrm{Ab}$ (Abelian groups): set $x : \mathbb{P}_{-i} \to \mathbb{V}_{\geq -i}$ to be as in (41).
The statement of the lemma follows by applying CLAIM 2 to $x$: the hypotheses of CLAIM 2 are met in view of the defining property (37) of $P_{-i}$, and the resulting element $y$ is the desired $f_i$. $\square$

Let $\varphi : \mathbb{V}_* \cong \mathbb{V}$ be any splitting. In view of the primitive Lefschetz decomposition (40), we can talk about the components $\varphi_{ij} : \mathbb{P}_{-i}(-j)[-2j] \to \mathbb{V}$. Of course, there are many splittings having components $\varphi_{ii} = f_i$.

We define the *first Deligne isomorphism* $\phi_{\mathrm{I}}(e)$ *associated with* $(\mathbb{V}, e)$ by taking the compositum of the following two isomorphism

$$\phi_{\mathrm{I}}(e) = \phi_{\mathrm{I}} : \mathbb{V}_* \xrightarrow[(40)]{\epsilon^{-1}} \bigoplus_{0 \leq j \leq i} \mathbb{P}_{-i}(-j)[-2j] \xrightarrow[\cong]{\sum e^j \circ f_i} \mathbb{V}. \tag{45}$$

By using the language of elements, if we denote by $p_{ij}$ the typical element in $P_{-i}(-j)$ and we form the typical element $v_* \in V_*$:

$$v_* = \sum_p v_p = \epsilon \left( \sum_p \sum_{-i+2j=p} p_{ij} \right) \in V_*, \tag{46}$$

then

$$\phi_{\mathrm{I}}(e) : v_* \longmapsto \sum_{0 \leq j \leq i} e^j \left( f_i(p_{ij}) \right). \tag{47}$$

In particular, we have that

$$\phi_{\mathrm{I}}(\epsilon(e^l p_{ij})) = e^l \phi_{\mathrm{I}}(\epsilon(p_{ij})), \qquad \forall\, 0 \leq l \leq i-j. \tag{48}$$

**Remark 2.4.2** Let us omit the shifts, translations and filtrations. Let $\varphi : \mathbb{V}_* \cong \mathbb{V}$ be a splitting and $\varphi_i : P_{-i} \to V$ be the resulting components. By (37), we have that

$$e^{i+t} \circ \varphi_i : P_{-i} \to V_{\leq i+2t-1},$$

for every $t > 0$. Lemma 2.4.1 yields $f_i : P_{-i} \to V$ with

$$e^{i+t} \circ f_i : P_{-i} \to V_{\leq i+t-1},$$

for every $t > 0$, i.e., an improvement by $t$ units with respect to an arbitrary splitting, even a good one. The paper [7] exploits this special property of $\phi_{\mathrm{I}}(e)$ in the context of a study of the geometry of the Hitchin fibration.

**Remark 2.4.3** By constuction (see (45) and Remark 2.1.1), the isomorphisms $\phi_{\mathrm{I}}(e)$ and $\omega_{\mathrm{I}}(e)$ are good (6). In general, the two differ from each other; see Examples 2.6.4 and 2.6.5. However, they agree on $\mathbb{V}_{-r} \oplus \mathbb{V}_r$: in fact, both induce the identity on the graded pieces, so that they must agree on $\mathbb{V}_{-r} = \mathbb{P}_{-r}$; by comparing the expression (45) for $\phi_{\mathrm{I}}(e)$ restricted to $\mathbb{V}_r$ with the corresponding one for $\omega_{\mathrm{I}}(e)$, i.e., (35) and (26), we see that they coincide. In particular, if $V_i = 0$ for every $|i| \neq r$, then $\omega_{\mathrm{I}}(e) = \phi_{\mathrm{I}}(e)$.

Let $\varphi : \mathbb{V}_* \cong \mathbb{V}$ be a splitting. The matrix $\tilde{e}(\varphi)$ of $e$ with respect to $\varphi$ is defined by setting:

$$\tilde{e}(\varphi) \;=\; \tilde{e} := \left(\varphi^{-1} \circ e \circ \varphi\right) \;:\; \mathbb{V}_* \longrightarrow \mathbb{V}_*[2](1), \qquad \tilde{e} = \sum_{pq} \tilde{e}_{pq} = \sum \tilde{e}^{\{d\}}. \tag{49}$$

By virtue of (14), we have that

$$\tilde{e}^{\{d\}} = 0, \quad \forall d > 2. \tag{50}$$

Let us assume that $\varphi$ is good. Then

$$\tilde{e}^{\{2\}} = \sum_p e, \quad e : \mathbb{V}_p \xrightarrow{e} \mathbb{V}_{p+2}(1). \tag{51}$$

Note that while $\tilde{e}^{\{2\}}$ is independent of $\varphi$, we have that $\tilde{e}(\varphi)^{\{d\}}$ depends on $\varphi$ for $d \leq 1$.

We have the refinement $\hat{e}(\varphi)$ of the matrix $\tilde{e}(\varphi)$ that takes into account the primitive Lefschetz decomposition (40):

$$\hat{e}(\varphi) \;=\; \hat{e} \;:=\; \left(\epsilon^{-1} \circ \tilde{e} \circ \epsilon\right) \;:\; \bigoplus_{0 \leq j \leq i} \mathbb{P}_{-i}(-j)[-2j] \longrightarrow \bigoplus_{0 \leq j \leq i} \mathbb{P}_{-i}(-j)[-2j][2](1). \tag{52}$$

By taking components, we have arrows

$$\hat{e}(\varphi)^{kl}_{ij} \;:\; \mathbb{P}_{-i}(-j)[-2j] \longrightarrow \mathbb{P}_{-k}(-l)[-2l][2](1). \tag{53}$$

Proposition 2.7 in [8] can be easily adapted to the present context and yield the following characterization of $\phi_{\mathrm{I}}(e)$. For a "visual", see [8], p.119.

**Lemma 2.4.4** *The splitting $\phi_{\mathrm{I}} : \mathbb{V}_* \cong \mathbb{V}$ is characterized among the good ones by the following two conditions:*

1. *for $0 \leq j < i$, we have $\hat{e}(\phi_{\mathrm{I}})^{kl}_{ij} = 0$ except for $\hat{e}(\phi_{\mathrm{I}})^{i,j+1}_{ij} = \mathrm{Id}$;*
2. *for $j = i$, we have the $\hat{e}(\phi_{\mathrm{I}})^{kl}_{ii} = 0$ except, possibly, for $l \leq i$.*

**Definition 2.4.5** We say that a splitting $\varphi : \mathbb{V}_* \cong \mathbb{V}$ is *e-good* if it induces the identity on the graded pieces and $\tilde{e}(\varphi)$ is homogeneous of degree two:

$$\tilde{e}(\varphi) = \tilde{e}(\varphi)^{\{2\}}. \tag{54}$$

Clearly, $\varphi$ is $e$-good if and only we have that

$$\hat{e}(\varphi)^{kl}_{ij} = 0 \qquad \text{except for } \hat{e}(\varphi)^{i,j+1}_{ij} = \mathrm{Id}, \ \forall\, 0 \leq j \leq i-1. \tag{55}$$

We also have that $\varphi$ is $e$-good if and only the composita:

$$\mathbb{P}_{-i} \xrightarrow{\subseteq} \mathbb{V} \xrightarrow{\quad e^{i+1} \quad} \mathbb{V}[2i+2](i+1) \tag{56}$$

are zero for every $i \geq 0$. In this case we say that the $i$-th graded primitive objects $P_{-i}$ are embedded into $V$ via $\varphi$ as bona-fide $i$-th primitive classes: i.e., killed by

$$e^{i+1} : P_{-i} \to V_{\leq i+1}(i+1),$$

and not just killed by the subsequent projection to $V_{i+1}(i+1)$.

**Remark 2.4.6** Lemma 2.4.4 implies that if $\varphi$ is $e$-good, then $\varphi = \phi_{\mathrm{I}}(e)$. In particular, if there exists an $e$-good splitting, then it is unique. However, $e$-good splittings do not exist in general: the reader can verify this in Examples 2.6.4 and 2.6.5; in the latter example, one can even take $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Proposition 2.6.3 shows that the existence of an $e$-good splitting is rare.

## 2.5 The second Deligne splitting $\phi_{\mathrm{II}}(e)$

Let $(\mathbb{V}, e)$ be as in the beginning of §2.4. In particular, $\mathbb{V}$ admits a splitting in $\mathscr{A}\mathscr{F}$ as in Proposition 2.3.2: in fact, we have three so far $\omega_{\mathrm{I}}(e)$, $\omega_{\mathrm{II}}(e)$ and $\phi_{\mathrm{I}}(e)$.

The first Deligne isomorphism $\phi_{\mathrm{I}}(e^o)$ (45) associated with $(\mathbb{V}^o, e^o)$ in $\mathscr{A}^o\mathscr{F}$ yields, by application of $(-)^o : \mathscr{A}^o\mathscr{F} \to \mathscr{A}\mathscr{F}$, the isomorphism in $\mathscr{A}\mathscr{F}$:

$$(\phi_{\mathrm{I}}(e^o))^o \,:\, \mathbb{V} \xrightarrow{\quad \cong \quad} \mathbb{V}_*. \tag{57}$$

We define the *second Deligne isomorphism associated with* $(\mathbb{V}, e)$ to be

$$\phi_{\mathrm{II}}(e) \,=\, \phi_{\mathrm{II}} \,:=\, ((\phi_{\mathrm{I}}(e^o))^o)^{-1} \,:\, \mathbb{V}_* \xrightarrow{\quad \cong \quad} \mathbb{V}. \tag{58}$$

In this context, the analogue of Lemma 2.4.1 reads as follows. Let

$$f'_{ij} \,:\, \mathbb{V} \xrightarrow{\qquad} \mathbb{P}_{-i}(-j)[-2j] \ \ (\overset{\epsilon}{\subseteq} \mathbb{V}_i) \tag{59}$$

be the components of (57) associated with (40) ($\epsilon$ as in (40)).

**Lemma 2.5.1** *For every $i \geq 0$, the arrow $f'_{ii}$ is the unique arrow $\mathbb{V} \to \mathbb{V}_i$ such that:*

1. *by taking the $i$-th graded pieces, $f'_{ii}$ induces the natural projection $\mathbb{V}_i \to \mathbb{P}_{-i}(-i)[-i]$;*
2. *for every $s > i$, the composition below is zero is (see [8], §3.1):*

$$\mathbb{V}_{\leq -s} \xrightarrow{\subseteq} \mathbb{V} \xrightarrow{\epsilon^s} \mathbb{V}[2s](s) \xrightarrow{\quad f'_{ii} \quad} \mathbb{P}_{-i}(-i)[-2i][2s](s). \tag{60}$$

By using Lemma 2.5.1 and the explicit formula (47) for $\phi_{\rm I}(e)$, it is easy to deduce the following explicit expression for the arrows $f'_{ii}$:

$$f'_{ii}(\phi_{\rm I}(v_*)) = p_{ii} \qquad (v_* \;=\; \textstyle\sum_{0 \leq j' \leq i'} p_{i'j'}). \tag{61}$$

In general, $\phi_{\rm I}(e) \neq \phi_{\rm II}(e)$ and this discrepancy is due to the fact that $f'_{ij}(\phi_{\rm I}(v_*)) \neq p_{ij}$. We now discuss how this discrepancy is measured exactly in terms the matrix $\hat{e}(\phi_{\rm I})$ (53) of $e$.

By combining (45) with (57), we see that there is the commutative diagram (the bottom identification is due to (40)):

$$\begin{tikzcd} \mathbb{V} \arrow[rr, "e^{i-j}"] \arrow[d, "f'_{ij}"] & & \mathbb{V}[2(i-j)](i-j) \arrow[d, "f'_{ii}"] \\ \mathbb{P}_{-i}(-j)[-2j] \arrow[rr, "e^{i-j}=\mathrm{Id}"] & & \mathbb{P}_{-i}(-i)[-2i][2i-2j](i-j). \end{tikzcd} \tag{62}$$

We fix $0 \leq j \leq i$. For every $0 \leq s \leq t$, we use (47) and (48) together with (61) and (62) with the goal of determining the value of

$$f'_{ij}\left(e^s f_t(p_{ts})\right). \tag{63}$$

Recalling that (57) induces the identity on the graded pieces, we deduce that:

1. if $t = i$ and $s < j$, then $f'_{ij}\left(e^s f_t(p_{ts})\right) = 0$;

   we can see this, as well as the assertions that follow, on the following diagram (we do not write $\epsilon$):

$$e^s f_t(p_{is}) = \phi_{\rm I}(p_{is}) \longmapsto e^{i-j}\phi_{\rm I}(p_{is}) = \phi_{\rm I}(e^{i-j}p_{is}) \stackrel{f'_{ii}}{\longmapsto} 0; \tag{64}$$

2. if $t = i$ and $s = j$ then $f'_{ij}\left(e^j f_i(p_{ij})\right) = p_{ij}$;

3. if $t \neq i$ and $s + i - j \leq t$, then $f'_{ij}\left(e^s f_t(p_{ts})\right) = 0$;

4. if $t \neq i$ and $\sigma := s + i - j - t \geq 1$, then

$$f'_{ij}\left(e^s f_t(p_{ts})\right) = f'_{ii}(e^\sigma e^t f_t(p_{ts})) \tag{65}$$

   which, recalling the definition (53) of $\hat{e}_{\rm I} = \hat{e}(\phi_{\rm I}(e))$, has the following form:

$$(\hat{e}_{\rm I}^\sigma)^{ii}_{tt}\,(q_{tt}) \qquad (\text{where } q_{tt} \;:=\; f'_{tt}(e^t f_t(p_{ts}))\,. \tag{66}$$

**Proposition 2.5.2** *The first Deligne isomorphism $\phi_{\rm I}(e)$ is $e$-good (55) if and only if*

$$\phi_{\rm I}(e) = \phi_{\rm II}(e).$$

*Proof.* In view of (57) and 58), we have that the two Deligne isomorphisms coincide if and only if, using the notation in (61), we have that $f'_{ij}(v_*) = p_{ij}$, for every $0 \leq j \leq i$. According to the four points above, the only obstruction to having this latter condition stems from (66) not being zero for some pair $(i,t)$ with $i \neq t$. By reasons of degree, i.e., by (14), since $i \neq t$, we have that $(\hat{e}_{\mathrm{I}})^{ii}_{tt}$ is in degrees $\leq 1$ . If $\phi_{\mathrm{I}}$ is $e$-good, then $\hat{e}_{\mathrm{I}}$ is of pure homogeneous degree 2, so that $e^{\sigma} e^t f_t(p_{ts})) = 0$ and we infer the desired equality.

Conversely, let us assume that the two Deligne isomorphisms coincide. By contradiction let us assume that $\phi_{\mathrm{I}}$ is not $e$-good. According to (55) there are integers $0 \leq t$ and $0 \leq l \leq k$ and a non zero arrow $l \leq t$. Among these non zero arrows $(\hat{e}_I)^{kl}_{tt}$, chose one, $(\hat{e}_I)^{k_o l_o}_{t_o t_o}$, for which the difference $k - l$ attains the minimum value. In the language of elements, what above ensures that there is $0 \neq p_{t_o} \in P_{-t_o}$ such that:

$$e^{t_o+1} f_{t_o}(p_{t_o}) = e^{l_o} f_{k_o} \left( \hat{e}^{k_o l_o}_{t_o t_o} \left( e^{t_o} p_{t_o} \right) \right) + \sum{}^{*} e^l f_k \left( \hat{e}^{kl}_{t_o t_o} \left( e^{t_o} p_{t_o} \right) \right), \tag{67}$$

where the first term on the r.h.s. is non-zero and $\sum^*$ is the sum over the non-zero terms with $(k,l) \neq (k_o, l_o)$. Since for these latter terms, $k - l \geq k_o - l_o$, we deduce that

$$f'_{k_o k_o} \left( e^{k_o - l_o} \sum{}^{*} \right) = 0 \in P_{-k_o}(-k_o). \tag{68}$$

On the other hand, since obviously $e^{k_o - l_o} e^{l_o} = e^{k_o}$, we have that:

$$q_{k_o} \ := \ f'_{k_o k_o} \left( e^{k_o - l_o} e^{l_o} f_{k_o} \left( \hat{e}^{k_o l_o}_{t_o t_o} \left( e^{t_o} p_{t_o} \right) \right) \right) \neq 0 \in P_{-k_o}(-k_o). \tag{69}$$

In view of (62), we have that

$$f'_{k_o, l_o - 1} \left( e^{t_o} f_{t_o}(p_{t_o}) \right) = q_{t_o} \neq 0. \tag{70}$$

Since we are assuming that the two Deligne isomorphisms coincide, by virtue of the first paragraph of this proof, we must have $k_o = t_o$ and $t_o = l_o - 1$. This contradicts $l_o \leq t_o$. $\square$

**Remark 2.5.3** In general, there is no $e$-good splitting; see Examples 2.6.4 and 2.6.5. In particular, neither of the two constructions $\phi_{\mathrm{I}}(e)$ and $\phi_{\mathrm{II}}(e)$ is self-dual.

## 2.6 The third Deligne splitting $\phi_{\mathrm{III}}(e)$

Let $(\mathbb{V}, e)$ be as in §2.4. In particular, $\mathbb{V}$ admits a splitting (34) in $\mathscr{A}\mathscr{F}$ as in Proposition 2.3.2; in fact, we have four, so far. We also assume that the Abelian category $\mathscr{A}$ is $\mathbb{Q}$-linear, i.e., that Hom-groups are rational vector spaces. The reason for this is that, in what follows, one needs to exploit the $\mathrm{sl}_2(\mathbb{Q})$-action arising from the given arrow $e$.

The goal of this section is to construct the third Deligne isomorphism associated with $(\mathbb{V}, e)$. Whereas we omit the detailed presentation of the algebra underlying this construction (see [8], Lemme 3.3 and Proposition 3.5), we review some of the key points, state its characterization and, along the way, indicate the necessary changes.

Let $\mathbb{V}$ and $\mathbb{W}$ be in $\mathscr{A}\mathscr{F}$ and set:

$$L^{[n]}_{(i,j)}(\mathbb{V}, \mathbb{W}) := \mathrm{Hom}_{\mathscr{A}\mathscr{F}}(\mathbb{V}(i), \mathbb{W}(j)[n]). \tag{71}$$

Up to the canonical isomorphism induced by the shift functors, the above depends only on the difference $m := (j - i)$ and we denote the resulting bi-functor by $L^{[n]}_{(m)}$.

Recalling the definition of the graded-type objects (5) and of degree of maps (13) (an arrow $(V_p, T) \to (V_q, T)[n](m)$ in $\mathscr{A}\mathscr{F}$ has degree $d := q - p$), we have the natural decomposition by homogeneous degrees:

$$L^{[n]}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*) = \bigoplus_{d=-2r}^{2r} L^{[n],\{d\}}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*). \tag{72}$$

The arrow $h := \sum_p p \,\mathrm{Id}_{(V_p,T)}$ induces the arrow:

$$h : L^{[n]}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*) \longrightarrow L^{[n]}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*), \quad u \longmapsto h \circ u; \tag{73}$$

this arrow is of homogeneous degree zero, i.e., $\{d\} \mapsto \{d\}$, with respect to (72).
By taking together the graded pieces of the arrow $e : \mathbb{V} \to \mathbb{V}[2](1)$, i.e., set $e' := \sum e_p$, with $e_p : V_p \to V_{p+2}(1)$, we obtain $e' \in L^{[0],\{2\}}_{(1)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*)$ which, in turn, induces the homogeneous degree two arrow:

$$e \;:\; L^{[n]}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*) \longrightarrow L^{[n]}_{(m+1)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*), \qquad u \longmapsto e(u) := e' \circ u - u \circ e'. \tag{74}$$

There is a canonical arrow of homogeneous degree $-2$ (this is where we need denominators ([8], p.121):

$$f \;:\; L^{[n]}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*) \longrightarrow L^{[n]}_{(m-1)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*). \tag{75}$$

The arrows $(h, e, f)$ in (73), (74) and (75) form an $\mathrm{sl}_2(\mathbb{Q})$-triple turning the rational vector spaces

$$L^{[n]}_j \;:\; \bigoplus_{d \in \mathbb{Z}^{\text{even/odd}}} L^{[n],\{d\}}_{j+d/2}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*) \tag{76}$$

into $\mathrm{sl}_2(\mathbb{Q})$-modules; in what above, $j$ is a fixed integer multiple of $1/2$ and the sum is over the integers $d$ with fixed parity, even if $j$ is integral, odd if $j$ is an half-integer. Recalling that the sum is finite, for $|d| \leq 2r$, we have that the corresponding HL statememt reads as follows: ($e^k$ the $k$-th iteration of $e$ (74))

$$e^k \;:\; L^{[n],(-k)}_{(j-k/2)} \xrightarrow{\quad\cong\quad} L^{[n],(k)}_{(j+k/2)}\,. \tag{77}$$

Let $\varphi \;:\; \mathbb{V}_* \cong \mathbb{V}$ be any good splitting (6) and let $\tilde{e}(\varphi)$ be the associated matrix of $e : \mathbb{V} \to \mathbb{V}[2](1)$ (49). The degree $d$ homogeneous part of $\tilde{e}(\varphi)$ satisfies:

$$\tilde{e}(\varphi)^{\{d\}} := \sum_{q-p=d} \tilde{e}(\varphi)_{pq} \;\in\; L^{[2-d],\{d\}}_{(1)} \tag{78}$$

and is subject to (50) and (51): it is zero for every $d > 2$ and it is the obvious arrow for $d = 2$.

The *the third Deligne isomorphism associated with* $(\mathbb{V}, e)$:

$$\phi_{\mathrm{III}}(e) \;:= \phi_{\mathrm{III}} \;:\; \mathbb{V}_* \xrightarrow{\quad\cong\quad} \mathbb{V}, \tag{79}$$

is the unique good splitting subject to the following conditions: ($e^{1-d}$ is the $(1-d)$-iteration of (74)):

$$e^{1-d}\left(\tilde{e}(\varphi)^{\{d\}}\right) \;=\; 0, \qquad \forall d \leq 1. \tag{80}$$

Let us illustrate how, any good splitting $\varphi : \mathbb{V}_* \cong \mathbb{V}$ can be modified recursively, via HL (77), to obtain a new good splitting subject to (80).

Let $d = 1$. The condition (80) reads $\tilde{e}^{\{1\}} = 0$. Set $\varphi_1 := \varphi(\mathrm{id} + \psi^{\{-1\}})$, where $\psi^{\{-1\}} \in L^{[1],\{-1\}}_{(0)}$ is a variable arrow. We conjugate $e$ and obtain:

$$\left(\mathrm{id} + \psi^{\{-1\}}\right)^{-1} \circ e \circ \left(\mathrm{id} + \psi^{\{-1\}}\right) \equiv \tilde{e}(\varphi)^{\{2\}} + e\left(\psi^{\{-1\}}\right) \quad \text{modulo degree} \leq 0. \tag{81}$$

Note that the last term on the left is in $L^{[1],\{1\}}_{(1)}$. We take the degree 1 part of the r.h.s of (81) and set it equal to zero

$$e\left(\psi^{\{-1\}}\right) = -\tilde{e}(\varphi)^{\{1\}} \qquad \left(\text{equality in } L^{[1],\{1\}}_{(1)}\right). \tag{82}$$

The HL (77) ensures that such a $\psi^{\{-1\}}$ exists and is unique. This determines $\varphi_1$.

Let $d = 0$. The condition (80) reads $e(\tilde{e}^{\{0\}}) = 0$. Set $\varphi_0 := \varphi_1(\mathrm{id} + \psi^{\{-2\}})$, where $\psi^{\{-2\}} \in L^{[1],\{-2\}}_{(0)}$ is a variable arrow. We conjugate $e$ and obtain

$$\left(\mathrm{id} + \psi^{\{-2\}}\right)^{-1} e \left(\mathrm{id} + \psi^{\{-2\}}\right) \equiv \tilde{e}(\varphi_1)^{\{2\}} + e\left(\psi^{\{-2\}}\right) \quad \text{modulo degree} \leq -1. \tag{83}$$

Note that the last term on the left is in $L^{[1],\{0\}}_{(1)}$. We take the degree 0 part of the r.h.s of (83) and set it equal to zero after application of $e$:

$$e^2\left(\psi^{\{-2\}}\right) = -e\left(\tilde{e}(\varphi_1)^{\{0\}}\right) \qquad \left(\text{equality in } L^{[1],\{2\}}_{(2)}\right). \tag{84}$$

The HL (77) ensures that such a $\psi^{\{-2\}}$ exists and is unique. This determines $\varphi_0$.

We repeat this procedure for all decreasing values of $d$ and, recalling that $\mathbb{V}$ has type $[-r, r]$, the procedure ends no later than $d = -2r$.

The unicity of the resulting arrow is verified easily as follows. Let $a, b$ be two good splittings subject to (80). Set

$$c \;:=\; b^{-1}a = \mathrm{Id} + \textstyle\sum_{l \geq 1} c^{\{-l\}} \;:\; \mathbb{V}_* \xrightarrow{\quad\cong\quad} \mathbb{V}. \tag{85}$$

We apply the procedure carried out above to $b$, modifying it to $b_1 := b(\mathrm{Id} + c^{\{-1\}})$. We have that $b_1 \equiv a$ modulo degree $\leq -2$, so that, in view of the fact that $a$ also satisfies (80), we must have that $c^{\{-1\}} = 0$. It follows that $b \equiv a$, modulo degree $\leq -1$. We repeat this procedure and kill all the $c^{\{-l\}}$.

In general, $\phi_{\mathrm{I}} \neq \phi_{\mathrm{III}}$, however, by [8], Proposition 3.6 (easily adapted to the present context), we have that: (see Lemma (2.4.1) for the definition of $f_i$)

$$\phi_{\mathrm{III}}(e)_{|\mathbb{P}_{-i}} = \phi_{\mathrm{I}}(e)_{|\mathbb{P}_{-i}} = f_i : \mathbb{P}_i \longrightarrow \mathbb{V}. \tag{86}$$

**Remark 2.6.1** Unlike the four previous splittings $\omega_{\mathrm{I,II}}(e)$ and $\phi_{\mathrm{I,II}}(e)$, the construction leading to $\phi_{\mathrm{III}}(e)$ is self-dual in the sense that we have:

$$\left(\left(\phi_{\mathrm{III}}\left(e^{o}\right)\right)^{o}\right)^{-1}=\phi_{\mathrm{III}}(e). \tag{87}$$

This is because an isomorphism $\varphi$ satisfies condition (80) if and only if $\varphi^o$ satisfies the analogous "opposite" conditions.

**Remark 2.6.2** In genereal, the five good splittings

$$\omega_{\mathrm{I}}(e), \quad \omega_{\mathrm{II}}(e), \quad \phi_{\mathrm{I}}(e), \quad \phi_{\mathrm{II}}(e), \quad \phi_{\mathrm{III}}(e) \tag{88}$$

are pairwise distinct; see Examples 2.6.4 and 2.6.5.

**Proposition 2.6.3** *If an $e$-good splitting* (2.4.5) *exists, then it is unique and it coincides with the third Deligne isomorphism. In this case we have:*

$$\omega_{I}(e)=\omega_{\mathrm{II}}(e)=\phi_{\mathrm{I}}(e)=\phi_{\mathrm{II}}(e)=\phi_{\mathrm{III}}(e). \tag{89}$$

*Proof.* Let $\varphi$ be $e$-good. Then $\tilde{e}(\varphi)^{\{d\}}=0$ for every $d\leq 1$. It follows that condition (80) is met by $\varphi$, so that $\varphi=\phi_{\mathrm{III}}(e)$ and we have proved uniqueness (see also Remark 2.4.6).
Assume that there is an $e$-good splitting, which, by the above, must coincide with $\phi_{\mathrm{III}}(e)$. By Remark 2.4.6, we have that $\phi_{\mathrm{III}}(e)=\phi_{\mathrm{I}}(e)$. Proposition 2.5.2 implies that $\phi_{\mathrm{I}}(e)=\phi_{\mathrm{II}}(e)$ (this equality can be also seen by using a duality argument similar to the one in (92)).
Let us compare $\phi_{\mathrm{I}}(e)$ with $\omega_{\mathrm{I}}(e)$. By Remark (2.4.3), the two agree on $\mathbb{V}_{-r}\oplus\mathbb{V}_{r}$.
By comparing the general description (25) of $\operatorname{Ker}\rho$ with the formula (47) for the embedding $\phi_{\mathrm{I}}$, we see, with the aid of (56), that

$$\operatorname{Ker}\rho=\left\{v\in V \mid v=\sum_{(i,j)\in I_{r}} e^{j} f_{i}(p_{ij})\right\}, \tag{90}$$

where $I_r$ is the set of the indices subject to $0\leq j\leq i$ and to $(i,j)\neq(r,0),(r,r)$. It follows that $\operatorname{Ker}\rho$ coincides with $\phi_{\mathrm{I}}(\sum_{I_r} P_{-i}(-j))$. By projecting onto $\sum_{|p|\neq r} V_p$, we deduce that $\phi_{\mathrm{I}}(e)$, restricted to $\sum_{|p|\neq r} V_p$ factors through $\operatorname{Ker}\rho\to V$. Now we repeat for $\operatorname{Ker}\rho$, what we have done above for $V$ and deduce, by descending induction on $r$, that

$$\phi_{\mathrm{I}}(e)=\omega_{\mathrm{I}}(e). \tag{91}$$

We conclude by using a duality argument:

$$\omega_{\mathrm{II}}(e)=((\omega_{\mathrm{I}}(e^{o}))^{o})^{-1}=((\phi_{\mathrm{I}}(e^{o}))^{o})^{-1}=\phi_{\mathrm{II}}(e)=\omega_{\mathrm{I}}(e), \tag{92}$$

where: the first equality is by definition; the second equality follows by the fact that there is an $e$-good splitting for $(\mathbb{V},e)$ if and only there is an $e^o$-good splitting for $(\mathbb{V}^o,e^o)$ and we have proved that $\omega_{\mathrm{I}}=\phi_{\mathrm{I}}$; the third equality is by definition; the final equality is (91). This concludes the proof. □

**Example 2.6.4** $V \cong \mathbb{Q}^3$ with basis $(v_{-2}, v_0, v_2)$, $e : (v_{-2}, v_0, v_2) \longmapsto (v_0, v_2, v_2)$,

$$V_{\leq -2} = V_{\leq -1} = \langle v_{-2} \rangle, \quad V_{\leq 0} = V_{\leq 1} = \langle v_{-2}, v_0 \rangle, \quad V_{\leq 2} = V, \tag{93}$$

$$V_{-2} = \langle [v_{-2}] \rangle, \quad V_0 = \langle [v_0] \rangle, \quad V_2 = \langle [v_2] \rangle. \tag{94}$$

The five splittings associated with $e$ discussed in this paper are:

- $\phi_{\mathrm{I}}(e) : ([v_{-2}], [v_0], [v_2]) \longmapsto (v_{-2}, v_0, v_2)$;
- $\phi_{\mathrm{II}}(e) : ([v_{-2}], [v_0], [v_2]) \longmapsto (v_{-2}, -v_{-2} + v_0, -v_0 + v_2)$;
- $\phi_{\mathrm{III}}(e) : ([v_{-2}], [v_0], [v_2]) \longmapsto (v_{-2}, -\frac{1}{3}v_{-2} + v_0, -\frac{2}{3}v_0 + v_2)$;
- $\omega_{\mathrm{I}}(e) : ([v_{-2}], [v_0], [v_2]) \longmapsto (v_{-2}, -v_{-2} + v_0, v_2)$;
- $\omega_{\mathrm{I}}(e) : ([v_{-2}], [v_0], [v_2]) \longmapsto (v_{-2}, -v_{-2} + v_0, -v_{-2} + v_2)$.

A direct calculation, or Proposition 2.6.3, shows that there is no $e$-good splitting.

**Example 2.6.5** Here is a class of examples from geometry where, unlike the previous example, $e$ is nilpotent.

Let $Y \times Z$ be the product of nonsingular complex projective varieties and let $r := \dim_{\mathbb{C}} Z$. Let $V := H(Y \times Z, \mathbb{Q}) = H(Y, \mathbb{Q}) \otimes H(Z, \mathbb{Q}) = \oplus_{r,s} H^r(Y, \mathbb{Q}) \otimes H^s(Z, \mathbb{Q})$. Let $F'_s V$ be the subspace spanned by the elements of the form $y_r z_\sigma$ with $\sigma \leq s$. Set $F := F'[r]$; this way $(V, F)$ has type $[-r, r]$. Let $\eta \in H^2(Y \times Z, \mathbb{Q})$ be the first Chern class of a line bundle on $Y \times Z$ which is ample when restricted to the fibers of the projection onto $Y$. Denote by $e : (V, F) \to (V, F[2])$ the map $v \mapsto \eta \cup v$.

By using the hard Lefschetz theorem on $Z$, one sees directly that the HL condition (33) holds for $((V, F), e)$. In addition to the five splittings considered in this paper, we also have the Künneth splitting $\kappa$. In general, the six splittings are pairwise distinct. The reader can verify this fact directly by taking $Y = Z$ to be an elliptic curve and the line bundle to be of the form $E \times \zeta + \zeta \times E + \mathfrak{P}$, where $\zeta \in E$ is a point and $\mathfrak{P}$ is a Poincaré bundle (this is to ensure that $\tilde{e}(\kappa)^{\{1\}} \neq 0$, so that, according to (80), we must have $\kappa \neq \phi_{\mathrm{III}}(e)$).

If we take $Y = \mathbb{P}^1 \times \mathbb{P}^2$, we are lead to examples where $\kappa = \phi_{\mathrm{III}}(e)$, but otherwise the isomorphisms of type $\phi$ and $\omega$ are pairwise distinct, and distinct from $\phi_{\mathrm{III}}(e)$. If we take $Y = Z = \mathbb{P}^1$, then we have $\kappa = \phi_{\mathrm{III}}(e)$, $\phi_{\mathrm{I}}(e) = \omega_{\mathrm{I}}(e)$ and $\phi_{\mathrm{II}}(e) = \omega_{\mathrm{II}}(e)$, but we have no further relation. In this case, if we take $\eta$ to be the class of the fiber of the projection onto $Z$, then we have an $e$-good splitting. In all the examples, for general $\eta$, there is no $e$-good splitting. A highly non-trivial example, where there is a good splitting is mentioned at the end of the introduction.

# 3 Appendix: a letter from P. Deligne

P. Deligne has sent the author a letter commenting on an earlier draft of this paper. The author is happy to include, with P. Deligne's kind permission, this letter in this appendix as it outlines the simple modifications necessary to obtain the splittings of this note in a Tannakian (tensor product) context.

March 16, 2012

Dear de Cataldo,

Thank you!

Let $\mathscr{A}$ be an Abelian category with a shift functor $A \to A(1)$ as in your text. Let $\mathscr{B}$ be the category of objects of $\mathscr{A}$ given with a finite increasing filtration $F$ and $e : A \to A[2](1)$ verifying HL.

**Corollary.** $\mathscr{B}$ is an abelian category.

*Proof.* (a) for objects $(A, F, e)$ of $\mathscr{B}$ of type $[-r, r]$, with $r > 0$, the "peeling off" 2.2

$$A = F_r \oplus \mathrm{Ker}\left(F_{r-1} \subseteq A \xrightarrow{e^r} A \longrightarrow A/F_{r-1}\right) \oplus e^r\left(F_r\right)$$

is functorial. By induction on $r$, it follows that

(b) the splitting $\omega_{\mathrm{I}}$ is functorial.

If $f : (A, F, e) \longrightarrow (A', F', e')$ is a morphism, (b) implies that $\mathrm{Gr}^F \mathrm{Ker}\,(f) \xrightarrow{\cong} \mathrm{Ker}\,\mathrm{Gr}^F(f)$, and dually for coKer. That $(\mathrm{Ker}\,(f), F, e)$ is in $\mathscr{B}$ follows. It is a kernel. Dually for cokernels. Morphisms are strictly compatible with filtrations, hence $\mathrm{CoIm}(f) \xrightarrow{\cong} \mathrm{Im}\,(f)$.

**Remark.** In your 2.2. you should assume $a < b$.

From now on, all categories are assumed to be $\mathbb{Q}$-linear.

Let $\mathscr{A}', \mathscr{A}''$ and $\mathscr{A}$ be as above and let $\otimes : \mathscr{A}' \times \mathscr{A}'' \longrightarrow \mathscr{A}$ be an exact biadditive functor, compatible with shifts: functorial isomorphisms

$$A'(1) \otimes A'' \xrightarrow{\cong} (A' \otimes A'')(1),$$

and

$$A' \otimes A''(1) \xrightarrow{\cong} (A' \otimes A'')(1)$$

are given, and the two resulting functorial isomorphisms

$$A'(1) \otimes A''(1) \longrightarrow (A' \otimes A'')(2)$$

coincide. Let $\mathscr{B}', \mathscr{B}''$ and $\mathscr{B}$ be the corresponding categories of triples $(A, F, e)$ as above. Given $(A', F', e')$ and $(A'', F'', e'')$, one defines the filtration $F$ (resp. morphism $e$) for $A' \otimes A''$ by

$$F_p = \sum_{p'+p''=p} F'_p \otimes F''_p$$

(so that $\mathrm{Gr}^{F'}(A') \otimes \mathrm{Gr}^{F''}(A'') \xrightarrow{\cong} \mathrm{Gr}^F(A)$), and

$$e = e' \otimes 1 + 1 \otimes e'$$

(so that the same formula holds for the graded $e, e', e''$).

**Proposition.** *This $\otimes$ sends $\mathscr{B}', \mathscr{B}''$ to $\mathscr{B}$.*

One has to check that for graded objects and morphisms of degree 2, $\otimes$ preserves HL. In the graded case, HL for $(A^*, e)$ is equivalent to the existence of $f^* : A^* \longrightarrow A^*(-1)$ of degree $-2$ such that $[e, f]$ id multiplicaiton by $n$ in degree $n$. Stability of this property is proved in the same way that tensor product of representations of Lie algebras are defined.

Suppose now that $\mathscr{A}$ is a Tannakian category and that the twist is the tensor product with an object $\mathbb{Q}(1)$ of rank one. The compatibility between $\otimes$ and shift we required amounts tothe symmetry automorphism of $\mathbb{Q}(1) \otimes \mathbb{Q}(1)$ being the identity.

**Corollary.** *If $\mathscr{A}$ is Tannakian, so is $\mathscr{B}$.*

*Proof.* If $\omega$ is a fiber functor on $\mathscr{A}$, then $(A, F, e) \mapsto \omega(A)$ is a fiber functor on $\mathscr{B}$.

In terms of actions on $SL(2)$, rather in terms of grading and of $e$ and $f$ above, I prefer to state the characteristic property of the (good) splitting $\phi_{\mathrm{III}}$ as follows: it is the filtered isomorphism, with graded the identity:

$$u : \mathrm{Gr}^F(A) \longrightarrow A$$

such that, with $e : \mathrm{Gr}^F(A) \longrightarrow \mathrm{Gr}^F(A)(1)$, and $f$ as above, if $X$ is defiend by

$$u^{-1}eu = e + X,$$

one has $[f, X] = 0$. This characterization makes it clear that this splitting is compatible with tensor products (in the sense of the proposition). It gives an equivalence of the category $\mathscr{B}$ of triples $(A, F, e)$ with the category of graded objects $A^*$, 0 outside of finitely many degrees, given with

$$e : A^* \longrightarrow A^*(1) \text{ and } f : A^* \longrightarrow A(-1)$$

of degree 2, resp. $-2$, with $[e, f] = n$ in degree $n$, and given with $X : (\oplus A^n) \longrightarrow (\oplus A^n)(1)$ such that $[f, X] = 0$.

In the Tannakian case, this equivalence is compatible with $\otimes$ [for $X$'s, $\otimes$ is defined by $X = X' \otimes 1 + 1 \otimes X''$].

Best,

P. Deligne.

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