

TOPOLOGICAL INVARIANTS AND MODULI OF GORENSTEIN SINGULARITIES

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ABSTRACT. We describe all connected components of the space of hyperbolic Gorenstein quasi-homogeneous surface singularities. We prove that any connected component is homeomorphic to a quotient of \mathbb{R}^d by a discrete group.

1. INTRODUCTION

In this paper, we study moduli spaces of hyperbolic Gorenstein quasi-homogeneous surface singularities (GQHSS). Quasi-homogeneous surface singularities and their graded affine coordinate rings were studied in great detail in the 1970's by Dolgachev, Milnor, Neumann, Pinkham and others. They described a correspondence between hyperbolic quasi-homogeneous surface singularities and (singular) complex line bundles on Riemann orbifolds, where a Riemann orbifold is a quotient \mathbb{H}/Γ of the hyperbolic plane \mathbb{H} by a Fuchsian group Γ . According to the work of Dolgachev [Dol83b], hyperbolic GQHSS of level m are in 1-to-1 correspondence with m -th roots of tangent bundles of Riemann orbifolds, i.e., with (singular) complex line bundles on Riemann orbifolds such that their m -th tensor power coincides with the tangent bundle. For more details of this correspondence, see section 2.1. In this paper, we complete the classification of hyperbolic Gorenstein quasi-homogeneous surface singularities by studying their parameter space.

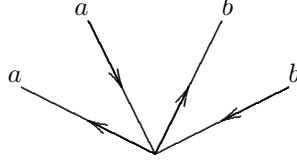
We determine the number and describe the topology of connected components of the space of all hyperbolic Gorenstein quasi-homogeneous surface singularities. We show that the space is connected if $g = 0$ or if $g > 1$ and m is odd and that the space has two connected components if $g > 1$ and m is even. We also determine the number of connected components in the case $g = 1$. Moreover, we prove that any component is homeomorphic to a quotient of \mathbb{R}^d by a discrete group action.

On the other hand, we obtain a description of the topology of the moduli space of higher spin structures on Riemann orbifolds. Because of their role in the modern models of 2D-gravitation, moduli spaces of this kind have enjoyed a great deal of interest recently, in particular, in relation to the Witten conjecture; see [Jar00, JKV01, FSZ10] and [Wit93]. Classification of classical spin structures on non-compact Riemann surfaces and the corresponding moduli space were studied in [Nat91, Nat04].

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Figure 1: $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) - 1$

The main technical tool is the following: We assign (Theorem 5.9) to a hyperbolic GQHSS of level m with corresponding orbifold P a function on the space of homotopy classes of simple contours on P with values in $\mathbb{Z}/m\mathbb{Z}$, the associated higher m -Arf function.

The higher m -Arf functions are described by simple geometric properties:

Definition: Let P be a Riemann orbifold and $p \in P$. Let $\pi(P, p)$ be the orbifold fundamental group of P (see Definition 4.3). We denote by $\pi^0(P, p)$ the set of all non-trivial elements of $\pi(P, p)$ that can be represented by simple contours. A (higher) m -Arf function is a function

$$\sigma : \pi^0(P, p) \rightarrow \mathbb{Z}/m\mathbb{Z}$$

satisfying the following conditions

1. $\sigma(bab^{-1}) = \sigma(a)$ for any elements $a, b \in \pi^0(P, p)$,
2. $\sigma(a^{-1}) = -\sigma(a)$ for any element $a \in \pi^0(P, p)$ that is not of order 2,
3. $\sigma(ab) = \sigma(a) + \sigma(b)$ for any elements a and b which can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b \rangle \neq 0$,
4. $\sigma(ab) = \sigma(a) + \sigma(b) - 1$ for any elements $a, b \in \pi^0(P, p)$ such that the element ab is in $\pi^0(P, p)$ and the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b \rangle = 0$ and a neighbourhood of the point p with the contours is homeomorphic to the one shown in Figure 1.
5. For any elliptic element c of order p we have $p \cdot \sigma(c) + 1 \equiv 0 \pmod{m}$.

In order to be able to state our main results we need to give some definitions.

Definition: (For a detailed discussion of Fuchsian groups, signatures and standard sets of generators, see Section 4.2.) Let Γ be a Fuchsian group such that the corresponding orbifold $P = \mathbb{H}/\Gamma$ is of signature $(g : p_1, \dots, p_r)$. Let $\sigma : \pi^0(P, p) \rightarrow \mathbb{Z}/m\mathbb{Z}$ be a higher m -Arf function. We define the *Arf invariant* $\delta = \delta(P, \sigma)$ of σ as follows: If $g > 1$ and m is even then we set $\delta = 0$ if there is a standard set of generators $\{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$ of the orbifold fundamental group $\pi(P, p)$ such that

$$\sum_{i=1}^g (1 - \sigma(a_i))(1 - \sigma(b_i)) \equiv 0 \pmod{2}$$

and we set $\delta = 1$ otherwise. If $g > 1$ and m is odd then we set $\delta = 0$. If $g = 0$ then we set $\delta = 0$. If $g = 1$ then we set

$$\delta = \gcd(m, p_1 - 1, \dots, p_r - 1, \sigma(a_1), \sigma(b_1)),$$

where $\{a_1, b_1, c_2, \dots, c_{r+1}\}$ is a standard set of generators of the orbifold fundamental group $\pi(P, p)$. The *type of the higher m -Arf function* (P, σ) is the tuple $(g, p_1, \dots, p_r, \delta)$, where δ is the Arf invariant of σ defined above.

Definition: We denote by $S^m(t) = S^m(g, p_1, \dots, p_r, \delta)$ the set of all GQHSS of level m such that the associated higher Arf function is of type $t = (g, p_1, \dots, p_r, \delta)$.

The following theorem summarizes the main results:

Theorem:

- 1) Two hyperbolic GQHSS are in the same connected component of the space of all hyperbolic GQHSS if and only if they are of the same type. In other words, the connected components of the space of all hyperbolic GQHSS are those sets $S^m(t)$ that are not empty.
- 2) The set $S^m(t)$ is not empty if and only if $t = (g, p_1, \dots, p_r, \delta)$ has the following properties:
 - (a) The orders p_1, \dots, p_r are prime with m and satisfy the condition

$$(p_1 \cdots p_r) \cdot \left(\sum_{i=1}^r \frac{1}{p_i} - (2g - 2) - r \right) \equiv 0 \pmod{m}.$$

- (b) If $g > 1$ and m is odd then $\delta = 0$.
 - (c) If $g = 1$ then δ is a divisor of $\gcd(m, p_1 - 1, \dots, p_r - 1)$.
 - (d) If $g = 0$ then $\delta = 0$.
- 3) Any connected component $S^m(t)$ of the space of all hyperbolic GQHSS of level m and signature $(g : p_1, \dots, p_r)$ is homeomorphic to a quotient of the space $\mathbb{R}^{6g-6+2r}$ by a discrete action of a certain subgroup of the modular group (see Section 6.3 for details).

The paper is organised as follows: In Section 2, we explore the connection between hyperbolic GQHSS, roots of tangent bundles of orbifolds and lifts of Fuchsian groups into the coverings G_m of $G = \mathrm{PSL}(2, \mathbb{R})$ (see Definition 2.2). In Section 3, we study algebraic properties of the covering groups G_m . We describe level functions induced by a decomposition of the covering G_m into sheets and choosing an enumeration of the sheets and study properties of these functions. In Section 4, we study lifts of Fuchsian groups into G_m . In Section 5, we define (higher) m -Arf functions. We prove that there is a 1-1-correspondence between the set of m -Arf functions and the set of functions associated to the lifts of Fuchsian groups into G_m via the enumeration of the covering sheets. Hence, these two sets are also in 1-1-correspondence with the set of all hyperbolic GQHSS of level m . Moreover, we show that the set of all m -Arf functions on an orbifold has a structure of an affine space, using an explicit description of the generalised Dehn generators of the group of homotopy classes of surface self-homeomorphisms. In the last section, we find topological invariants of higher Arf functions and prove that they describe the connected components of the moduli space. Furthermore, we show using a version of the theorem of Fricke and Klein [Nat78, Zie81] that any connected component is homeomorphic to a quotient of \mathbb{R}^d by a discrete group action.

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2. GORENSTEIN SINGULARITIES AND LIFTS OF FUCHSIAN GROUPS

2.1. Singularities and automorphy factors. In this section, we recall the results of Dolgachev, Milnor, Neumann and Pinkham [Dol75, Dol77, Mil75, Neu77, Pin77] on the graded affine coordinate rings, which correspond to quasi-homogeneous surface singularities.

Definition 2.1. A *negative unramified automorphy factor* (U, Γ, L) is a complex line bundle L over a simply-connected Riemann surface U together with a discrete co-compact subgroup $\Gamma \subset \mathrm{Aut}(U)$ acting compatibly on U and on the line bundle L , such that the following two conditions are satisfied:

- 1) The action of Γ is free on L^* , the complement of the zero-section in L .
- 2) Let $\tilde{\Gamma} \triangleleft \Gamma$ be a normal subgroup of finite index, which acts freely on U , and let $E \rightarrow P$ be the complex line bundle $E = L/\tilde{\Gamma}$ over the compact Riemann surface $P = U/\tilde{\Gamma}$. Then E is a negative line bundle, i.e., the self-intersection number $P \cdot P$ is negative.

A simply-connected Riemann surface U can be \mathbb{CP}^1 , \mathbb{C} , or \mathbb{H} , the real hyperbolic plane. We call the corresponding automorphy factor and the corresponding singularity *spherical*, *Euclidean*, resp. *hyperbolic*.

Remark. There always exists a normal freely acting subgroup of Γ of finite index [Dol83b]. In the hyperbolic case the existence follows from the theorem of Fox-Bundgaard-Nielsen. If the second assumption in the last definition holds for some normal freely acting subgroup of finite index, then it holds for any such subgroup, see [Dol83b].

Remark. In this paper we are going to study the largest class, the class of hyperbolic GQHSS. See a remark at the end of the paper for more information about the other two classes of GQHSS.

The simplest example of such complex line bundle with a group action is the (co)tangent bundle of $U = \mathbb{CP}^1$, resp. $U = \mathbb{H}$, equipped with the canonical action of a subgroup of $\text{Aut}(U)$.

Let (U, Γ, L) be a negative unramified automorphy factor and $\tilde{\Gamma}$ a normal subgroup of Γ as above. Since the bundle $E = L/\tilde{\Gamma}$ is negative, one can contract the zero section of E to get a complex surface with one isolated singularity corresponding to the zero section. There is a canonical action of the group $\Gamma/\tilde{\Gamma}$ on this surface. The quotient is a complex surface $X(U, \Gamma, L)$ with an isolated singular point $o(U, \Gamma, L)$, which depends only on the automorphy factor (U, Γ, L) .

The following theorem summarizes the results of Dolgachev, Milnor, Neumann and Pinkham:

Theorem 2.1. *The surface $X(U, \Gamma, L)$ associated to a negative unramified automorphy factor (U, Γ, L) is a quasi-homogeneous affine algebraic surface with a normal isolated singularity. Its affine coordinate ring is the graded \mathbb{C} -algebra of generalised Γ -invariant automorphic forms*

$$A = \bigoplus_{m \geq 0} H^0(U, L^{-m})^\Gamma.$$

All normal isolated quasi-homogeneous surface singularities (X, x) are obtained in this way, and the automorphy factors with $(X(U, \Gamma, L), o(U, \Gamma, L))$ isomorphic to (X, x) are uniquely determined by (X, x) up to isomorphism.

We now recall the definition of Gorenstein singularities and the characterization of the corresponding automorphy factors.

An isolated singularity of dimension n is Gorenstein if and only if it is Cohen-Macaulay and there exists a nowhere vanishing holomorphic n -form on a punctured neighbourhood of the singular point. A normal isolated surface singularity is Gorenstein if and only if there exists a nowhere vanishing holomorphic 2-form on a punctured neighbourhood of the singular point. For example all isolated singularities of complete intersections are Gorenstein.

In [Dol83b], Dolgachev proved the following theorem obtained independently by W. Neumann (see also [Dol83a]).

Theorem 2.2. *A quasi-homogeneous surface singularity is Gorenstein if and only if for the corresponding automorphy factor (U, Γ, L) there is an integer m (called the level or the exponent of the automorphy factor) such that the m -th tensor power L^m is Γ -equivariantly isomorphic to the tangent bundle T_U of the surface U .*

Let (U, Γ, L) be an automorphy factor of level m , which corresponds to a Gorenstein singularity. The isomorphism $L^m \cong T_U$ induces an isomorphism $E^m \cong T_P$. A simple computation with Chern numbers shows that the possible values of the exponent are $m = -1$ or $m = -2$ for $U = \mathbb{C}\mathbb{P}^1$, whereas $m = 0$ for $U = \mathbb{C}$ and m is a positive integer for $U = \mathbb{H}$.

2.2. Hyperbolic automorphy factors and lifts of Fuchsian groups. We consider the universal cover $\tilde{G} = \widetilde{\text{PSL}}(2, \mathbb{R})$ of the Lie group

$$G = \text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R}) / \{\pm 1\},$$

the group of orientation-preserving isometries of the hyperbolic plane. Here our model of the hyperbolic plane is the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ and the action of an element $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PSL}(2, \mathbb{R})$ on \mathbb{H} is by

$$z \mapsto \frac{az + b}{cz + d}.$$

As a topological space $G = \text{PSL}(2, \mathbb{R})$ is homeomorphic to the open solid torus $\mathbb{S}^1 \times \mathbb{C}$, its fundamental group is infinite cyclic. Therefore, for each natural number m , there is a unique connected m -fold covering

$$G_m = \tilde{G} / (m \cdot Z(\tilde{G}))$$

of G , where \tilde{G} is the universal covering of G and $Z(\tilde{G})$ is the centre of \tilde{G} . The centre $Z(\tilde{G})$ of \tilde{G} coincides with the pre-image of the identity element of G under the covering map $\tilde{G} \rightarrow G$. For $m = 2$, we obtain $G_2 = \text{SL}(2, \mathbb{R})$.

Here is another description of the covering groups G_m of G which fixes a group structure. Let $\text{Hol}(\mathbb{H}, \mathbb{C}^*)$ be the set of all holomorphic functions $\mathbb{H} \rightarrow \mathbb{C}^*$.

Proposition 2.3. *The m -fold covering group G_m of G can be described as*

$$\{(g, \delta) \in G \times \text{Hol}(\mathbb{H}, \mathbb{C}^*) \mid \delta^m(z) = g'(z) \text{ for all } z \in \mathbb{H}\}$$

with multiplication $(g_2, \delta_2) \cdot (g_1, \delta_1) = (g_2 \cdot g_1, (\delta_2 \circ g_1) \cdot \delta_1)$.

Proof. Let X be the subspace of $G \times \text{Hol}(\mathbb{H}, \mathbb{C}^*)$ in question. One can check that the space X is connected and that the map $X \rightarrow G$ given by $(\gamma, \delta) \mapsto \gamma$ is an m -fold covering of G . Hence, the coverings $X \rightarrow G$ and $G_m \rightarrow G$ are isomorphic. One can check that the operation described above defines a group structure on X and that the covering map $X \rightarrow G$ is a homomorphism with respect to this group structure. \square

Remark. This description of G_m is inspired by the notion of automorphic differential forms of fractional degree, introduced by J. Milnor in [Mil75]. For a more detailed discussion see [LV80], section 1.8.

Definition 2.2. A *lift* of the Fuchsian group Γ into G_m is a subgroup Γ^* of G_m such that the restriction of the covering map $G_m \rightarrow G$ to Γ^* is an isomorphism $\Gamma^* \rightarrow \Gamma$.

Using the description of the m -fold covering group G_m of G in Proposition 2.3, we obtain the following result:

Proposition 2.4. *There is a 1-1-correspondence between the lifts of Γ into G_m and hyperbolic Gorenstein automorphy factors of level m associated to the Fuchsian group Γ .*

Proof. Theorem 2.2 implies that a hyperbolic Gorenstein automorphy factor of level m (associated to a Fuchsian group Γ) is an action of the Fuchsian group Γ on the trivial complex line bundle $\mathbb{H} \times \mathbb{C}$ over the hyperbolic plane \mathbb{H} given by

$$g \cdot (z, t) = (g(z), \delta_g(z) \cdot t),$$

where $\delta_g : \mathbb{H} \rightarrow \mathbb{C}^*$ is a holomorphic function; for any $g \in \Gamma$, we have $\delta_g^m = g'$ and, for any $g_1, g_2 \in \Gamma$, we have $\delta_{g_2 \circ g_1} = (\delta_{g_2} \circ g_1) \cdot \delta_{g_1}$. Comparing this description of hyperbolic Gorenstein automorphy factors with Proposition 2.3 yields the result. \square

3. LEVEL FUNCTIONS ON COVERING GROUPS OF $\mathrm{PSL}(2, \mathbb{R})$

3.1. Classification of elements in $G = \mathrm{PSL}(2, \mathbb{R})$ and its covering groups. Elements of G can be classified with respect to the fixed point behavior of their action on \mathbb{H} . An element is called *hyperbolic* if it has two fixed points, which lie on the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ of \mathbb{H} . One of the fixed points of a hyperbolic element is attracting, the other fixed point is repelling. The *axis* $\ell(g)$ of a hyperbolic element g is the geodesic between the fixed points of g , oriented from the repelling fixed point to the attracting fixed point. The element g preserves the geodesic $\ell(g)$. We call a hyperbolic element with attracting fixed point α and repelling fixed point β *positive* if $\alpha < \beta$. The *shift parameter* of a hyperbolic element g is the minimal displacement $\inf_{x \in \mathbb{H}} d(x, g(x))$. An element is called *parabolic* if it has one fixed point, which is on the boundary $\partial\mathbb{H}$. We call a parabolic element g with fixed point α *positive* if $g(x) > x$ for all $x \in \mathbb{R} \setminus \{\alpha\}$. An element that is neither hyperbolic nor parabolic is called *elliptic*. It has one fixed point that is in \mathbb{H} . Given a base-point $x \in \mathbb{H}$ and a real number φ , let $\rho_x(\varphi) \in G$ denote the rotation through angle φ counter-clockwise about the point x . Any elliptic element is of the form $\rho_x(\varphi)$, where x is the fixed point. Thus, we obtain a 2π -periodic homomorphism $\rho_x : \mathbb{R} \rightarrow G$ (with respect to the additive structure on \mathbb{R}). Elements of \tilde{G} , resp. G_m , can be classified with respect to the fixed point behavior of action on \mathbb{H} of their image in G . We say that an element of \tilde{G} , resp. G_m , is *hyperbolic*, *parabolic*, resp. *elliptic*, if its image in G has this property.

3.2. Central elements in covering groups of $G = \mathrm{PSL}(2, \mathbb{R})$. The homomorphism $\rho_x : \mathbb{R} \rightarrow G$ lifts to a unique homomorphism $r_x : \mathbb{R} \rightarrow \tilde{G}$ into the universal covering group. Since $\rho_x(2\pi\ell) = \mathrm{id}$ for $\ell \in \mathbb{Z}$, it follows that the lifted element $r_x(2\pi\ell)$ belongs to the centre $Z(\tilde{G})$ of \tilde{G} . Note that the element $r_x(2\pi\ell)$ depends continuously on x . But the centre of \tilde{G} is discrete, so this element must remain constant, thus $r_x(2\pi\ell)$ does not depend on x . We obtain that $Z(\tilde{G}) = \{r_x(2\pi\ell) \mid \ell \in \mathbb{Z}\}$. Let $u = r_x(2\pi)$ for some (and hence for any) x in \mathbb{H} . The element u is one of the two generators of the centre of \tilde{G} since any other element of the centre is of the form $r_x(2\pi\ell) = (r_x(2\pi))^\ell = u^\ell$.

3.3. Definition of a level function. Let Δ be the set of all elliptic elements of order 2 in G . Let Ξ be the complement in G of the set Δ . The space G is homeomorphic to the open solid torus $\mathbb{S}^1 \times \mathbb{C}$. Jankins and Neumann gave an explicit homeomorphism (see [JN85], Appendix). The image of the set Δ under this homeomorphism is $\{*\} \times \mathbb{C}$. From this description, it follows in particular that the subset Ξ is simply-connected. The pre-image $\tilde{\Xi}$ of the subset Ξ in \tilde{G} consists of infinitely many connected components, each of them is homeomorphic to Ξ . Each connected component of the subset $\tilde{\Xi}$ contains one and only one pre-image of the identity element of G , i.e., one and only one element of the centre of \tilde{G} .

Definition 3.1. If an element of \tilde{G} is contained in the same connected component of the set $\tilde{\Xi}$ as the central element u^k , $k \in \mathbb{Z}$, we say that the element is *of level k* and set the *level function* s

on this element to be equal to k . For pre-images of elliptic elements of order 2 we set $s(r_x(\xi)) = k$ for $\xi = \pi + 2\pi k$.

Remark. For elliptic elements we have $s(r_x(\xi)) = k \iff \xi \in (-\pi + 2\pi k, \pi + 2\pi k]$.

Definition 3.2. We define the *level function* s_m on elements of G_m with values in $\mathbb{Z}/m\mathbb{Z}$ by

$$s_m(C \bmod (m \cdot Z(\tilde{G}))) = s(C) \bmod m \text{ for } C \in \tilde{G}.$$

Definition 3.3. The canonical lift of an element \tilde{C} in G into \tilde{G} is an element \tilde{C} in \tilde{G} such that $\pi(\tilde{C}) = \tilde{C}$ and $s(\tilde{C}) = 0$. The canonical lift of an element \tilde{C} in G into G_m is an element \tilde{C} in G_m such that $\pi(\tilde{C}) = \tilde{C}$ and $s_m(\tilde{C}) = 0$.

3.4. Properties of the level function. In this subsection, we study the behavior of the level function s_m under inversion (Lemma 3.1), conjugation (Lemma 3.2) and multiplication in some special cases (Lemma 3.3). We shall obtain further statements about the behavior of the level function under multiplication in Corollary 4.7.

In this section, we shall repeatedly use the following fact: Connected components of the set $\tilde{\Xi}$ are separated from each other by connected components of the set $\tilde{\Delta}$ of all pre-images of (elliptic) elements of order 2. If a path γ in \tilde{G} avoids all pre-images of elements of order 2, i.e., avoids $\tilde{\Delta}$, then it means that the path γ remains in the same component of the set $\tilde{\Xi}$ and therefore the level function s is constant along γ .

Lemma 3.1. *The equation $s(A^{-1}) = -s(A)$ is satisfied for any element A in \tilde{G} with the exception of the pre-images of elliptic elements of order 2. The equation $s_m(A^{-1}) = -s_m(A)$ is satisfied for any element A in G_m with the exception of the pre-images of elliptic elements of order 2.*

Proof. We shall prove the statement about the level function s on \tilde{G} , the statement about the level function s_m on G_m follows immediately. Let $A \in \tilde{G}$ and let $k = s(A)$, then A is in the same connected component of $\tilde{\Xi}$ as u^k . Let γ be the path in $\tilde{\Xi}$ that connects A with u^k . Let the path δ be given by $\delta(t) = (\gamma(t))^{-1}$. The path δ connects A^{-1} with u^{-k} . Since the path γ remains in the same component of $\tilde{\Xi}$, it avoids $\tilde{\Delta}$. Consequently, the path δ also avoids $\tilde{\Delta}$, i.e., it remains in the same component of $\tilde{\Xi}$. Thus the element A^{-1} is in the same connected component of $\tilde{\Xi}$ as u^{-k} , i.e., $s(A^{-1}) = -k = -s(A)$. \square

Lemma 3.2. *For any elements A and B in \tilde{G} , we have $s(BAB^{-1}) = s(A)$. For any elements A and B in G_m , we have $s_m(BAB^{-1}) = s_m(A)$.*

Proof. We shall prove the statement about the level function s on \tilde{G} , the statement about the level function s_m on G_m follows immediately. An element $B \in \tilde{G}$ can be connected to the unit element in \tilde{G} via a path β . The path γ given by $\gamma(t) = \beta(t) \cdot A \cdot (\beta(t))^{-1}$ connects the elements A and $B \cdot A \cdot B^{-1}$. If A is not in $\tilde{\Delta}$ then any conjugate $\gamma(t)$ of A is not in $\tilde{\Delta}$, hence the path γ remains in the same component of the set $\tilde{\Xi}$. If A is in $\tilde{\Delta}$ then any conjugate $\gamma(t)$ of A is also in $\tilde{\Delta}$, hence the path γ remains in the same component of the set $\tilde{\Delta}$. In both cases s is constant along γ , in particular $s(B \cdot A \cdot B^{-1}) = s(A)$. \square

Lemma 3.3. *If the axes of two hyperbolic elements A and B in \tilde{G} intersect, then*

$$s(AB) = s(A) + s(B).$$

If the axes of two hyperbolic elements A and B in G_m intersect, then

$$s_m(AB) = s_m(A) + s_m(B).$$

Proof. Let ℓ_A , resp. ℓ_B , be the axes of A , resp. B . Let x be the intersection point of ℓ_A and ℓ_B . Any hyperbolic transformation with the axis ℓ_A is a product of a rotation by π at some point $y \neq x$ on ℓ_A and a rotation by π at the point x . Similarly any hyperbolic transformation with the axis ℓ_B is a product of a rotation by π at the point x and a rotation by π at some point $z \neq x$ on ℓ_B . Hence the product of any hyperbolic transformation with the axis ℓ_A and any hyperbolic transformation with the axis ℓ_B is a product of a rotation by π at a point $y \neq x$ on ℓ_A and a rotation by π at a point $z \neq x$ on ℓ_B , i.e., it is a hyperbolic transformation with an axis going through the points y and z . Thus the product of two hyperbolic elements with distinct but intersecting axes is always a hyperbolic element.

We shall prove the statement about the level function s on \tilde{G} , the statement about the level function s_m on G_m follows immediately. Assume without loss of generality that the elements $A, B \in \tilde{G}$ satisfy the conditions $s(A) = s(B) = 0$. We want to show that $s(AB) = 0$. Let us deform the elements A and B . If we are decreasing the shift parameters while keeping the same axes, then the product tends to the identity element. On the other hand, we have just explained that the product remains hyperbolic, i.e., stays in $\tilde{\Xi}$. Therefore, the value of s on the product remains constant, i.e., $s(AB) = s(\text{id}) = 0$. \square

4. LEVEL FUNCTIONS ON LIFTS OF FUCHSIAN GROUPS

4.1. Lifting elliptic cyclic subgroups. A lift of the Fuchsian group Γ into G_m is a subgroup Γ^* of G_m such that the restriction of the covering map $G_m \rightarrow G$ to Γ^* is an isomorphism $\Gamma^* \rightarrow \Gamma$. In this subsection, we shall discuss the lifts of cyclic Fuchsian groups generated by an elliptic element.

Lemma 4.1. *Let Γ be an elliptic cyclic Fuchsian group of order p . The group Γ is generated by an element $\gamma = \rho_x(2\pi/p)$ for some $x \in \mathbb{H}$.*

- 1) *Let us assume that p and m are relatively prime. Then the lift Γ^* of Γ into G_m exists and is unique. There is a unique element $n \in \mathbb{Z}/m\mathbb{Z}$ such that $p \cdot n + 1 \equiv 0$ modulo m . The lift Γ^* is generated by the pre-image $\tilde{\gamma}$ of $\gamma = \rho_x(2\pi/p)$ in G_m such that $s_m(\tilde{\gamma}) = n$.*
- 2) *If p and m are not relatively prime, then the group Γ cannot be lifted into G_m .*

Proof. To lift Γ into G_m , we have to find an element $\tilde{\gamma}$ in the pre-image of γ in G_m such that $\tilde{\gamma}^p = 1$. The pre-image of γ in G_m can be described as the coset $\{u^n \cdot r_x(2\pi/p) \mid n \in \mathbb{Z}/m\mathbb{Z}\}$. For the element $r_x(2\pi/p)$, we obtain $(r_x(2\pi/p))^p = r_x(2\pi) = u$. Hence, for an element $u^n \cdot r_x(2\pi/p)$ we obtain

$$(u^n \cdot r_x(2\pi/p))^p = u^{np} (r_x(2\pi/p))^p = u^{np+1}.$$

Therefore, $(u^n \cdot r_x(2\pi/p))^p = 1$ if and only if $n \cdot p + 1 \equiv 0 \pmod{m}$. There exists an $n \in \mathbb{Z}/m\mathbb{Z}$ with $n \cdot p + 1 \equiv 0 \pmod{m}$ if and only if the numbers p and m are relatively prime. Hence for not relatively prime p and m it is impossible to lift Γ into G_m . For relatively prime p and m , there is a unique lift of Γ into G_m generated by $u^n \cdot r_x(2\pi/p)$ with $n \cdot p + 1 \equiv 0 \pmod{m}$. \square

4.2. Finitely generated Fuchsian groups. In this section, we are going to describe finitely generated (co-compact) Fuchsian groups using standard sets of generators. The following definitions follow [Zie81]:

Definition 4.1. A *Riemann factor surface* or *Riemann orbifold* (P, Q) of signature

$$(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$$

is a topological surface P of genus g with l_h holes and l_p punctures and a set

$$Q = \{(x_1, p_1), \dots, (x_{l_e}, p_{l_e})\}$$

of points x_i in P equipped with orders p_i such that $p_i \in \mathbb{Z}$, $p_i \geq 2$ and $x_i \neq x_j$ for $i \neq j$. The set Q is called the *marking* of the Riemann orbifold (P, Q) .

Definition 4.2. Let $(P, Q = \{(x_1, p_1), \dots, (x_{l_e}, p_{l_e})\})$ be a Riemann orbifold. Two curves γ_0 and γ_1 , which do not pass through exceptional points $x_i \in Q$, are called *Q-homotopic* if γ_0 can be deformed into γ_1 by a finite sequence of the following processes:

- (a) Homotopic deformations with fixed starting point such that during the deformation no exceptional point is encountered.
- (b) Omitting a subcurve of γ_i which does not contain the starting point of γ_i and is of the form $\delta^{\pm p_i}$, where δ is a curve on P which bounds a disk that contains exactly one exceptional point x_i in the interior.
- (c) Inserting into γ_i a subcurve which does not contain the starting point of γ_i and is of the form $\delta^{\pm p_i}$, where δ is a curve on P which bounds a disk that contains exactly one exceptional point x_i in the interior.

Two curves γ_0 and γ_1 which do not pass through exceptional points $x_i \in Q$ are called *freely Q-homotopic* if the base point may be moved during the deformations.

Definition 4.3. Let $(P, Q = \{(x_1, p_1), \dots, (x_{l_e}, p_{l_e})\})$ be a Riemann orbifold and let p be in $P \setminus \{x_1, \dots, x_{l_e}\}$. Then, the set of Q -homotopy classes of curves starting and ending in p forms a group. This group is called the *Q-fundamental group* or the *orbifold fundamental group* and denoted by $\pi^Q(P, p)$ or $\pi^{\text{orb}}(P, p)$ or simply $\pi(P, p)$.

Definition 4.4. Let Γ be a Fuchsian group. The quotient $P = \mathbb{H}/\Gamma$ is a surface and the projection $\Psi : \mathbb{H} \rightarrow P$ is a branched cover. Let Q consist of the branching points and the corresponding orders. Then, (P, Q) is a Riemann orbifold. We say that the Riemann orbifold (P, Q) is defined by Γ .

Proposition 4.2. Let Γ be a Fuchsian group, (P, Q) the corresponding Riemann orbifold and $p \in P$ not an exceptional point. Then, $\pi(P, p) \cong \Gamma$.

Definition 4.5. A *canonical system of curves* on a Riemann orbifold

$$(P, Q = \{(x_1, p_1), \dots, (x_{l_e}, p_{l_e})\})$$

of signature $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ is a set of simply closed curves

$$\{\tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g, \tilde{c}_{g+1}, \dots, \tilde{c}_n\}$$

based at a point $p \in P$, where $n = g + l_h + l_p + l_e$, with the following properties:

- 1) The contour \tilde{c}_{g+i} encloses a hole in P for $i = 1, \dots, l_h$, a puncture for $i = l_h + 1, \dots, l_h + l_p$ and the marking point x_i for $i = l_h + l_p + 1, \dots, n - g$.
- 2) Any two curves only intersect at the point p .
- 3) A neighbourhood of the point p with the curves is homeomorphic to the one shown in Figure 2.
- 4) The system of curves cuts the surface P into $l_h + l_p + l_e + 1$ connected components of which $l_p + l_e$ are homeomorphic to a disc with a hole, $l_h + 1$ are discs. The last disc has boundary $\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1} \tilde{b}_1^{-1} \dots \tilde{a}_g \tilde{b}_g \tilde{a}_g^{-1} \tilde{b}_g^{-1} \tilde{c}_g \dots \tilde{c}_n$.

If $\{\tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g, \tilde{c}_{g+1}, \dots, \tilde{c}_n\}$ is a canonical system of curves, then we call the corresponding set $\{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_n\}$ of elements in the orbifold fundamental group $\pi(P, p)$ a *standard set of generators* or a *standard basis* of $\pi(P, p)$.

Definition 4.6. A *sequential set of signature* $(0; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ with $l_h + l_p + l_e = 3$ is a triple of elements (C_1, C_2, C_3) in G such that the element C_i is hyperbolic for $i = 1, \dots, l_h$, parabolic for $i = l_h + 1, \dots, l_h + l_p$ and elliptic of order $p_{i-l_h-l_p}$ for $i = l_h + l_p + 1, \dots, l_h + l_p + l_e = 3$,

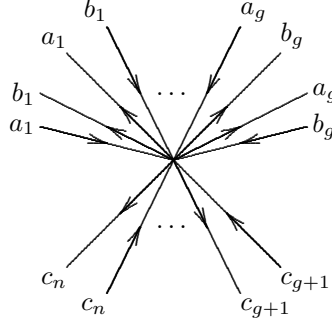
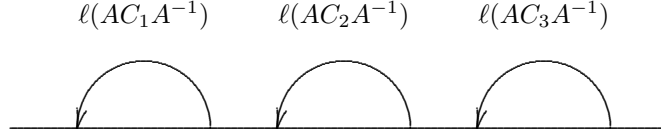


Figure 2: Canonical system of curves

Figure 3: Axes of a sequential set of signature $(0; 3, 0, 0)$

their product is $C_1 \cdot C_2 \cdot C_3 = 1$, and for some element $A \in G$ the elements $\{C'_i = AC_i A^{-1}\}_{i=1,2,3}$ are positive, have finite fixed points and satisfy $C'_1 < C'_2 < C'_3$. (Figure 3 illustrates the position of the axes of the elements C'_i for a sequential set of signature $(0; 3, 0, 0)$, i.e., when all elements are hyperbolic.)

Definition 4.7. A *sequential set of signature* $(0; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ is a tuple of elements $(C_1, \dots, C_{l_h+l_p+l_e})$ in G such that the element C_i is hyperbolic for $i = 1, \dots, l_h$, parabolic for $i = l_h + 1, \dots, l_h + l_p$ and elliptic of order $p_{i-l_h-l_p}$ for $i = l_h + l_p + 1, \dots, l_h + l_p + l_e$, and for any $i \in \{2, \dots, l_h + l_p + l_e - 1\}$ the triple $(C_1 \cdots C_{i-1}, C_i, C_{i+1} \cdots C_{l_h+l_p+l_e})$ is a sequential set.

Definition 4.8. A *sequential set of signature* $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ is a tuple of elements

$$(A_1, \dots, A_g, B_1, \dots, B_g, C_{g+1}, \dots, C_{g+l_h+l_p+l_e})$$

in G such that

- (1) the elements $A_1, \dots, A_g, B_1, \dots, B_g$ are hyperbolic,
- (2) the element C_{g+i} is hyperbolic for $i = 1, \dots, l_h$, parabolic for $i = l_h + 1, \dots, l_h + l_p$, and elliptic of order $p_{i-l_h-l_p}$ for $i = l_h + l_p + 1, \dots, l_h + l_p + l_e$,
- (3) and the tuple

$$(A_1, B_1 A_1^{-1} B_1^{-1}, \dots, A_g, B_g A_g^{-1} B_g^{-1}, C_{g+1}, \dots, C_{g+l_h+l_p+l_e})$$

is a sequential set of signature $(0; 2g + l_h, l_p, l_e : p_1, \dots, p_{l_e})$.

The relation between sequential sets, standard bases, canonical systems of curves and Fuchsian groups was studied in [Nat72]. Details for the case of Fuchsian groups with elliptic elements can be found in [Zie81]. We recall here the results:

Theorem 4.3. Let V be a sequential set of signature $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$. For $i = 1, \dots, l_e$ let $y_i \in \mathbb{H}$ be the fixed point of the corresponding elliptic element of order p_i in V . Let $P =$

\mathbb{H}/Γ and let $\Psi : \mathbb{H} \rightarrow P$ be the natural projection. Let $Q = \{(\Psi(y_1), p_1), \dots, (\Psi(y_{l_e}), p_{l_e})\}$. Then the sequential set V generates a Fuchsian group Γ such that the Riemann factor surface $(P = \mathbb{H}/\Gamma, Q)$ is of signature $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$. The natural projection $\Psi : \mathbb{H} \rightarrow P$ maps the sequential set V to a canonical system of curves on the factor surface (P, Q) .

Theorem 4.4. *Let Γ be a Fuchsian group such that the factor surface $P = \mathbb{H}/\Gamma$ is of signature $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ with $l_h + l_p + l_e = n$. Let p be a point in P which does not belong to the marking. Let $\Psi : \mathbb{H} \rightarrow P$ be the natural projection. Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \rightarrow \pi(P, p)$ be the induced isomorphism. Let*

$$v = \{\tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g, \tilde{c}_{g+1}, \dots, \tilde{c}_n\} =: \{\tilde{d}_1, \dots, \tilde{d}_{g+n}\}$$

be a canonical system of curves on P , then

$$V = \Phi^{-1}(v) = \{\Phi^{-1}([\tilde{d}_1]), \dots, \Phi^{-1}([\tilde{d}_{g+n}])\}$$

is a sequential set of signature $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$.

4.3. Lifting Fuchsian groups of genus 0.

Lemma 4.5. *Let $(0; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ with $l_h + l_p + l_e = n$ be the signature of the sequential set $(\bar{C}_1, \dots, \bar{C}_n)$. For $i = 1, \dots, n$, let \tilde{C}_i be the canonical lift of \bar{C}_i into \tilde{G} , resp. G_m . Let u be the generator of the centre $Z(\tilde{G})$, resp. $Z(G_m)$, defined in section 3.2. The element u is given by the element $r_x(\pi)$, resp. its projection into G_m . Then, the elements $\tilde{C}_1, \dots, \tilde{C}_n$ satisfy the following relations: $\tilde{C}_{l_h+l_p+i}^{p_i} = u$ for $i = 1, \dots, l_e$ and $\tilde{C}_1 \cdots \tilde{C}_n = u^{n-2}$.*

Proof. A canonical lift $\tilde{C}_{l_h+l_p+i}$ of an elliptic element $\bar{C}_{l_h+l_p+i}$ is of the form $r_x(2\pi/p_i)$ for some x . Hence, $\tilde{C}_{l_h+l_p+i}^{p_i} = (r_x(2\pi/p_i))^{p_i} = r_x(2\pi) = u$. We will now use a geometric approach due to Milnor [Mil75]. Let Π be the canonical fundamental polygon for the group generated by the elements $\bar{C}_1, \dots, \bar{C}_n$ such that the generators \bar{C}_i can be described as products $\bar{C}_i = \sigma_i \sigma_{i+1}$ of reflexions $\sigma_1, \dots, \sigma_n$ in the edges of the polygon Π (suitably numbered). Then, $\sigma_i^2 = \text{id}$ and, therefore,

$$\bar{C}_1 \cdots \bar{C}_n = (\sigma_1 \sigma_2)(\sigma_2 \sigma_3) \cdots (\sigma_{n-1} \sigma_n)(\sigma_n \sigma_1) = \text{id}.$$

Lifting the elements \bar{C}_i to their canonical lifts \tilde{C}_i in \tilde{G} , it follows that the product $\tilde{C}_1 \cdots \tilde{C}_n$ belongs to the centre $Z(\tilde{G})$. As we vary the polygon Π continuously, this central element must also vary continuously. But $Z(\tilde{G})$ is a discrete group, so $\tilde{C}_1 \cdots \tilde{C}_n$ must remain constant. In particular, we can shrink the polygon Π down towards a point x . In the course of this continuous deformation of the fundamental polygon Π , the hyperbolic and parabolic elements of the sequential set will become elliptic. As we continue shrinking the polygon Π towards the point x , the angles of the polygon tend to the angles β_1, \dots, β_n of some Euclidean n -sided polygon. Thus, the element $\tilde{C}_i \in \tilde{G}$ tends towards the limit $r_x(2\beta_i)$, while the product $\tilde{C}_1 \cdots \tilde{C}_n$ tends towards the product $r_x(2\beta_1) \cdots r_x(2\beta_n) = r_x(2\beta_1 + \cdots + 2\beta_n)$. Therefore, using the formula $\beta_1 + \cdots + \beta_n = (n-2)\pi$ for the sum of the angles of a Euclidean n -sided polygon, we see that the constant product $\tilde{C}_1 \cdots \tilde{C}_n$ must be equal to $r_x(2(n-2)\pi) = u^{n-2}$. Thus $\tilde{C}_{l_h+l_p+i}^{p_i} = u$ and $\tilde{C}_1 \cdots \tilde{C}_n = u^{n-2}$. Projecting into G_m we get the corresponding statement in G_m . \square

Lemma 4.6. *Let (C_1, \dots, C_n) be an n -tuple of elements in G_m such that their images $(\bar{C}_1, \dots, \bar{C}_n)$ in G form a sequential set of signature $(0; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ with $l_h + l_p + l_e = n$. Then, $C_1 \cdots C_n = e$ if and only if $\sum_{i=1}^n s_m(C_i) \equiv -(n-2)$ modulo m .*

Proof. For $i = 1, \dots, n$, let \tilde{C}_i be the canonical lift of \bar{C}_i into G_m . The elements C_i can be written in the form $C_i = \tilde{C}_i \cdot u^{s_m(C_i)}$; therefore, the product $C_1 \cdots C_n$ is equal to

$$(\tilde{C}_1 \cdots \tilde{C}_n) \cdot u^{s_m(C_1) + \cdots + s_m(C_n)}.$$

Using Lemma 4.5, we obtain that this product is equal to $u^{n-2+s_m(C_1)+\dots+s_m(C_n)}$. Hence, the product $C_1 \cdots C_n$ is equal to e if and only if the exponent of u in the last equation is divisible by m , i.e., if $\sum_{i=1}^n s_m(C_i) \equiv -(n-2)$ modulo m . \square

Corollary 4.7. *Let (C_1, C_2, C_3) be a triple of elements in G_m with $C_1 \cdot C_2 \cdot C_3 = e$. Let \bar{C}_i be the image of the element C_i in G . Let $(\bar{C}_1, \bar{C}_2, \bar{C}_3)$ be a sequential set of signature*

$$(0; l_h, l_p, l_e : p_1, \dots, p_{l_e}),$$

with $l_h + l_p + l_e = 3$. Then,

$$s_m(C_1 \cdot C_2) = s_m(C_1) + s_m(C_2) + 1, \text{ if the element } C_3 \text{ is not of order 2, and}$$

$$s_m(C_1 \cdot C_2) = -s_m(C_1) - s_m(C_2) - 1, \text{ if the element } C_3 \text{ is of order 2.}$$

Proof. According to Lemma 4.6 the elements C_i satisfy

$$s_m(C_1) + s_m(C_2) + s_m(C_3) \equiv -1 \pmod{m}.$$

On the other hand, $C_1 C_2 C_3 = e$ implies $C_1 C_2 = C_3^{-1}$; hence,

$$s_m(C_1 C_2) = s_m(C_3^{-1}) = -s_m(C_3) = s_m(C_1) + s_m(C_2) + 1,$$

if the element C_3 is not of order 2, and

$$s_m(C_1 C_2) = s_m(C_3^{-1}) = s_m(C_3) = -s_m(C_1) - s_m(C_2) - 1,$$

if the element C_3 is of order 2. \square

4.4. Lifting sets of generators of Fuchsian groups.

Lemma 4.8. *Let Γ be a Fuchsian group of signature $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ generated by the sequential set $\bar{V} = \{\bar{A}_1, \bar{B}_1, \dots, \bar{A}_g, \bar{B}_g, \bar{C}_{g+1}, \dots, \bar{C}_n\}$, where $n = g + l_h + l_p + l_e$. Let $V = \{A_1, B_1, \dots, A_g, B_g, C_{g+1}, \dots, C_n\}$ be a set of lifts of the elements of the sequential set \bar{V} into G_m , i.e., the image of A_i, B_i , resp. C_j , in G is \bar{A}_i, \bar{B}_i , resp. \bar{C}_j . Then, the subgroup Γ^* of G_m generated by V is a lift of Γ into G_m if and only if*

$$[A_1, B_1] \cdots [A_g, B_g] \cdot C_{g+1} \cdots C_n = e, \quad C_{g+l_h+l_p+i}^{p_i} = e \quad \text{for } i = 1, \dots, l_e.$$

Proof. For any choice of the set of lifts V , the restriction of the covering map $G_m \rightarrow G$ to the group Γ^* generated by V is a homomorphism with image Γ . If the conditions of the lemma hold true, then the group Γ^* satisfies the same relations as the group Γ ; hence, this homomorphism is injective. \square

Lemma 4.9. *Let $\{A_1, B_1, \dots, A_g, B_g, C_{g+1}, \dots, C_n\}$ be a tuple of elements in G_m such that the images $\{\bar{A}_1, \bar{B}_1, \dots, \bar{A}_g, \bar{B}_g, \bar{C}_{g+1}, \dots, \bar{C}_n\}$ in G form a sequential set of signature*

$$(g; l_h, l_p, l_e : p_1, \dots, p_{l_e}),$$

with $g + l_h + l_p + l_e = n$. Then,

$$\prod_{i=1}^g [A_i, B_i] \cdot \prod_{j=g+1}^n C_j = e \iff \sum_{j=g+1}^n s_m(C_j) \equiv (2-2g) - (n-g) \pmod{m}.$$

(in the case $n = g$, this means $2-2g \equiv 0 \pmod{m}$) and, for any $i = 1, \dots, l_e$,

$$C_{g+l_h+l_p+i}^{p_i} = e \iff p_i \cdot s_m(C_{g+l_h+l_p+i}) + 1 \equiv 0 \pmod{m}.$$

Proof. The case $g = 0$ was discussed in Lemma 4.6. We shall now reduce the general case to the case $g = 0$. By definition of sequential sets, the set

$$(\bar{A}_1, \bar{B}_1 \bar{A}_1^{-1} \bar{B}_1^{-1}, \dots, \bar{A}_g, \bar{B}_g \bar{A}_g^{-1} \bar{B}_g^{-1}, \bar{C}_{g+1}, \dots, \bar{C}_n)$$

is a sequential set of signature $(0; 2g + l_h, l_p, l_e)$; hence,

$$\prod_{i=1}^g [A_i, B_i] \cdot \prod_{i=g+1}^n C_i = \prod_{i=1}^g (A_i \cdot B_i A_i^{-1} B_i^{-1}) \cdot \prod_{i=g+1}^n C_i = e$$

if and only if

$$\begin{aligned} \sum_{i=1}^g (s_m(A_i) + s_m(B_i A_i^{-1} B_i^{-1})) + \sum_{i=g+1}^n s_m(C_i) &\equiv -(2g + (n - g) - 2) \\ &\equiv (2 - 2g) - (n - g) \pmod{m}. \end{aligned}$$

Invariance of the level function s_m under conjugation (Lemma 3.2) implies that

$$s_m(B_i A_i^{-1} B_i^{-1}) = s_m(A_i^{-1}).$$

Since A_i is not an element of order 2, $s_m(A_i^{-1}) = -s_m(A_i)$ and, hence,

$$s_m(A_i) + s_m(B_i A_i^{-1} B_i^{-1}) = s_m(A_i) - s_m(A_i) = 0.$$

The last statement of the lemma follows from Lemma 4.1. \square

Proposition 4.10. *Let Γ be a Fuchsian group of signature $(g : p_1, \dots, p_r)$. Let*

$$\bar{V} = \{\bar{A}_1, \bar{B}_1, \dots, \bar{A}_g, \bar{B}_g, \bar{C}_{g+1}, \dots, \bar{C}_n\}$$

be a sequential set that generates Γ . Then, there exist lifts of Γ into G_m if and only if the signature $(g : p_1, \dots, p_r)$ satisfies the following liftability conditions: $\gcd(p_i, m) = 1$ for $i = 1, \dots, r$ and

$$(p_1 \cdots p_r) \cdot \left(\sum_{i=1}^r \frac{1}{p_i} - (2g - 2) - r \right) \equiv 0 \pmod{m}.$$

Moreover, if the liftability conditions are satisfied, then any set of lifts $\{A_i, B_i\}$ of $\{\bar{A}_i, \bar{B}_i\}$ into G_m can be extended in a unique way to a set $\{A_i, B_i, C_j\}$ of lifts of $\{\bar{A}_i, \bar{B}_i, \bar{C}_j\}$ that generates a lift of Γ into G_m ; hence, there are m^{2g} different lifts of Γ into G_m .

Proof. In this proof all congruences will be modulo m . Let us first assume that there exists a lift of Γ into G_m . Let $\{A_i, B_i, C_j\}$ be a set of lifts of \bar{V} as in Lemmas 4.8 and 4.9. Let $n_i = s_m(C_{g+i})$. Then according to Lemma 4.9 we have $p_i \cdot n_i + 1 \equiv 0$ for $i = 1, \dots, r$ and

$$(2g - 2) + r + \sum_{i=1}^r n_i \equiv 0.$$

The congruence $p_i \cdot n_i + 1 \equiv 0$ implies that p_i is prime with m for $i = 1, \dots, r$. Furthermore, since

$$(p_1 \cdots p_r) \cdot n_i \equiv \frac{p_1 \cdots p_r}{p_i} \cdot (p_i \cdot n_i) \equiv \frac{p_1 \cdots p_r}{p_i} \cdot (-1) \equiv -(p_1 \cdots p_r) \cdot \frac{1}{p_i},$$

we obtain that

$$(p_1 \cdots p_r) \cdot \left(\sum_{i=1}^r \frac{1}{p_i} - (2g - 2) - r \right) \equiv -(p_1 \cdots p_r) \cdot \left(\sum_{i=1}^r n_i + (2g - 2) + r \right) \equiv 0.$$

Now, let us assume that the liftability conditions are satisfied. We want to construct a lift of Γ into G_m . Since p_i is prime with m , we can choose $n_i \in \mathbb{Z}/m\mathbb{Z}$ such that $p_i \cdot n_i + 1 \equiv 0$ for $i = 1, \dots, r$. Then,

$$(p_1 \cdots p_r) \cdot n_i \equiv -(p_1 \cdots p_r) \cdot \frac{1}{p_i}$$

and, hence,

$$(p_1 \cdots p_r) \cdot \left((2g-2) + r + \sum_{i=1}^r n_i \right) \equiv (p_1 \cdots p_r) \cdot \left((2g-2) + r - \sum_{i=1}^r \frac{1}{p_i} \right) \equiv 0.$$

Since $p_1 \cdots p_r$ is prime with m , we conclude that $(2g-2) + r + \sum_{i=1}^r n_i \equiv 0$, i.e., $\sum_{i=1}^r n_i \equiv (2-2g) - r$. Let $V = \{A_i, B_i, C_j\}$ be any set of lifts of \bar{V} such that $s_m(C_{g+i}) = n_i$ for $i = 1, \dots, r$. We have $p_i \cdot n_i + 1 \equiv 0$ for $i = 1, \dots, r$ and $\sum_{i=1}^r n_i \equiv (2-2g) - r$. Hence, according to Lemma 4.9, the set V generates a lift of Γ into G_m . Since Lemma 4.9 does not impose any conditions on the values $s_m(A_i)$ and $s_m(B_i)$ for $i = 1, \dots, g$, any of m^{2g} choices of these $2g$ values leads to a different lift of Γ into G_m . \square

5. HIGHER ARF FUNCTIONS

In [NP05, NP09], we introduced the notion of a higher Arf function and used it to study moduli spaces of higher spin bundles on Riemann surfaces. In this section, we will introduce higher Arf functions on orbifolds, and study their connection with Gorenstein automorphy factors.

5.1. Definition of higher Arf functions on orbifolds. In this subsection, we will define higher Arf functions on orbifolds (compare with subsection 4.1 in [NP09]).

Let Γ be a Fuchsian group of signature $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ and $P = \mathbb{H}/\Gamma$ the corresponding orbifold. Let $p \in P$. Let $\Psi : \mathbb{H} \rightarrow P$ be the natural projection. Let $\pi(P, p)$ be the orbifold fundamental group of P (see definition 4.3). Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \rightarrow \pi(P, p)$ be the induced isomorphism. Let Γ^* be a lift of Γ in G_m .

Definition 5.1. Let us consider a function $\hat{\sigma}_{\Gamma^*} : \pi(P, p) \rightarrow \mathbb{Z}/m\mathbb{Z}$ such that the following diagram commutes

$$\begin{array}{ccc} \Gamma & \xrightarrow{\cong} & \Gamma^* \\ \Phi \downarrow & & \downarrow s_m|_{\Gamma^*} \\ \pi(P, p) & \xrightarrow{\hat{\sigma}_{\Gamma^*}} & \mathbb{Z}/m\mathbb{Z} \end{array}$$

Lemma 5.1. *Let α, β and γ be simple contours in P intersecting pairwise in exactly one point p . Let a, b and c be the corresponding elements of $\pi(P, p)$. We assume that a, b and c satisfy the relations $a, b, c \neq 1$ and $abc = 1$. Let $\langle \cdot, \cdot \rangle$ be the intersection form on $\pi(P, p)$. Then for $\hat{\sigma} = \hat{\sigma}_{\Gamma^*}$*

1. *If the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b \rangle \neq 0$, then $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b)$.*
2. *If ab is in $\pi^0(P, p)$ and the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b \rangle = 0$ and a neighbourhood of the point p with the contours is homeomorphic to the one shown in Figure 4, then $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) + 1$ if the element ab is not of order 2, and $\hat{\sigma}(ab) = -\hat{\sigma}(a) - \hat{\sigma}(b) - 1$ if the element ab is of order 2.*

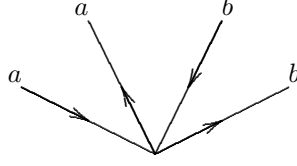


Figure 4: $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) + 1$

3. If ab is in $\pi^0(P, p)$ and the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b \rangle = 0$ and a neighbourhood of the point p with the contours is homeomorphic to the one shown in Figure 1, then $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) - 1$.
4. For any standard set of generators $v = \{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_n\}$ of $\pi(P, p)$, we have

$$\sum_{i=g+1}^n \hat{\sigma}(c_i) \equiv (2 - 2g) - (n - g) \pmod{m}.$$

5. For any elliptic element $c_{g+l_h+l_p+i}$, $i = 1, \dots, l_e$, we have $p_i \cdot \hat{\sigma}(c_{g+l_h+l_p+i}) + 1 \equiv 0 \pmod{m}$.

Proof. According to Theorem 4.4, either the set $V = \{\Phi^{-1}(a), \Phi^{-1}(b), \Phi^{-1}(c)\}$ or the set $V^{-1} = \{\Phi^{-1}(a^{-1}), \Phi^{-1}(b^{-1}), \Phi^{-1}(c^{-1})\}$ is sequential. This sequential set can be of signature $(0 : *, *, *)$ or $(1 : *)$.

- If V is a sequential set of signature $(1 : *)$, then according to Lemma 3.3 we obtain $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b)$.
- If V is a sequential set of signature $(0 : *, *, *)$, then according to Corollary 4.7 we obtain

$$\begin{aligned} \hat{\sigma}(ab) &= \hat{\sigma}(a) + \hat{\sigma}(b) + 1 \text{ if } ab \text{ is not of order 2,} \\ \hat{\sigma}(ab) &= -\hat{\sigma}(a) - \hat{\sigma}(b) - 1 \text{ if } ab \text{ is of order 2.} \end{aligned}$$

- If V^{-1} is a sequential set of signature $(0 : *, *, *)$, then according to Corollary 4.7 we obtain,

$$\begin{aligned} \hat{\sigma}(b^{-1}a^{-1}) &= \hat{\sigma}(a^{-1}) + \hat{\sigma}(b^{-1}) + 1 \text{ if } ab \text{ is not of order 2,} \\ \hat{\sigma}(b^{-1}a^{-1}) &= -\hat{\sigma}(a^{-1}) - \hat{\sigma}(b^{-1}) - 1 \text{ if } ab \text{ is of order 2.} \end{aligned}$$

Therefore, for the element ab not of order 2, we obtain

$$\begin{aligned} \hat{\sigma}(ab) &= -\hat{\sigma}((ab)^{-1}) = -\hat{\sigma}(b^{-1}a^{-1}) = -(\hat{\sigma}(a^{-1}) + \hat{\sigma}(b^{-1}) + 1) \\ &= -\hat{\sigma}(a^{-1}) - \hat{\sigma}(b^{-1}) - 1 = \hat{\sigma}(a) + \hat{\sigma}(b) - 1 \end{aligned}$$

and, for the element ab of order 2, we obtain

$$\hat{\sigma}(ab) = \hat{\sigma}((ab)^{-1}) = \hat{\sigma}(b^{-1}a^{-1}) = -\hat{\sigma}(a^{-1}) - \hat{\sigma}(b^{-1}) - 1 = \hat{\sigma}(a) + \hat{\sigma}(b) - 1.$$

To prove properties 4 and 5 of $\hat{\sigma}$, we apply Lemma 4.9. □

We now formalize the properties of the function $\hat{\sigma}$ in the following definition:

Definition 5.2. We denote by $\pi^0(P, p)$ the set of all non-trivial elements of $\pi(P, p)$ that can be represented by simple contours. An m -Arf function is a function

$$\sigma : \pi^0(P, p) \rightarrow \mathbb{Z}/m\mathbb{Z}$$

satisfying the following conditions

1. $\sigma(bab^{-1}) = \sigma(a)$ for any elements $a, b \in \pi^0(P, p)$,
2. $\sigma(a^{-1}) = -\sigma(a)$ for any element $a \in \pi^0(P, p)$ that is not of order 2,

3. $\sigma(ab) = \sigma(a) + \sigma(b)$ for any elements a and b which can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b \rangle \neq 0$,
4. $\sigma(ab) = \sigma(a) + \sigma(b) - 1$ for any elements $a, b \in \pi^0(P, p)$ such that the element ab is in $\pi^0(P, p)$ and the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b \rangle = 0$ and a neighbourhood of the point p with the contours is homeomorphic to the one shown in Figure 1.
5. For any elliptic element c of order p , we have $p \cdot \sigma(c) + 1 \equiv 0 \pmod{m}$.

The following property of m -Arf functions follows immediately from Properties 4 and 2 in Definition 5.2:

Proposition 5.2. *Let a and b be elements of $\pi^0(P, p)$ such that the element ab is in $\pi^0(P, p)$ and the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b \rangle = 0$ and a neighbourhood of the point p with the contours is homeomorphic to the one shown in Figure 4. Then,*

$$\begin{aligned}\sigma(c_1c_2) &= \sigma(c_1) + \sigma(c_2) + 1, \text{ if } c_1c_2 = c_3^{-1} \text{ is not of order 2,} \\ \sigma(c_1c_2) &= -\sigma(c_1) - \sigma(c_2) - 1, \text{ if } c_1c_2 = c_3^{-1} \text{ is of order 2.}\end{aligned}$$

Lemma 5.3. *Let Γ be a hyperbolic polygon group of signature $(0 : p_1, \dots, p_r)$, $r > 3$. Let $\{c_1, \dots, c_r\}$ be a standard set of generators of Γ . Then, the element c_1c_2 is not elliptic.*

Proof. Let Π be the canonical fundamental polygon for the group generated by the elements c_1, \dots, c_r such that the generators c_i can be described as products $c_i = \sigma_i\sigma_{i+1}$ of reflexions $\sigma_1, \dots, \sigma_r$ in the edges of the polygon Π (suitably numbered). Then,

$$c_1c_2 = (\sigma_1\sigma_2)(\sigma_2\sigma_3) = \sigma_1\sigma_3.$$

The product of two reflexions $\sigma_1\sigma_3$ is an elliptic element if and only if the axes of the reflexions intersect in \mathbb{H} . Since $r > 3$, the sides of the polygon Π that correspond to the reflexions σ_1 and σ_3 are not next to each other. Let us assume that the axes intersect and let Q be the hyperbolic polygon enclosed between by the axes and the polygon Π . All angles of the polygon Π are acute. One angle of the polygon Q is the angle between the intersecting axes, two angles are larger than $\pi/2$, all other angles of Q are larger than π . Hence, the sum of the angles of Q is larger than that it should be for a hyperbolic polygon. \square

Proposition 5.4. *For any standard set of generators*

$$v = \{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_n\}$$

of $\pi(P, p)$ with $n = g + l_h + l_p + l_e$, we have

$$\sum_{j=g+1}^n \sigma(c_j) \equiv (2 - 2g) - (n - g) \pmod{m}.$$

Proof. We discuss the case $g = 0$ first and, then, we reduce the general case to the case $g = 0$.

- Let $g = 0$. We prove that the statement is true for lifts of sequential sets of signature $(0 : p_1, \dots, p_r)$ by induction on r . In the case $r = 3$, Proposition 5.2 implies

$$\begin{aligned}\sigma(c_1c_2) &= \sigma(c_1) + \sigma(c_2) + 1 \text{ if } c_1c_2 = c_3^{-1} \text{ is not of order 2,} \\ \sigma(c_1c_2) &= -\sigma(c_1) - \sigma(c_2) - 1 \text{ if } c_1c_2 = c_3^{-1} \text{ is of order 2.}\end{aligned}$$

If the element c_3 is not of order 2, then Property 2 of m -Arf functions implies $\sigma(c_1c_2) = \sigma(c_3^{-1}) = -\sigma(c_3)$. Combining $\sigma(c_1c_2) = \sigma(c_1) + \sigma(c_2) + 1$ and $\sigma(c_1c_2) = -\sigma(c_3)$, we obtain $\sigma(c_1) + \sigma(c_2) + \sigma(c_3) = -1$. If the element c_3 is of order 2, then $\sigma(c_1c_2) = \sigma(c_3^{-1}) =$

$\sigma(c_3)$. Combining $\sigma(c_1c_2) = -\sigma(c_1) - \sigma(c_2) - 1$ and $\sigma(c_1c_2) = \sigma(c_3)$, we obtain that $\sigma(c_1) + \sigma(c_2) + \sigma(c_3) = -1$. Let us now assume that the statement is true for $r \leq k-1$ and consider the case $r = k$. By our assumption,

$$\sigma(c_1 \cdot c_2) + \sigma(c_3) + \cdots + \sigma(c_k) = 2 - (k-1) = (2-k) + 1.$$

Moreover, according to Lemma 5.3, the element c_1c_2 cannot be of order 2. Hence, by Proposition 5.2, we have $\sigma(c_1c_2) = \sigma(c_1) + \sigma(c_2) + 1$. The last two equations imply that

$$\sigma(c_1) + \cdots + \sigma(c_k) = 2 - k.$$

- We now consider the general case. The set

$$(a_1, b_1a_1^{-1}b_1^{-1}, \dots, a_g, b_ga_g^{-1}b_g^{-1}, c_{g+1}, \dots, c_{g+l_h+l_p+l_e})$$

is a standard set of generators of an orbifold of signature $(0 : 2g + l_h, l_p, l_e : p_1, \dots, p_{l_e})$; hence,

$$\begin{aligned} & \sum_{i=1}^g (\sigma(a_i) + \sigma(b_i a_i^{-1} b_i^{-1})) + \sum_{i=g+1}^{g+l_h+l_p+l_e} \sigma(c_i) \\ &= 2 - (2g + l_h + l_p + l_e) = (2 - 2g) - (l_h + l_p + l_e). \end{aligned}$$

Properties 1 and 2 of m -Arf functions imply that $\sigma(b_i a_i^{-1} b_i^{-1}) = \sigma(a_i^{-1}) = -\sigma(a_i)$ and, hence, $\sigma(a_i) + \sigma(b_i a_i^{-1} b_i^{-1}) = 0$. \square

Definition 5.3. Let $\hat{\sigma}_{\Gamma^*} : \pi(P, p) \rightarrow \mathbb{Z}/m\mathbb{Z}$ be the function associated to a lift Γ^* as in Definition 5.1, then the function $\sigma_{\Gamma^*} = \hat{\sigma}_{\Gamma^*}|_{\pi^0(P, p)}$ is an m -Arf function according to Lemmata 5.1, 3.1 and 3.2. We call the function σ_{Γ^*} the m -Arf function associated to the lift Γ^* .

5.2. Higher Arf functions and self-homeomorphisms of orbifolds. Let Γ be a Fuchsian group of signature $(g : p_1, \dots, p_r)$ and $P = \mathbb{H}/\Gamma$ the corresponding orbifold. Let $p \in P$. Let $\Psi : \mathbb{H} \rightarrow P$ be the natural projection. Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \rightarrow \pi^0(P, p)$ be the induced isomorphism. Let Γ^* be a lift of Γ in G_m . Consider the following transformations of a standard set of generators $v = \{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$ of $\pi^0(P, p)$ to another standard set of generators $v' = \{a'_1, b'_1, \dots, a'_g, b'_g, c'_{g+1}, \dots, c'_{g+r}\}$:

1. $a'_1 = a_1 b_1$.
2. $a'_1 = (a_1 a_2) a_1 (a_1 a_2)^{-1}$,
 $b'_1 = (a_1 a_2) a_1^{-1} a_2^{-1} b_1 (a_1 a_2)^{-1}$,
 $a'_2 = a_1 a_2 a_1^{-1}$,
 $b'_2 = b_2 a_2^{-1} a_1^{-1}$.
3. $a'_g = (b_g^{-1} c_{g+1}) b_g^{-1} (b_g^{-1} c_{g+1})^{-1}$,
 $b'_g = (b_g^{-1} c_{g+1} b_g) c_{g+1}^{-1} b_g a_g b_g^{-1} (b_g^{-1} c_{g+1} b_g)^{-1}$,
 $c'_{g+1} = b_g^{-1} c_{g+1} b_g$.
4. $a'_k = a_{k+1}$, $b'_k = b_{k+1}$,
 $a'_{k+1} = (c_{k+1}^{-1} c_k) a_k (c_{k+1}^{-1} c_k)^{-1}$,
 $b'_{k+1} = (c_{k+1}^{-1} c_k) b_k (c_{k+1}^{-1} c_k)^{-1}$.
5. $c'_k = c_{k+1}$, $c'_{k+1} = c_{k+1}^{-1} c_k c_{k+1}$.

Here, $c_i = [a_i, b_i]$ for $i = 1, \dots, g$; in (4), we consider $k \in \{1, \dots, g\}$; in (5), we consider

$$k \in \{g+1, \dots, g+r\}$$

such that $\text{ord}(c_k) = \text{ord}(c_{k+1})$. If a'_i, b'_i , resp. c'_i , is not described explicitly, this means $a'_i = a_i$, $b'_i = b_i$, resp. $c'_i = c_i$.

We will call these transformations generalised Dehn twists. Each generalised Dehn twist induces a homotopy class of self-homeomorphisms of the orbifold P , which maps elliptic fixed points to elliptic fixed points of the same order. The group of all homotopy classes of self-homeomorphisms of the orbifold P is generated by the homotopy classes of generalised Dehn twists as described above (compare [Zie73]).

Now, we will compute the values of an m -Arf function σ on the standard set of generators v' from the values of σ on the standard set of generators v for each of the generalised Dehn twists described above.

Lemma 5.5. *Let $\sigma : \pi^0(P, p) \rightarrow \mathbb{Z}/m\mathbb{Z}$ be an m -Arf function. Let D be a generalised Dehn twist of the type described above. Suppose that D maps the standard set of generators*

$$v = \{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$$

into the standard set of generators

$$v' = D(v) = \{a'_1, b'_1, \dots, a'_g, b'_g, c'_{g+1}, \dots, c'_{g+r}\}.$$

Let $\alpha_i, \beta_i, \gamma_i$, resp. $\alpha'_i, \beta'_i, \gamma'_i$, be the values of σ on the elements of v , resp. v' . Then, for the Dehn twists of types 1–5, we obtain

1. $\alpha'_1 = \alpha_1 + \beta_1$.
2. $\beta'_1 = \beta_1 - \alpha_1 - \alpha_2 - 1$, $\beta'_2 = \beta_2 - \alpha_2 - \alpha_1 - 1$.
3. $\alpha'_g = -\beta_g$, $\beta'_g = \alpha_g - \gamma_{g+1} - 1$.
4. $\alpha'_k = \alpha_{k+1}$, $\beta'_k = \beta_{k+1}$, $\alpha'_{k+1} = \alpha_k$, $\beta'_{k+1} = \beta_k$.
5. $\gamma'_k = \gamma_{k+1}$, $\gamma'_{k+1} = \gamma_k$.

Proof. We assume that the Dehn twist D belongs to one of the types described in the definition above. In the following computations, we illustrate the position of the contours on the surface with figures showing the position of the axes of the corresponding elements in Γ . Let $\{A_1, B_1, \dots, A_g, B_g, C_{g+1}, \dots, C_{g+r}\}$ be the sequential set corresponding to the standard set of generators v . In the first case, according to Property 3 of m -Arf functions, we obtain $\sigma(a'_1) = \sigma(a_1 b_1) = \sigma(a_1) + \sigma(b_1)$.

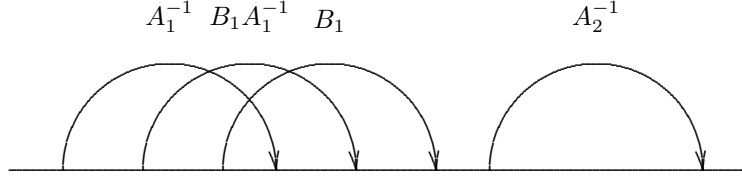
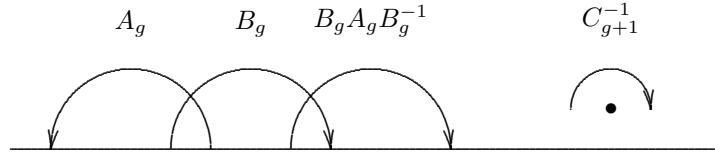
In the second case, according to Property 1, we obtain

$$\begin{aligned} \sigma(a'_1) &= \sigma((a_1 a_2) a_1 (a_1 a_2)^{-1}) = \sigma(a_1), \\ \sigma(b'_1) &= \sigma((a_1 a_2) a_1^{-1} a_2^{-1} b_1 (a_1 a_2)^{-1}) = \sigma(a_1^{-1} a_2^{-1} b_1) \\ &= \sigma(a_1 (a_1^{-1} a_2^{-1} b_1) a_1^{-1}) = \sigma(a_2^{-1} b_1 a_1^{-1}). \end{aligned}$$

The mutual position of the axes of the elements A_2^{-1} and $B_1 A_1^{-1}$ is as in Figure 5; hence, Property 4 implies $\sigma(b'_1) = \sigma(a_2^{-1} \cdot (b_1 a_1^{-1})) = \sigma(a_2^{-1}) + \sigma(b_1 a_1^{-1}) - 1$. According to Property 3, we have $\sigma(b_1 a_1^{-1}) = \sigma(b_1) + \sigma(a_1^{-1})$. Thus, using Property 2, we obtain

$$\sigma(b'_1) = \sigma(a_2^{-1}) + \sigma(b_1) + \sigma(a_1^{-1}) - 1 = \sigma(b_1) - \sigma(a_1) - \sigma(a_2) - 1.$$

Similarly, we show that $\sigma(a'_2) = \sigma(a_2)$ and $\sigma(b'_2) = \sigma(b_2) - \sigma(a_2) - \sigma(a_1) - 1$.


 Figure 5: Axes of $B_1 A_1^{-1}$ and A_2^{-1}

 Figure 6: Axis of $B_g A_g B_g^{-1}$ and fixed point of C_{g+1}^{-1}

In the third case, we obtain according to Properties 2 and 1

$$\begin{aligned}\sigma(a'_g) &= \sigma((b_g^{-1} c_{g+1}) b_g^{-1} (b_g^{-1} c_{g+1})^{-1}) = \sigma(b_g^{-1}) = -\sigma(b_g), \\ \sigma(b'_g) &= \sigma((b_g^{-1} c_{g+1} b_g) c_{g+1}^{-1} b_g a_g b_g^{-1} (b_g^{-1} c_{g+1} b_g)^{-1}) = \sigma(c_{g+1}^{-1} b_g a_g b_g^{-1}), \\ \sigma(c'_{g+1}) &= \sigma(b_g^{-1} c_{g+1} b_g) = \sigma(c_{g+1}).\end{aligned}$$

The actions of the elements C_{g+1}^{-1} and $B_g A_g B_g^{-1}$ on \mathbb{H} are illustrated in Figure 6. According to Properties 4 and 1, we obtain

$$\sigma(b'_g) = \sigma(c_{g+1}^{-1} \cdot (b_g a_g b_g^{-1})) = \sigma(c_{g+1}^{-1}) + \sigma(b_g a_g b_g^{-1}) - 1 = \sigma(c_{g+1}^{-1}) + \sigma(a_g) - 1.$$

In the fourth and fifth cases, computations are easy; we use only Property 1 of m -Arf functions. \square

5.3. Correspondence between higher Arf functions and hyperbolic Gorenstein automorphy factors. Let Γ be a Fuchsian group of signature $(g : p_1, \dots, p_r)$ and $P = \mathbb{H}/\Gamma$ the corresponding orbifold. Let $p \in P$. Let $\Psi : \mathbb{H} \rightarrow P$ be the natural projection. Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \rightarrow \pi^0(P, p)$ be the induced isomorphism.

Lemma 5.6. *The difference $\sigma_1 - \sigma_2 : \pi^0(P, p) \rightarrow \mathbb{Z}/m\mathbb{Z}$ of two Arf functions σ_1 and σ_2 induces a linear function $\ell : H_1(P; \mathbb{Z}/m\mathbb{Z}) \rightarrow \mathbb{Z}/m\mathbb{Z}$.*

Proof. The proof is analogous to the proof of the corresponding statement for higher Arf functions on Fuchsian groups without torsion (see Lemma 4.5 in [NP09]). The main observation is the fact that according to Lemma 5.5 the action of the generalised Dehn twists on the tuples of values of a higher Arf function on elements of a standard set of generators are by affine-linear maps; therefore, the action on the tuples of differences of values of two higher Arf functions is by linear maps. \square

Corollary 5.7. *The set $\text{Arf}^{P,m}$ of all m -Arf functions on $\pi^0(P, p)$ has the structure of an affine space, i.e., the set $\{\sigma - \sigma_0 \mid \sigma \in \text{Arf}^{P,m}\}$ is a free module over $\mathbb{Z}/m\mathbb{Z}$ for any $\sigma_0 \in \text{Arf}^{P,m}$.*

Corollary 5.8. *An m -Arf function is uniquely determined by its values on the elements of some standard set of generators of $\pi^0(P, p)$.*

Theorem 5.9. *Let Γ be a Fuchsian group of signature $(g : p_1, \dots, p_r)$ and $P = \mathbb{H}/\Gamma$ the corresponding orbifold. Let $p \in P$. There is a 1-1-correspondence between*

- 1) *hyperbolic Gorenstein automorphy factors of level m associated to the Fuchsian group Γ .*
- 2) *lifts of Γ into G_m .*
- 3) *m -Arf functions $\sigma : \pi^0(P, p) \rightarrow \mathbb{Z}/m\mathbb{Z}$.*

Proof. According to Proposition 2.4, there is a 1-1-correspondence between hyperbolic Gorenstein automorphy factors of level m associated to the Fuchsian group Γ and the lifts of Γ into G_m . In Definition 5.1, we attached to any lift Γ^* of Γ into G_m an m -Arf function σ_{Γ^*} on P . On the other hand, we can attach to any m -Arf function σ a subset of G_m

$$\Gamma_\sigma^* = \{g \in G_m \mid \pi(g) \in \Gamma, s_m(g) = \sigma(\Phi(\pi(g)))\},$$

where $\pi : G_m \rightarrow G$ is the covering map. It remains to prove that this subset of G_m is actually a lift of Γ . Let

$$v = \{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\} = \{d_1, \dots, d_{2g+r}\}$$

be a standard set of generators of $\pi(P, p)$ and let

$$\bar{V} = \{\Phi^{-1}(d_1), \dots, \Phi^{-1}(d_{2g+r})\}$$

be the corresponding sequential set. Let $\{D_j\}_{j=1, \dots, 2g+r}$ be a lift of the sequential set \bar{V} , i.e., $\pi(D_j) = \Phi^{-1}(d_j)$, such that $s_m(D_j) = \sigma(d_j)$. Then, we obtain according to Proposition 5.4 that

$$\sum_{i=g+1}^{g+r} s_m(C_i) = \sum_{i=g+1}^{g+r} \sigma(c_i) \equiv (2 - 2g) - r \pmod{m}.$$

Hence, by Lemma 4.9, we obtain

$$\prod_{i=1}^g [A_i, B_i] \cdot \prod_{i=g+1}^{g+r} C_i = e.$$

This, and the fact that, for any $i = 1, \dots, r$,

$$p_i \cdot s_m(C_{g+i}) + 1 = p_i \cdot \sigma(c_{g+i}) + 1 \equiv 0 \pmod{m},$$

imply, according to Lemma 4.8, that the subgroup Γ^* of G_m generated by V is a lift of Γ into G_m . Let us compare the corresponding Arf function σ_{Γ^*} with the Arf function σ . We have $\sigma_{\Gamma^*}(d_j) = s_m(D_j) = \sigma(d_j)$ for all j , i.e., the Arf functions σ_{Γ^*} and σ coincide on the standard set of generators v . Thus, by Lemma 5.6 the Arf functions σ_{Γ^*} and σ coincide on the whole $\pi^0(P, p)$. From the definition of σ_{Γ^*} and Γ_σ^* , we see that this implies that $\Gamma^* = \Gamma_\sigma^*$; hence, Γ_σ^* is indeed a lift of Γ into G_m . It is clear from the definitions that the mappings $\Gamma^* \mapsto \sigma_{\Gamma^*}$ and $\sigma \mapsto \Gamma_\sigma^*$ are inverse to each other. \square

Corollary 5.10. *Let P be a Riemann orbifold of signature $(g : p_1, \dots, p_r)$. Let*

$$v = \{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$$

be a standard set of generators of $\pi(P, p)$. An m -Arf function on $\pi^0(P, p)$ exists if and only if the signature $(g : p_1, \dots, p_r)$ satisfies the liftability conditions described in Proposition 4.10. Moreover, if the liftability conditions are satisfied then any possible tuple of $2g$ values in $\mathbb{Z}/m\mathbb{Z}$

can be realised in a unique way as a set of values on a_i, b_i of an m -Arf function on $\pi^0(P, p)$, hence there are m^{2g} different m -Arf functions on $\pi^0(P, p)$.

Proof. The statement follows immediately from Theorem 5.9 and Proposition 4.10. \square

6. MODULI SPACES OF GORENSTEIN SINGULARITIES

We study the moduli space of Gorenstein quasi-homogeneous surface singularities (GQHSS). Using Proposition 2.4, we define the moduli space of GQHSS of level m as the space of conjugacy classes of subgroups Γ^* in G_m such that the restriction of the covering map $G_m \rightarrow G = \mathrm{PSL}(2, \mathbb{R})$ to Γ^* is an isomorphism between Γ^* and a Fuchsian group Γ . The projection $\Gamma^* \mapsto \Gamma$ from the moduli space of GQHSS of level m to the moduli space of Riemann orbifolds is a finite ramified covering.

6.1. Topological classification of higher Arf functions. There is a 1-1-correspondence (see Theorem 5.9) between automorphy factors of level m and m -Arf functions on $\pi^0(P, p)$. This correspondence allows us to reduce the problem of finding the number of connected components of the moduli space of GQHSS of level m to the problem of finding the number of orbits of the action of the group of self-homeomorphisms on the set of m -Arf functions. We describe the orbit of an m -Arf function under the action of the group of homotopy classes of surface self-homeomorphisms.

Let P be a Riemann orbifold of signature $(g : p_1, \dots, p_r)$. Let $p \in P$.

Definition 6.1. Let $\sigma : \pi^0(P, p) \rightarrow \mathbb{Z}/m\mathbb{Z}$ be an m -Arf function. We define the *Arf invariant* $\delta = \delta(P, \sigma)$ of σ as follows: If $g > 1$ and m is even, then we set $\delta = 0$ if there is a standard set of generators $\{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$ of the orbifold fundamental group $\pi(P, p)$ such that

$$\sum_{i=1}^g (1 - \sigma(a_i))(1 - \sigma(b_i)) \equiv 0 \pmod{2}$$

and we set $\delta = 1$ otherwise. If $g > 1$ and m is odd, then we set $\delta = 0$. If $g = 0$, then we set $\delta = 0$. If $g = 1$, then we set $\delta = \gcd(m, p_1 - 1, \dots, p_r - 1, \sigma(a_1), \sigma(b_1))$, where $\{a_1, b_1, c_2, \dots, c_{r+1}\}$ is a standard set of generators of the fundamental group $\pi(P, p)$.

Remark. It is not hard to see that δ does not change under the transformations described in Lemma 5.5, i.e., it is indeed an invariant of the Arf function.

Proof. Let $D, v, v', \alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i$ be as in Lemma 5.5. Let us first consider the case $g > 1$: For a Dehn twist of type 1, we have

$$\begin{aligned} (1 - \alpha'_1)(1 - \beta'_1) &= (1 - (\alpha_1 + \beta_1))(1 - \beta_1) \\ &= (1 - \alpha_1)(1 - \beta_1) - \beta_1(1 - \beta_1) \equiv (1 - \alpha_1)(1 - \beta_1) \pmod{2}. \end{aligned}$$

For a Dehn twist of type 2, we have

$$\begin{aligned} &(1 - \alpha'_1)(1 - \beta'_1) + (1 - \alpha'_2)(1 - \beta'_2) \\ &= (1 - \alpha_1)(1 - \beta_1 + \alpha_1 + \alpha_2 + 1) + (1 - \alpha_2)(1 - \beta_2 + \alpha_1 + \alpha_2 + 1) \\ &= (1 - \alpha_1)(1 - \beta_1) + (1 - \alpha_2)(1 - \beta_2) + (2 - (\alpha_1 + \alpha_2))((\alpha_1 + \alpha_2) + 1) \\ &\equiv (1 - \alpha_1)(1 - \beta_1) + (1 - \alpha_2)(1 - \beta_2) \pmod{2}. \end{aligned}$$

For a Dehn twist of type 3, since m is even and $p_1 \cdot \gamma_{g+1} + 1 \equiv 0 \pmod{m}$, we have that γ_{g+1} is odd. Then, $\gamma_{g+1} + 1 \equiv 0 \pmod{2}$ and $1 + \beta_g \equiv 1 - \beta_g \pmod{2}$ imply

$$\begin{aligned} (1 - \alpha'_g)(1 - \beta'_g) &= (1 + \beta_g)(1 - \alpha_g + (\gamma_{g+1} + 1)) \\ &\equiv (1 + \beta_g)(1 - \alpha_g) \equiv (1 - \beta_g)(1 - \alpha_g) \pmod{2}. \end{aligned}$$

Dehn twists of type 4 do not change δ since they only permute (α_k, β_k) with $(\alpha_{k+1}, \beta_{k+1})$. Dehn twists of type 5 do not change δ since they only permute γ_i .

Let us now consider the case $g = 1$: Dehn twists of types 2 and 4 involve pairs a_i, b_i and a_j, b_j , i.e., they are not applicable in the case $g = 1$. Dehn twists of type 5 do not change δ since they do not change α_i, β_i . For a Dehn twist of type 1, we obtain $\alpha'_1 = \alpha_1 + \beta_1$ and $\beta'_1 = \beta_1$. Thus,

$$\gcd(\alpha'_1, \beta'_1) = \gcd(\alpha_1 + \beta_1, \beta_1) = \gcd(\alpha_1, \beta_1)$$

and, therefore, $\gcd(m, p_1 - 1, \dots, p_r - 1, \alpha'_1, \beta'_1) = \gcd(m, p_1 - 1, \dots, p_r - 1, \alpha_1, \beta_1)$. For a Dehn twist of type 3, we obtain $\alpha'_1 = -\beta_1$ and $\beta'_1 = \alpha_1 - \gamma_2 - 1$. Let d be a common divisor of $m, p_1 - 1, \dots, p_r - 1, \alpha_1, \beta_1$, i.e

$$m \equiv \alpha_1 \equiv \beta_1 \equiv 0 \pmod{d}, \quad p_1 \equiv \dots \equiv p_r \equiv 1 \pmod{d}.$$

We know that $p_1 \cdot \gamma_2 + 1 \equiv 0 \pmod{m}$, but $m \equiv 0 \pmod{d}$; hence, $p_1 \cdot \gamma_2 + 1 \equiv 0 \pmod{d}$. Since $p_1 \equiv 1 \pmod{d}$, we obtain that $\gamma_2 + 1 \equiv 0 \pmod{d}$. Hence, d is a common divisor of

$$m, p_1 - 1, \dots, p_r - 1, \alpha'_1 = -\beta_1, \beta'_1 = \alpha_1 - (\gamma_2 + 1).$$

Similarly, every common divisor of

$$m, p_1 - 1, \dots, p_r - 1, \alpha'_1, \beta'_1$$

is a common divisor of $m, p_1 - 1, \dots, p_r - 1, \alpha_1, \beta_1$. Thus, the greatest common divisors coincide. \square

Definition 6.2. By the *type of the m -Arf function* (P, σ) , we mean the tuple $(g, p_1, \dots, p_r, \delta)$, where δ is the Arf invariant of σ defined above.

Lemma 6.1. *Let $\sigma : \pi^0(P, p) \rightarrow \mathbb{Z}/m\mathbb{Z}$ be an m -Arf function.*

(a) *If $g > 1$, then there is a standard set of generators*

$$v = \{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$$

of $\pi(P, p)$ such that $(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, \xi, 1, \dots, 1)$ with $\xi \in \{0, 1\}$. If m is odd then the set of generators can be chosen in such a way that $\xi = 1$, i.e., so that $(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, 1, 1, \dots, 1)$.

(b) *If $g = 1$, then there is a standard set of generators $v = \{a_1, b_1, c_2, \dots, c_{g+1}\}$ of $\pi(P, p)$ such that $(\sigma(a_1), \sigma(b_1)) = (\delta, 0)$, where δ is the Arf invariant of σ .*

Proof. The proof is along the lines of the proofs of Lemma 5.1 and Lemma 5.2 in [NP09]. Using generalised Dehn twists of types 1, 2 and 4, we can show that a set of generators can be chosen in the desired way. The last step in the proof of Lemma 5.1 in [NP09] was to show that, if m and $\sigma(c_{g+i})$ were even, then we could transform a set of generators with

$$(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, 0, 1, \dots, 1)$$

into a set of generators with $(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, 1, 1, \dots, 1)$. However, in the situation we are considering now, we know that $\sigma(c_{g+i})$ satisfies the equation $p_i \cdot \sigma(c_{g+i}) + 1 \equiv 0$ modulo m . Therefore, if m is even then $\sigma(c_{g+i})$ must be odd. Hence, this last reduction step is not possible in the case considered here. \square

Theorem 6.2. *A tuple $t = (g, p_1, \dots, p_r, \delta)$ is the type of a hyperbolic m -Arf function on a Riemann orbifold of signature $(g : p_1, \dots, p_r)$ if and only if it satisfies the following conditions:*

(a) *The liftability conditions: The orders p_1, \dots, p_r are prime with m and satisfy the condition*

$$(p_1 \cdots p_r) \cdot \left(\sum_{i=1}^r \frac{1}{p_i} - (2g - 2) - r \right) \equiv 0 \pmod{m}.$$

(b) *If $g > 1$ then $\delta \in \{0, 1\}$.*

(c) *If $g > 1$ and m is odd then $\delta = 0$.*

(d) *If $g = 1$ then δ is a divisor of $\gcd(m, p_1 - 1, \dots, p_r - 1)$.*

(e) *If $g = 0$ then $\delta = 0$.*

Proof. Let us first assume that the tuple t is a type of a hyperbolic m -Arf function on an orbifold of signature $(g : p_1, \dots, p_r)$. Then, according to Corollary 5.10, the signature $(g : p_1, \dots, p_r)$ satisfies the liftability conditions. If $g > 1$ and m is odd, then, according to Lemma 6.1, there is a standard set of generators

$$\{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$$

of $\pi^0(P, p)$ such that

$$(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, 1, 1, \dots, 1).$$

Hence, $\delta(P, \sigma) = 0$ by definition. If $g = 1$, then δ is a divisor of $m, p_1 - 1, \dots, p_r - 1$ by definition. If $g = 0$, then $\delta = 0$ by definition.

Now, let us assume that $t = (g, p_1, \dots, p_r, \delta)$ satisfies the conditions (a)-(e). Let P be a Riemann orbifold of signature $(g : p_1, \dots, p_r)$ and let

$$\{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$$

be a standard set of generators of $\pi^0(P, p)$. According to Corollary 5.10, any tuple of $2g$ values in $\mathbb{Z}/m\mathbb{Z}$ can be realised as a set of values on a_i, b_i of an m -Arf function on $\pi^0(P, p)$. In particular, if $g > 1$, then, for any $\delta \in \{0, 1\}$, there exists an m -Arf function σ^δ such that $(\sigma^\delta(a_1), \sigma^\delta(b_1), \dots, \sigma^\delta(a_g), \sigma^\delta(b_g)) = (0, 1 - \delta, 1, \dots, 1)$ and, if $g = 1$, then, for any divisor δ of $m, p_1 - 1, \dots, p_r - 1$, there exists an m -Arf function σ^δ such that $(\sigma^\delta(a_1), \sigma^\delta(b_1)) = (\delta, 0)$. Let $g > 1$. If $\delta = 0$, then the equation $\delta(\sigma^0) = 0$ is satisfied by definition. If $\delta = 1$ and m is even, it remains to prove that $\delta(\sigma^1) = 1$. To this end, we recall that $\sum_{i=1}^g (1 - \sigma(a_i))(1 - \sigma(b_i)) \pmod{2}$ is preserved under the Dehn twists and, hence, is equal to 1 modulo 2 for any standard set of generators. Now, let $g = 1$. Then, $\delta(\sigma^\delta) = \gcd(m, p_1 - 1, \dots, p_r - 1, \delta, 0) = \delta$, since δ is a divisor of $\gcd(m, p_1 - 1, \dots, p_r - 1)$. \square

6.2. Teichmüller spaces of Fuchsian groups. We recall the results on the moduli spaces of Fuchsian groups from [Zie81]. Let $\Gamma_{g;p_1, \dots, p_r}$ be the group generated by the elements

$$v = \{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$$

with defining relations

$$\prod_{i=1}^g [a_i, b_i] \prod_{i=g+1}^{g+r} c_i = 1, \quad c_{g+1}^{p_1} = \dots = c_{g+r}^{p_r} = 1.$$

We denote by $\tilde{T}_{g;p_1, \dots, p_r}$ the set of monomorphisms $\psi : \Gamma_{g;p_1, \dots, p_r} \rightarrow \text{Aut}(\mathbb{H})$ such that

$$\psi(v) = \{a_1^\psi, b_1^\psi, \dots, a_g^\psi, b_g^\psi, c_{g+1}^\psi, \dots, c_{g+r}^\psi\}$$

is a sequential set of signature $(g; p_1, \dots, p_r)$. For $(g; p_1, \dots, p_r)$ to be a signature of a group of hyperbolic isometries, we have to assume that $\sum_{i=1}^r \frac{1}{p_i} < r + (2g - 2)$. The group $\text{Aut}(\mathbb{H})$ acts on $\tilde{T}_{g;p_1, \dots, p_r}$ by conjugation. We set $T_{g;p_1, \dots, p_r} = \tilde{T}_{g;p_1, \dots, p_r} / \text{Aut}(\mathbb{H})$. We parametrise the space $\tilde{T}_{g;p_1, \dots, p_r}$ by the fixed points and shift parameters of the elements of the sequential sets $\psi(v)$. We use here the following analogue of a version [Nat78, Nat04] of the Theorem of Fricke and Klein [FK65]:

Theorem 6.3. *The space $T_{g;p_1, \dots, p_r}$ is diffeomorphic to an open domain in the space $\mathbb{R}^{6g-6+2r}$ which is homeomorphic to $\mathbb{R}^{6g-6+2r}$.*

For an element $\psi : \Gamma_{g;p_1, \dots, p_r} \rightarrow \text{Aut}(\mathbb{H})$ of $\tilde{T}_{g;p_1, \dots, p_r}$, we write

$$\widetilde{\text{Mod}}^\psi = \widetilde{\text{Mod}}_{g;p_1, \dots, p_r}^\psi = \{\alpha \in \text{Aut}(\Gamma_{g;p_1, \dots, p_r}) \mid \psi \circ \alpha \in \tilde{T}_{g;p_1, \dots, p_r}\}.$$

One can show that $\widetilde{\text{Mod}}^\psi$ does not depend on ψ ; hence, we write $\widetilde{\text{Mod}}$ instead of $\widetilde{\text{Mod}}^\psi$. Let $I\widetilde{\text{Mod}}$ be the subgroup of all inner automorphisms of $\Gamma_{g;p_1, \dots, p_r}$ and let

$$\text{Mod}_{g;p_1, \dots, p_r} = \text{Mod} = \widetilde{\text{Mod}} / I\widetilde{\text{Mod}}.$$

We now recall the description of the moduli space of Riemann orbifolds

Theorem 6.4. *The group $\text{Mod} = \text{Mod}_{g;p_1, \dots, p_r}$ and the group of homotopy classes of orientation preserving self-homeomorphisms of the orbifold of signature $(g; p_1, \dots, p_r)$ are naturally isomorphic. The group $\text{Mod}_{g;p_1, \dots, p_r}$ acts naturally on $T_{g;p_1, \dots, p_r}$ by diffeomorphisms. This action is discrete. The quotient set of $T_{g;p_1, \dots, p_r}$ by the action of $\text{Mod}_{g;p_1, \dots, p_r}$ can be identified naturally with the moduli space $M_{g;p_1, \dots, p_r}$ of Riemann orbifolds of signature $(g; p_1, \dots, p_r)$.*

6.3. Connected components of the moduli space.

Definition 6.3. We denote by $S^m(t) = S^m(g, p_1, \dots, p_r, \delta)$ the set of all GQHSS of level m and signature $(g; p_1, \dots, p_r)$ such that the associated m -Arf function is of type $t = (g, p_1, \dots, p_r, \delta)$.

Theorem 6.5. *Let $t = (g, p_1, \dots, p_r, \delta)$ be a tuple that satisfies the conditions of Theorem 6.2, i.e., the space $S^m(t)$ is not empty. Then, the space $S^m(t)$ is homeomorphic to*

$$T_{g;p_1, \dots, p_r} / \text{Mod}_{g;p_1, \dots, p_r}^m(t),$$

where $T_{g;p_1, \dots, p_r}$ is homeomorphic to $\mathbb{R}^{6g-6+2r}$ and $\text{Mod}_{g;p_1, \dots, p_r}^m(t)$ acts on $T_{g;p_1, \dots, p_r}$ as a subgroup of finite index in the group $\text{Mod}_{g;p_1, \dots, p_r}$.

Proof. Let us consider an element ψ of the space $T_{g;p_1, \dots, p_r}$, i.e., consider a homomorphism $\psi : \Gamma_{g;p_1, \dots, p_r} \rightarrow \text{Aut}(\mathbb{H})$. To the homomorphism ψ , we attach an orbifold $P_\psi = \mathbb{H} / \psi(\Gamma_{g;p_1, \dots, p_r})$, a standard set of generators

$$v_\psi = \psi(v) = \{a_1^\psi, b_1^\psi, \dots, a_g^\psi, b_g^\psi, c_{g+1}^\psi, \dots, c_{g+r}^\psi\}$$

of $\pi(P_\psi, p)$ and an m -Arf function σ_ψ on this surface given by

$$\begin{aligned} (\sigma_\psi(a_1^\psi), \sigma_\psi(b_1^\psi)) &= (\delta, 0) \quad \text{if } g = 1, \\ (\sigma_\psi(a_1^\psi), \sigma_\psi(b_1^\psi), \sigma_\psi(a_2^\psi), \sigma_\psi(b_2^\psi), \dots, \sigma_\psi(a_g^\psi), \sigma_\psi(b_g^\psi)) \\ &= (0, 1 - \delta, 1, \dots, 1) \quad \text{if } g > 1. \end{aligned}$$

By Theorem 5.9, the m -Arf function σ_ψ on the orbifold P_ψ corresponds to a lift of $\psi(\Gamma_{g;p_1, \dots, p_r})$ into G_m . This correspondence defines a map $T_{g;p_1, \dots, p_r} \rightarrow S^m(t)$. According to Theorem 6.2,

this map is surjective. Let $\text{Mod}_{g;p_1,\dots,p_r}^m(t)$ be the subgroup of $\text{Aut}(P_\psi) = \text{Mod}_{g;p_1,\dots,p_r}$ that preserves the m -Arf function σ_ψ . For any point in $S^m(t)$, its pre-image in $T_{g;p_1,\dots,p_r}$ consists of an orbit of the subgroup $\text{Mod}_{g;p_1,\dots,p_r}^m(t)$. Thus, $S^m(t) = T_{g;p_1,\dots,p_r} / \text{Mod}_{g;p_1,\dots,p_r}^m(t)$. \square

Summarizing the results of Theorems 6.2 and 6.5, we obtain the following:

Theorem 6.6.

- 1) Two hyperbolic GQHSS are in the same connected component of the space of all hyperbolic GQHSS if and only if they are of the same type. In other words, the connected components of the space of all hyperbolic GQHSS are those sets $S^m(t)$ that are not empty.
- 2) The set $S^m(t)$ is not empty if and only if $t = (g, p_1, \dots, p_r, \delta)$ has the following properties:
 - (a) The orders p_1, \dots, p_r are prime with m and satisfy the condition

$$(p_1 \cdots p_r) \cdot \left(\sum_{i=1}^r \frac{1}{p_i} - (2g - 2) - r \right) \equiv 0 \pmod{m}.$$

- (b) If $g > 1$ and m is odd, then $\delta = 0$.
 - (c) If $g = 1$, then δ is a divisor of $\text{gcd}(m, p_1 - 1, \dots, p_r - 1)$.
 - (d) If $g = 0$, then $\delta = 0$.
- 3) Any connected component $S^m(t)$ of the space of all hyperbolic GQHSS of level m and signature $(g : p_1, \dots, p_r)$ is homeomorphic to

$$\mathbb{R}^{6g-6+2r} / \text{Mod}_{g;p_1,\dots,p_r}^m(t),$$

where $\text{Mod}_{g;p_1,\dots,p_r}^m(t)$ is a subgroup of finite index in the group $\text{Mod}_{g;p_1,\dots,p_r}$ and acts discretely on $\mathbb{R}^{6g-6+2r}$.

7. CONCLUDING REMARKS

- 1) **Higher Spin Structures on General Riemann Orbifolds:** Combining the results in this paper on moduli spaces of higher spin structures on compact Riemann orbifolds with the results on moduli spaces of higher spin structures on Riemann surfaces with holes and punctures in [NP09], we obtain a description of moduli spaces of higher spin structures on Riemann orbifolds with holes and punctures.
- 2) **\mathbb{Q} -Gorenstein singularities:** A normal isolated singularity of dimension at least 2 is \mathbb{Q} -Gorenstein if there is a natural number r such that the divisor $r \cdot \mathcal{K}_X$ is defined on a punctured neighbourhood of the singular point by a function. Here, \mathcal{K}_X is the canonical divisor of X . According to [Pra07], hyperbolic \mathbb{Q} -Gorenstein quasi-homogeneous surface singularities are in 1-to-1 correspondence with groups of the form $C^* \times \Gamma^*$, where C^* is a lift of a finite cyclic group of order r into G_m and Γ^* is a lift of a Fuchsian group Γ into G_m . The lift of a finite cyclic group is unique; hence, hyperbolic \mathbb{Q} -Gorenstein quasi-homogeneous surface singularities are in 1-to-1 correspondence with lifts of a Fuchsian group into G_m . Thus, the moduli space of hyperbolic \mathbb{Q} -Gorenstein quasi-homogeneous surface singularities coincides with the moduli space of hyperbolic Gorenstein quasi-homogeneous surface singularities as described in Theorem 6.6.
- 3) **Spherical and Euclidean Automorphy Factors:** For a spherical Gorenstein automorphy factor $(\mathbb{CP}^1, \Gamma, L)$, the group of automorphisms is $\text{Aut}(U) = \text{Aut}(\mathbb{CP}^1) = \text{PSU}(2)$. The finite subgroups of $\text{SU}(2)$ are the cyclic groups, the dihedral groups and the symmetry groups of the regular polyhedra, i.e., the tetrahedral, octahedral and icosahedral groups. The corresponding singularities are A_k, D_k, E_6, E_7, E_8 . For a Euclidean Gorenstein automorphy factor (\mathbb{C}, Γ, L) ,

the group Γ is contained in the translation subgroup of $\text{Aut}(\mathbb{C})$ and can be identified with a sublattice $\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau$ of the additive group \mathbb{C} , where $\tau \in \mathbb{C}$ and $\text{Im}(\tau) > 0$, see [Dol83b]. The corresponding singularities are $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. All GQHSS other than $A_k, D_k, E_6, E_7, E_8, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ belong to the class of hyperbolic GQHSS, which is studied in this paper.

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