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POLARIZATIONS ON LIMITING MIXED HODGE STRUCTURES

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ABSTRACT. We construct polarizations of mixed Hodge structures on the relative log de Rham cohomology groups of a projective log deformation. To this end, we study the behavior of weight and Hodge filtrations under the cup product and construct a trace morphism for a projective log deformation.

INTRODUCTION

0.1. In [22] Steenbrik introduced the notion of the log deformation and constructed mixed Hodge structures on the relative log de Rham cohomology groups of a projective log deformation. In this article, we construct natural polarizations on these mixed Hodge structures in the sense of Cattani-Kaplan-Schmid [2, Definition (2.26)].

A typical example of log deformations is the singular fiber of a semistable reduction over the unit disc. For the case of a projective semistable reduction over the unit disc, the mixed Hodge structure on the relative log de Rham cohomology groups of the singular fiber is considered as the limits of Hodge structures on the cohomology groups of general fibers, and called the limiting mixed Hodge structures. These mixed Hodge structures were constructed by two different methods, the transcendental method in [20] and the algebro-geometric method in [21]. In fact, Schmid's nilpotent orbit theorem and SL_2-orbit theorem imply that a variation of polarized Hodge structures on the punctured disc degenerates to a *polarized* mixed Hodge structure ([20, (6.16) Theorem]). For the case of a projective semistable reduction over the unit disc, the Hodge structures on the cohomology groups of fibers induce variations of polarized Hodge structures on the punctured disc. By applying Schmid's result above to these variations of polarized Hodge structures, we obtained the limiting mixed Hodge structures. Here we note that these mixed Hodge structures are canonically polarized by their construction. On the other hand, Steenbrink constructed mixed Hodge structures on the relative log de Rham cohomology groups of the singular fiber of a projective semistable reduction over the unit disc by algebro-geometric methods. The coincidence between Steenbrink's mixed Hodge structures and Schmid's mixed Hodge structures was proved in [19, 4.2.5 Remarque] and in [23, (A.1)] independently. The motivation of this article is to construct the polarizations on the limiting mixed Hodge structures by algebro-geometric methods in Steenbrink's approach. Once we obtain polarizations in Theorem 8.16 below, the remaining task is to prove that our polarizations coincide with the ones given in [20, (6.16) Theorem] for the case of a projective semistable reduction.

The main result of this article concerns the question whether the mixed Hodge structures on the relative log de Rham cohomology groups of a projective log deformation yield nilpotent orbits. In fact, Kashiwara-Kawai [13, Proposition 1.2.2] and Cattani-Kaplan [1, Theorem (3.13)] show that a polarized mixed Hodge structure yields a nilpotent orbit and vice versa. Therefore the main result of this article implies that the relative log de Rham cohomology groups of a

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projective log deformation give us nilpotent orbits. Thus, it is expected that a projective log deformation yields polarized log Hodge structures on the standard log point (see Kato-Usui [15, 2.5]) as a by-product of our main result. This question is treated in the forthcoming article [9].

0.2. Let $Y \longrightarrow *$ be a projective log deformation of pure dimension n. In order to put mixed Hodge structures on the log de Rham cohomology groups $\mathrm{H}^q(Y, \omega_{Y/*})$, we replace $\omega_{Y/*}$ by the weak cohomological mixed Hodge complex K defined in [6, (5.4)]. Here we note that the complex K carries a multiplicative structure which is compatible with the wedge product on $\omega_{Y/*}$. Then our aim is, more precisely, to construct polarizations on $\mathrm{H}^q(Y, K)$ for all q. To this end, we follow a way similar to the case of compact Kähler complex manifolds. First, we construct a cup product on $\mathrm{H}^*(Y, K)$ by using the multiplicative structure on K. Second, we define a trace morphism $\mathrm{H}^{2n}(Y, K) \longrightarrow \mathbb{C}$. Third, we study the property of the cup product with the class of an ample invertible sheaf in $\mathrm{H}^2(Y, K)$. Finally, we prove a kind of positivity for the bilinear form as a conclusion. The key ingredient for our argument is the comparison morphism $\varphi : A \longrightarrow K$, where A denotes the cohomological mixed Hodge complex constructed by Steenbrink in [22, Section 5] (cf. [21, Section 4]). By the fact that φ induces isomorphisms of mixed Hodge structures between $\mathrm{H}^q(Y, A)$ and $\mathrm{H}^q(Y, K)$ for all q, we can apply the results by Guillén-Navarro Aznar [11, (5.1)Théorème], or by Morihiko Saito [19, 4.2.5 Remarque] to prove the positivity.

This article is organized as follows: In Section 1, we fix the notation and the sign convention used in this article. Section 2 treats the Čech complex of a co-cubical complex. We give the definition of a product morphism for the Čech complexes of two co-cubical complexes. In Section 3, we study the residue morphisms for the log de Rham complex and for the Koszul complex of a log deformation. Section 4 is devoted to the study of the Gysin morphism for a log deformation. In Section 5, we first recall the definition of the complex K in [6] for the case of a log deformation. We slightly modify the definition and the notation in [6]. Then we recall results of [6] in Theorem 5.9. Next, we study several properties of the Gysin morphism of the complex K for the later use. Furthermore, we recall the definition of the complex A in Steenbrink [22] and in Fujisawa-Nakayama [8]. Here we also modify the definition of A slightly. Theorem 5.21 restates the results of [22] and of [8]. Then we construct the comparison morphism φ from A to K mentioned above. We prove that the morphism φ induces isomorphisms between $\mathrm{H}^q(Y, A)$ and $\mathrm{H}^q(Y, K)$ for all q in this section. In Section 6, the multiplicative structures on the complex K and on other related complexes are studied. The multiplicative structure on K induces the cup product on $\mathrm{H}^*(Y, K)$ mentioned above. We prove that the cup product on $\mathrm{H}^*(Y, K)$ satisfies the expected properties for the weight filtration W and the Hodge filtration F. In Section 7, the trace morphism $\mathrm{Tr} : \mathrm{H}^{2n}(Y, K_{\mathbb{C}}) \longrightarrow \mathbb{C}$ mentioned above is defined by using the E_2-degeneracy of the spectral sequence $E_r^{p,q}(K_{\mathbb{C}}, W)$. The cup product on $\mathrm{H}^*(Y, K)$ and the trace morphism induce the bilinear form Q_K on $\mathrm{H}^*(Y, K)$. Combining all these together in Section 8, we prove the main results of this article by applying the results on bigraded polarized Hodge-Lefschetz modules in [11].

0.3. The results of this article have been already announced with few proofs in [7]. There we restrict ourselves to the case of a semistable reduction over the unit disc for simplicity. In this paper we will give the complete proofs for the results. Moreover, we modify some definitions slightly and correct several mistakes in [7].

1. Preliminaries

In this section, we collect several definitions which will be used in this article constantly. We follow [3, 1.3] and [16, Notation] for sign conventions. We recall some of them for the later use.

1.1. The cardinality of a finite set A is denoted by $|A|$.

1.2. For the shift of an increasing filtration W, we use the notation by Deligne [5, Définition (1,1,2), (1,1,3)]. Namely, we set

$$W[k]_m = W_{m-k}$$

for every k, m. This notation is different from that used by Cattani-Kaplan-Schmid [2, p.475]. For the shift of a decreasing filtration F, we follow the standard notation, that is,

$$F[k]^p = F^{p+k}$$

for every k, p.

1.3. Let $f : K \longrightarrow L$ be a morphism of complexes. The complex $(C(f), d)$, called the mapping cone of f, is defined by

$$\begin{aligned} C(f)^p &= K^{p+1} \oplus L^p \\ d(x,y) &= (-dx, f(x) + dy) \quad x \in K^{p+1}, y \in L^p \end{aligned}$$

as in [12]. Two morphisms of complexes

$$\begin{aligned} \alpha(f) &: L \longrightarrow C(f) \\ \beta(f) &: C(f) \longrightarrow K[1] \end{aligned}$$

are defined by

$$\begin{aligned} \alpha(f)(y) &= (0, y) \qquad y \in L^p \\ \beta(f)(x,y) &= -x \qquad x \in K^{p+1},\ y \in L^p \end{aligned}$$

for every integer p. (See e.g. [3, (1.3.3)], [16, Notation (4)].)

For every integer m, we set a morphism

$$\zeta_m : C(f)[m]^p \longrightarrow C(f[m])^p$$

by $\zeta_m(x,y) = ((-1)^m x, y)$ for an element $(x,y) \in C(f)[m]^p = K^{p+m+1} \oplus L^{p+m}$. It is easy to see that this defines an isomorphism of complexes $\zeta_m : C(f)[m] \longrightarrow C(f[m])$. Then the diagram

$$\begin{array}{ccccc} L[m] & \xrightarrow{\alpha(f)[m]} & C(f)[m] & \xrightarrow{\beta(f)[m]} & K[m+1] \\ \| & & \downarrow{\scriptstyle \zeta_m} & & \downarrow{\scriptstyle (-1)^m \mathrm{id}} \\ L[m] & \xrightarrow[\alpha(f[m])]{} & C(f[m]) & \xrightarrow[\beta(f[m])]{} & K[m+1] \end{array} \tag{1.3.1}$$

is commutative.

Let

$$0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} M \longrightarrow 0 \tag{1.3.2}$$

be an exact sequence of complexes, that is,

$$0 \longrightarrow K^p \xrightarrow{f} L^p \xrightarrow{g} M^p \longrightarrow 0$$

is exact for every p. We define a morphism of complexes

$$\delta(f,g) : C(f) \longrightarrow M$$

by sending $(x,y) \in C(f)^p = K^{p+1} \oplus L^p$ to $\delta(f,g)(x,y) = g(y) \in M^p$. It is well known that this morphism $\delta(f,g)$ is a quasi-isomorphism. Therefore the diagram

$$M \xleftarrow{\delta(f,g)} C(f) \xrightarrow{\beta(f)} K[1]$$

gives us a morphism

$$\gamma(f,g) : M \longrightarrow K[1] \tag{1.3.3}$$

in the derived category. We can easily check that the morphism

$$\mathrm{H}^p(\gamma(f,g)) : \mathrm{H}^p(M) \longrightarrow \mathrm{H}^{p+1}(K)$$

induced by the morphism (1.3.3) coincides with the classical connecting homomorphism induced by the short exact sequence (1.3.2).

Because we have a commutative diagram

$$\begin{array}{ccccc} M[m] & \xleftarrow[\delta(f,g)[m]]{} & C(f)[m] & \xrightarrow[\beta(f)[m]]{} & K[m+1] \\ \| & & \downarrow \zeta_m & & \downarrow (-1)^m \mathrm{id} \\ M[m] & \xleftarrow{\delta(f[m],g[m])} & C(f[m]) & \xrightarrow{\beta(f[m])} & K[m+1] \end{array}$$

by (1.3.1), we have the equality

$$\gamma(f,g)[m] = (-1)^m \gamma(f[m], g[m]) \tag{1.3.4}$$

for every m.

1.4. For two integers a, b, we identify two complexes $K[a] \otimes L[b]$ and $(K \otimes L)[a+b]$ as follows (see [3, (1.3.6)]). The morphism

$$K[a] \otimes L[b] \longrightarrow (K \otimes L)[a+b] \tag{1.4.1}$$

is given by

$$x \otimes y \mapsto (-1)^{pb} x \otimes y$$

on the component $K[a]^p \otimes L[b]^q = K^{p+a} \otimes L^{q+b}$. This gives us the identification expected. For a morphism of complexes

$$f : K_1 \otimes K_2 \longrightarrow K_3$$

the morphism of complexes

$$f[a,b] : K_1[a] \otimes K_2[b] \longrightarrow K_3[a+b] \tag{1.4.2}$$

is the composite of the identification (1.4.1) and the morphism

$$f[a+b] : (K_1 \otimes K_2)[a+b] \longrightarrow K_3[a+b]$$

for every a, b.

1.5. Let K be a complex equipped with an increasing filtration W. For an integer m, the exact sequence

$$0 \longrightarrow \mathrm{Gr}^W_{m-1} K \longrightarrow W_m K / W_{m-2} K \longrightarrow \mathrm{Gr}^W_m K \longrightarrow 0$$

induces the morphism

$$\mathrm{Gr}^W_m K \longrightarrow \mathrm{Gr}^W_{m-1} K[1]$$

in the derived category as (1.3.3). It is called the Gysin morphism of the filtered complex (K, W) and denoted by $\gamma_m(K, W)$. By (1.3.4), we have

$$\gamma_m(K[l], W) = (-1)^l \gamma_m(K, W)[l] \tag{1.5.1}$$

for every l.

The morphism

$$\mathrm{H}^{p+q}(\gamma_{-p}(K,W)) : \mathrm{H}^{p+q}(\mathrm{Gr}^W_{-p} K) \longrightarrow \mathrm{H}^{p+q}(\mathrm{Gr}^W_{-p-1} K[1]) = \mathrm{H}^{p+q+1}(\mathrm{Gr}^W_{-p-1} K)$$

coincides with the morphism

$$d_1 : E_1^{p,q}(K,W) \longrightarrow E_1^{p+1,q}(K,W)$$

of the E_1-terms of the spectral sequences associated to the filtered complex (K, W) under the identification $E_1^{p,q}(K, W) \simeq \mathrm{H}^{p+q}(\mathrm{Gr}_{-p}^W K)$.

1.6. Let (K, d) be a complex equipped with an increasing filtration W and $f : K \longrightarrow K[1]$ a morphism of complexes satisfying the conditions $f^2 = 0$ and $f(W_m K) \subset W_{m-1}K[1]$ for every m. Since we can easily check $(d + f)^2 = 0$, we obtain a complex $(K, d + f)$ which is denoted by K' for a while. The same W defines an increasing filtration on K'. We have the identity

$$\mathrm{Gr}_m^W K = \mathrm{Gr}_m^W K'$$

as complexes for every m. Moreover the morphism f induces a morphism of complexes

$$\mathrm{Gr}_m^W(f) : \mathrm{Gr}_m^W K \longrightarrow \mathrm{Gr}_{m-1}^W K[1]$$

for every m. The following Proposition is easy to check.

Proposition 1.7. *In the situation above, we have*

$$\gamma_m(K', W) = \gamma_m(K, W) + \mathrm{Gr}_m^W(f)$$
$$: \mathrm{Gr}_m^W K' = \mathrm{Gr}_m^W K \longrightarrow \mathrm{Gr}_{m-1}^W K[1] = \mathrm{Gr}_{m-1}^W K'[1]$$

for every integer m.

1.8. Let K_1, K_2, K_3 be complexes. Assume that a morphism of complexes

$$\varphi : K_1 \otimes K_2 \longrightarrow K_3$$

is given. Then φ induces the morphism

$$\mathrm{H}^p(K_1) \otimes \mathrm{H}^q(K_2) \longrightarrow \mathrm{H}^{p+q}(K_3)$$

for every p, q. This morphism is denoted by $\mathrm{H}^{p,q}(\varphi)$ in this article.

For the case where K_1, K_2, K_3 carry increasing filtrations W, if the morphism φ satisfies the condition

$$\varphi(W_a K_1 \otimes W_b K_2) \subset W_{a+b} K_3$$

for all integers a, b, then the morphism φ induces the morphisms

$$\mathrm{Gr}_{a,b}^W \varphi : \mathrm{Gr}_a^W K_1 \otimes \mathrm{Gr}_b^W K_2 \longrightarrow \mathrm{Gr}_{a+b}^W K_3$$

for all a and b.

2. Čech complexes of co-cubical complexes

2.1. Let Λ be a non-empty set. For a positive integer n, Λ^n denotes the n-times product set of Λ. We set $\prod \Lambda = \coprod_{n>0} \Lambda^n$. We consider Λ as a subset of $\prod \Lambda$. We use a symbol λ for an element of $\prod \Lambda$. For an element $\lambda \in \Lambda^{n+1}$, we set $d(\lambda) = n$.

An element $\lambda \in \Lambda^{k+1}$ is denoted by

$$\lambda = (\lambda(0), \lambda(1), \ldots, \lambda(k)) \in \Lambda^{k+1}$$

more explicitly, and the subset

$$\{\lambda(0), \lambda(1), \ldots, \lambda(k)\} \subset \Lambda$$

is denoted by $\underline{\lambda}$. We note that $|\underline{\lambda}| \leq d(\lambda) + 1$ and that the equality holds if and only if all $\lambda(i)$'s are distinct. We set

$$\Lambda^{k+1,\circ} = \{\lambda \in \Lambda^{k+1}; |\underline{\lambda}| = k + 1\} \subset \Lambda^{k+1}$$

for $k \geq 0$ and $\prod^\circ \Lambda = \coprod_{k \geq 0} \Lambda^{k+1,\circ} \subset \prod \Lambda$.

For an element $\lambda \in \Lambda^{k+1}$, we set

$$\lambda_i = (\lambda(0), \lambda(1), \ldots, \lambda(i-1), \lambda(i+1), \ldots, \lambda(k)) \in \Lambda^k \tag{2.1.1}$$

for $i = 0, 1, \dots, k$. If $\lambda \in \Lambda^{k+1,\circ}$, then $\lambda_i \in \Lambda^{k,\circ}$ for all i.

We define a map

$$h_i : \Lambda^{k+1} \longrightarrow \Lambda^{i+1} \tag{2.1.2}$$

for $0 \le i \le k$ by

$$h_i(\lambda)(j) = \lambda(j) \quad 0 \le j \le i$$

for $\lambda \in \Lambda^{k+1}$. Similarly, a map

$$t_i : \Lambda^{k+1} \longrightarrow \Lambda^{k-i+1} \tag{2.1.3}$$

for $0 \le i \le k$ by

$$t_i(\lambda)(j) = \lambda(j+i) \quad 0 \le j \le k-i$$

for $\lambda \in \Lambda^{k+1}$.

We trivially have

$$h_i(\Lambda^{k+1,\circ}) \subset \Lambda^{i+1,\circ}$$
$$t_i(\Lambda^{k+1,\circ}) \subset \Lambda^{k-i+1,\circ}$$

for all i, k.

2.2. For a finite subset $\underline{\lambda}$ of Λ, the free $\mathbb{Z}$-module of rank $|\underline{\lambda}|$ generated by $\{e_\lambda\}_{\lambda \in \underline{\lambda}}$ is denoted by $\mathbb{Z}^{\underline{\lambda}}$, that is, we have

$$\mathbb{Z}^{\underline{\lambda}} = \bigoplus_{\lambda \in \underline{\lambda}} \mathbb{Z} e_\lambda$$

by definition. By setting

$$\varepsilon(\underline{\lambda}) = \bigwedge^{|\underline{\lambda}|} \mathbb{Z}^{\underline{\lambda}},$$

we obtain a free $\mathbb{Z}$-module $\varepsilon(\underline{\lambda})$ of rank 1. We note $\varepsilon(\emptyset) = \mathbb{Z}$ by definition. There exists the canonical isomorphism

$$\vartheta(\underline{\lambda}) : \varepsilon(\underline{\lambda}) \otimes \varepsilon(\underline{\lambda}) \longrightarrow \mathbb{Z} \tag{2.2.1}$$

which sends $e_{\lambda_1} \wedge e_{\lambda_2} \wedge \cdots \wedge e_{\lambda_k} \otimes e_{\lambda_1} \wedge e_{\lambda_2} \wedge \cdots \wedge e_{\lambda_k}$ to 1. For two finite subsets $\underline{\lambda}, \underline{\mu}$ of Λ with $\underline{\lambda} \cap \underline{\mu} = \emptyset$, we define a morphism

$$\chi(\underline{\lambda}, \underline{\mu}) : \varepsilon(\underline{\lambda}) \otimes \varepsilon(\underline{\mu}) \longrightarrow \varepsilon(\underline{\lambda} \cup \underline{\mu}) \tag{2.2.2}$$

by $\chi(\underline{\lambda}, \underline{\mu})(v \otimes w) = v \wedge w$.

For $\lambda \in \Lambda^{k+1,\circ}$, we set

$$e_\lambda = e_{\lambda(0)} \wedge e_{\lambda(1)} \wedge \cdots \wedge e_{\lambda(k)} \in \varepsilon(\underline{\lambda}),$$

which is a base of $\varepsilon(\underline{\lambda})$ over $\mathbb{Z}$. For a subset $\underline{\mu}$ of Λ with $\underline{\lambda} \cap \underline{\mu} = \emptyset$, we define an isomorphism

$$e_\lambda \wedge : \varepsilon(\underline{\mu}) \longrightarrow \varepsilon(\underline{\lambda} \cup \underline{\mu})$$

by sending $v \in \varepsilon(\underline{\mu})$ to $e_\lambda \wedge v \in \varepsilon(\underline{\lambda} \cup \underline{\mu})$. In particular, we obtain an isomorphism

$$e_\lambda \wedge : \mathbb{Z} \longrightarrow \varepsilon(\underline{\lambda}) \tag{2.2.3}$$

for the case of $\underline{\mu} = \emptyset$.

2.3. The set of all subsets of Λ is denoted by $S(\Lambda)$. Moreover $S_n(\Lambda)$ denotes the set of all subsets $\underline{\lambda} \subset \Lambda$ with $|\underline{\lambda}| = n$ for $n \ge 0$. For two subsets $\underline{\lambda}, \underline{\mu}$ with $\underline{\lambda} \subset \underline{\mu}$, the inclusion $\underline{\lambda} \hookrightarrow \underline{\mu}$ is denoted by $\iota_{\underline{\mu}, \underline{\lambda}}$.

The set $S(\Lambda)$ admits an order by the inclusion of subsets. We denote by $\mathcal{S}(\Lambda)$ the category associated to the ordered set $S(\Lambda)$ as in [14, p.14]. The subset of $S(\Lambda)$ consisting of all the non-empty subsets of Λ is denoted by $S^+(\Lambda)$ and the category associated to the ordered set $S^+(\Lambda)$ by $\mathcal{S}^+(\Lambda)$.

2.4. Let $\mathcal{C}$ be a category. A cubical object in $\mathcal{C}$ indexed by the category $\mathcal{S}(\Lambda)$ (resp. $\mathcal{S}^+(\Lambda)$) is a contravariant functor from the category $\mathcal{S}(\Lambda)$ (resp. $\mathcal{S}^+(\Lambda)$) to $\mathcal{C}$. On the other hand, a co-cubical object in $\mathcal{C}$ indexed by the category $\mathcal{S}(\Lambda)$ (resp. $\mathcal{S}^+(\Lambda)$) is a covariant functor from the category $\mathcal{S}(\Lambda)$ (resp. $\mathcal{S}^+(\Lambda)$) to $\mathcal{C}$. A morphism of (co-)cubical objects is a morphism of functors as usual. We use terminology such as (co-)cubical module, (co-)cubical complex, and so on, as in the obvious meaning.

2.5. Now we fix a commutative $\mathbb{Q}$-algebra κ. Let $\mathcal{A}$ be the category of κ-modules, or the category of the κ-sheaves on a topological space.

Let Λ be a non-empty set. For a co-cubical complex K in $\mathcal{A}$ indexed by the category $\mathcal{S}^+(\Lambda)$, we set

$$\mathcal{C}(K)^{k,l} = \prod_{\lambda \in \Lambda^{k+1,\circ}} K(\underline{\lambda})^l$$

for integers k, l. An element $f \in \mathcal{C}(K)^{k,l}$ is a collection

$$f = (f_\lambda)_{\lambda \in \Lambda^{k+1,\circ}} \qquad f_\lambda \in K(\underline{\lambda})^l$$

by definition. The morphism $\delta = \delta_K : \mathcal{C}(K)^{k,l} \longrightarrow \mathcal{C}(K)^{k+1,l}$ is defined by

$$\delta(f)_\lambda = \sum_{i=0}^{k+1} (-1)^i K(\iota_{\underline{\lambda}, \underline{\lambda_i}})(f_{\lambda_i})$$

for $f \in \mathcal{C}(K)^{k,l}$ and for $\lambda \in \Lambda^{k+2,\circ}$ as usual. On the other hand, we define a morphism

$$\partial : \mathcal{C}(K)^{k,l} \longrightarrow \mathcal{C}(K)^{k,l+1}$$

by

$$\partial(f)_\lambda = df_\lambda$$

for $\lambda \in \Lambda^{k+1,\circ}$. By setting

$$\mathcal{C}(K)^p = \bigoplus_{k+l=p} \mathcal{C}(K)^{k,l}$$

$$d = \delta + (-1)^k \partial$$

we obtain a complex $(\mathcal{C}(K), d)$ in $\mathcal{A}$, which is simply denoted by $\mathcal{C}(K)$ for short. Here we follow the sign convention in [17, p.24]. We call it the Čech complex of a co-cubical complex K. The construction above is functorial in the usual sense.

2.6. Let (K, W) be an increasingly filtered co-cubical complex in $\mathcal{A}$ indexed by $\mathcal{S}^+(\Lambda)$. Then increasing filtrations W and δW on the complex $\mathcal{C}(K)$ are defined by

$$W_m \mathcal{C}(K)^{k,l} = \prod_{\lambda \in \Lambda^{k+1,\circ}} W_m K(\underline{\lambda})^l$$

$$W_m \mathcal{C}(K)^p = \bigoplus_{k+l=p} W_m \mathcal{C}(K)^{k,l}$$

$$(\delta W)_m \mathcal{C}(K)^{k,l} = \prod_{\lambda \in \Lambda^{k+1,\circ}} W_{m+k} K(\underline{\lambda})^l$$

$$(\delta W)_m \mathcal{C}(K)^p = \bigoplus_{k+l=p} (\delta W)_m \mathcal{C}(K)^{k,l}$$

for every m. We easily see the equality

$$\mathrm{Gr}_m^{\delta W} \mathcal{C}(K) = \bigoplus_{k \geq 0} \prod_{\lambda \in \Lambda^{k+1,\circ}} \mathrm{Gr}_{m+k}^W K(\underline{\lambda})[-k] \tag{2.6.1}$$

for every m.

For a decreasingly filtered co-cubical complex (K, F) in $\mathcal{A}$ indexed by $\mathcal{S}^+(\Lambda)$, we similarly define decreasing filtrations F and δF on $\mathcal{C}(K)$. These constructions satisfy the functoriality in the obvious meaning.

Lemma 2.7. *We have*

$$\gamma_m(\mathcal{C}(K), \delta W) = \bigoplus_{k\geq 0} \prod_{\lambda\in\Lambda^{k+1,\circ}} (-1)^k \gamma_{m+k}(K(\underline{\lambda}), W)[-k] + \mathrm{Gr}^W_{m+k}\,\delta$$

for every m.

Proof. Applying Proposition 1.7 and the equality (1.5.1), we obtain the conclusion. □

2.8. Let K, L be two co-cubical complexes in $\mathcal{A}$ indexed by $\mathcal{S}^+(\Lambda)$. A co-cubical complex $K\otimes L$ is defined by

$$(K\otimes L)(\underline{\lambda}) = K(\underline{\lambda}) \otimes L(\underline{\lambda})$$

for $\underline{\lambda} \in S^+(\Lambda)$. For $\underline{\lambda}, \underline{\mu} \in S^+(\Lambda)$ with $\underline{\lambda} \subset \underline{\mu}$, the morphism

$$(K\otimes L)(\iota_{\underline{\mu},\underline{\lambda}}) : (K\otimes L)(\underline{\lambda}) \longrightarrow (K\otimes L)(\underline{\mu})$$

is defined by $(K\otimes L)(\iota_{\underline{\mu},\underline{\lambda}}) = K(\iota_{\underline{\mu},\underline{\lambda}}) \otimes L(\iota_{\underline{\mu},\underline{\lambda}})$. For the case where K and L carry increasing filtrations W,

$$W_m(K\otimes L)(\underline{\lambda}) = \sum_{a+b=m} W_aK(\underline{\lambda}) \otimes W_bL(\underline{\lambda})$$

defines a filtration W on $K \otimes L$.

Now, we will define a morphism

$$\tau : \mathcal{C}(K) \otimes \mathcal{C}(L) \longrightarrow \mathcal{C}(K\otimes L),$$

which is a straightforward generalization of the cup product on the singular cohomology groups of a topological space in terms of the Čech cohomology.

Definition 2.9. We define a morphism

$$\tau_{k,l} : \mathcal{C}(K)^{k,p-k} \otimes \mathcal{C}(L)^{l,q-l} \longrightarrow \mathcal{C}(K\otimes L)^{k+l,p+q-k-l}$$

by setting

$$\tau_{k,l}(f\otimes g)_\lambda = K(\iota_{\underline{\lambda},\underline{h_k(\lambda)}})(f_{h_k(\lambda)}) \otimes L(\iota_{\underline{\lambda},\underline{t_k(\lambda)}})(g_{t_k(\lambda)}) \in K(\underline{\lambda})^{p-k} \otimes L(\underline{\lambda})^{q-l}$$

for $f \in \mathcal{C}(K)^{k,p-k}, g \in \mathcal{C}(L)^{l,q-l}$ and for $\lambda \in \Lambda^{k+l+1,\circ}$, where h_k and t_k are the maps defined in (2.1.2) and in (2.1.3) respectively.

By setting

$$\tau = \tau_{K,L} = \sum_{k,l\geq 0} (-1)^{(p-k)l} \tau_{k,l}$$

we obtain a morphism

$$\tau : \mathcal{C}(K)^p \otimes \mathcal{C}(L)^q \longrightarrow \mathcal{C}(K\otimes L)^{p+q}$$

for all p, q.

The following lemmas can be checked by easy and direct computation.

Lemma 2.10. *The morphism τ defines a morphism of complexes $\tau : \mathcal{C}(K)\otimes\mathcal{C}(L) \longrightarrow \mathcal{C}(K\otimes L)$.*

Lemma 2.11. *For three co-cubical complexes K_1, K_2, K_3 in $\mathcal{A}$ indexed by $\mathcal{S}^+(\Lambda)$, the diagram*

$$\begin{array}{ccc} \mathcal{C}(K_1)\otimes\mathcal{C}(K_2)\otimes\mathcal{C}(K_3) & \xrightarrow{\tau_{K_1,K_2}\otimes\mathrm{id}} & \mathcal{C}(K_1\otimes K_2)\otimes\mathcal{C}(K_3) \\ {\scriptstyle \mathrm{id}\otimes\tau_{K_2,K_3}}\downarrow & & \downarrow{\scriptstyle \tau_{K_1\otimes K_2,K_3}} \\ \mathcal{C}(K_1)\otimes\mathcal{C}(K_2\otimes K_3) & \xrightarrow[\tau_{K_1,K_2\otimes K_3}]{} & \mathcal{C}(K_1\otimes K_2\otimes K_3) \end{array}$$

is commutative.

Lemma 2.12. *Let $(K,W),(L,W)$ be co-cubical filtered complexes. Then the morphism τ above satisfies*

$$\tau(W_a\,\mathcal{C}(K)\otimes W_b\,\mathcal{C}(L))\subset W_{a+b}\,\mathcal{C}(K\otimes L)$$
$$\tau((\delta W)_a\,\mathcal{C}(K)\otimes(\delta W)_b\,\mathcal{C}(L))\subset(\delta W)_{a+b}\,\mathcal{C}(K\otimes L)$$

for all a,b. We have the same formulas for decreasing filtrations.

3. Residue morphisms

In this section, we first fix the notation for log deformations. Then we give the definition of the residue morphism in our case, and prove several results on it. In the last part of this section, we study the residue morphism for the Koszul complexes of a log deformation.

3.1. Let $Y \longrightarrow *$ be a log deformation (for the definition, see [8, Definition 2.15], [22, Definition (3,8)]). We assume that all the irreducible components of the log deformation Y are smooth as in [8]. The log structure on Y is denoted by M_Y. The morphism of monoid sheaves $\mathbb{N}_Y \longrightarrow M_Y$ is induced by the morphism of log complex analytic spaces $Y \longrightarrow *$. The image of $1\in\mathbb{N}=\Gamma(*,\mathbb{N})$ by the morphism above is denoted by $t\in\Gamma(Y,M_Y)$.

3.2. We describe the irreducible decomposition of Y by $Y=\bigcup_{\lambda\in\Lambda}Y_\lambda$. We set

$$Y_{\underline{\lambda}}=\bigcap_{\lambda\in\underline{\lambda}}Y_\lambda$$

for $\underline{\lambda}\in S^+(\Lambda)$. We set $Y_\emptyset=Y$. For $\underline{\lambda},\underline{\mu}\in S(\Lambda)$ with $\underline{\lambda}\subset\underline{\mu}$, we have the canonical closed immersion

$$a_{\underline{\lambda},\underline{\mu}}:Y_{\underline{\mu}}\longrightarrow Y_{\underline{\lambda}} \tag{3.2.1}$$

which satisfies the natural functorial property with respect to $\underline{\lambda}$ and $\underline{\mu}$ trivially. The morphism $a_{\emptyset,\underline{\lambda}}:Y_{\underline{\lambda}}\longrightarrow Y$ is denoted by $a_{\underline{\lambda}}$ for short. We omit the symbol $(a_{\underline{\lambda}})_*$ and $(a_{\underline{\lambda},\underline{\mu}})_*$ for complexes of sheaves on Y and $Y_{\underline{\lambda}}$ as usual. Then we have

$$M_Y/\mathcal{O}_Y^*=\bigoplus_{\lambda\in\Lambda}\mathbb{N}_{Y_\lambda} \tag{3.2.2}$$

by definition.

For $\underline{\lambda}\in S^+(\Lambda)$, the induced log structure $a_{\underline{\lambda}}^*M_Y$ is simply denoted by $M_{Y_{\underline{\lambda}}}$. Unless otherwise mentioned, $Y_{\underline{\lambda}}$ is considered as a log complex manifold with the log structure $M_{Y_{\underline{\lambda}}}$. The log de Rham complex of $Y_{\underline{\lambda}}$ is denoted by $\omega_{Y_{\underline{\lambda}}}$. The closed immersion $a_{\underline{\lambda},\underline{\mu}}$ in (3.2.1) is a morphism of log complex analytic spaces for $\underline{\lambda},\underline{\mu}\in S(\Lambda)$ with $\underline{\lambda}\subset\underline{\mu}$. Thus the data $\{Y_{\underline{\lambda}}\}_{\underline{\lambda}\in S^+(\Lambda)}$ form a cubical log complex manifold indexed by the category $\mathcal{S}^+(\Lambda)$, denoted by $Y_\bullet$.

We have the morphism $a_{\underline{\lambda}}^{-1}M_Y \longrightarrow M_{Y_{\underline{\lambda}}}$ of monoid sheaves. The image of $t \in \Gamma(Y, M_Y)$ by the morphism above is denoted by the same letter t in $\Gamma(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})$. We have

$$\begin{aligned} M_{Y_{\underline{\lambda}}}/\mathcal{O}^*_{Y_{\underline{\lambda}}} &= a_{\underline{\lambda}}^{-1}(M_Y/\mathcal{O}^*_Y) \\ &= a_{\underline{\lambda}}^{-1}(\bigoplus_{\mu\in\Lambda}\mathbb{N}_{Y_\mu}) = \bigoplus_{\mu\in\Lambda}\mathbb{N}_{Y_{\underline{\lambda}}\cap Y_\mu} = \bigoplus_{\mu\in\Lambda}\mathbb{N}_{Y_{\underline{\lambda}\cup\{\mu\}}}, \end{aligned}$$

by (3.2.2). We have the canonical projection

$$\mathrm{red}_{Y_{\underline{\lambda}}} : M_{Y_{\underline{\lambda}}} \longrightarrow M_{Y_{\underline{\lambda}}}/\mathcal{O}^*_{Y_{\underline{\lambda}}} = \bigoplus_{\mu\in\Lambda}\mathbb{N}_{Y_{\underline{\lambda}\cup\{\mu\}}}$$

for $\underline{\lambda} \in S(\Lambda)$.

For $\underline{\sigma} \in S(\Lambda)$, the monoid subsheaf

$$M^{\underline{\sigma}}_{Y_{\underline{\lambda}}} = \mathrm{red}^{-1}_{Y_{\underline{\lambda}}}(\bigoplus_{\mu\in\underline{\sigma}}\mathbb{N}_{Y_{\underline{\lambda}\cup\{\mu\}}})$$

of $M_{Y_{\underline{\lambda}}}$ equipped with the restriction of the structure morphism $M_{Y_{\underline{\lambda}}} \longrightarrow \mathcal{O}_{Y_{\underline{\lambda}}}$ defines a log structure on $Y_{\underline{\lambda}}$. The complex analytic space $Y_{\underline{\lambda}}$ equipped with the log structure $M^{\underline{\sigma}}_{Y_{\underline{\lambda}}}$ is denoted by $Y^{\underline{\sigma}}_{\underline{\lambda}}$. According to this definition, $M_{Y_{\underline{\lambda}}}$ coincides with $M^{\Lambda}_{Y_{\underline{\lambda}}}$, and $Y_{\underline{\lambda}}$ with $Y^{\Lambda}_{\underline{\lambda}}$. For an element $\lambda \in \Lambda$, we use the notation $M^{\lambda}_{Y_{\underline{\lambda}}}, Y^{\lambda}_{\underline{\lambda}}$ instead of $M^{\{\lambda\}}_{Y_{\underline{\lambda}}}, Y^{\{\lambda\}}_{\underline{\lambda}}$ for simplicity. The log de Rham complex of $Y^{\underline{\sigma}}_{\underline{\lambda}}$ is denoted by $\omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}}$, which is a subcomplex of $\omega_{Y_{\underline{\lambda}}}$ in the trivial way. For $\underline{\sigma}, \underline{\tau} \in S(\Lambda)$ with $\underline{\sigma} \subset \underline{\tau}$, we have $M^{\underline{\sigma}}_{Y_{\underline{\lambda}}} \subset M^{\underline{\tau}}_{Y_{\underline{\lambda}}}$, and the inclusion $\omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}} \subset \omega_{Y^{\underline{\tau}}_{\underline{\lambda}}}$ as subcomplexes of $\omega_{Y_{\underline{\lambda}}}$.

3.3. For $\underline{\sigma} \in S(\Lambda)$, we define an increasing filtration $W(\underline{\sigma})$ on $\omega_{Y_{\underline{\lambda}}}$ by

$$W(\underline{\sigma})_m\omega^p_{Y_{\underline{\lambda}}} = \mathrm{Image}(\omega^m_{Y_{\underline{\lambda}}} \otimes_{\mathcal{O}_{Y_{\underline{\lambda}}}} \omega^{p-m}_{Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}}} \longrightarrow \omega^p_{Y_{\underline{\lambda}}})$$

for every non-negative integer m. The filtration $W(\Lambda)$ is denoted by W for short. The morphism

$$\bigwedge^m \mathrm{dlog} : \bigwedge^m M^{\mathrm{gp}}_{Y_{\underline{\lambda}}} \longrightarrow \omega^m_{Y_{\underline{\lambda}}}$$

induces a morphism

$$\bigwedge^m M^{\mathrm{gp}}_{Y_{\underline{\lambda}}} \otimes_{\mathbb{Z}} \omega^{p-m}_{Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}}} \longrightarrow W(\underline{\sigma})_m\omega^p_{Y_{\underline{\lambda}}}$$

for every p. By composing the morphism above and the projection

$$W(\underline{\sigma})_m\omega^p_{Y_{\underline{\lambda}}} \longrightarrow \mathrm{Gr}^{W(\underline{\sigma})}_m \omega^p_{Y_{\underline{\lambda}}},$$

the morphism

$$\bigwedge^m M^{\mathrm{gp}}_{Y_{\underline{\lambda}}} \otimes_{\mathbb{Z}} \omega^{p-m}_{Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}}} \longrightarrow \mathrm{Gr}^{W(\underline{\sigma})}_m \omega^p_{Y_{\underline{\lambda}}}$$

is obtained. We can easily see that the morphism above factors through the surjection

$$\bigwedge^m M^{\mathrm{gp}}_{Y_{\underline{\lambda}}} \otimes_{\mathbb{Z}} \omega^{p-m}_{Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}}} \longrightarrow \bigwedge^m (M^{\mathrm{gp}}_{Y_{\underline{\lambda}}}/(M^{\Lambda\setminus\underline{\sigma}}_{Y_{\underline{\lambda}}})^{\mathrm{gp}}) \otimes_{\mathbb{Z}} \omega^{p-m}_{Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}\cup\underline{\sigma}}}$$

by the definition of $W(\underline{\sigma})$. If $m = |\underline{\sigma}|$, we obtain a morphism

$$\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \omega^{p-m}_{Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}\cup\underline{\sigma}}} \longrightarrow \mathrm{Gr}^{W(\underline{\sigma})}_m \omega^p_{Y_{\underline{\lambda}}} \tag{3.3.1}$$

by using $\bigwedge^m (M^{\mathrm{gp}}_{Y_{\underline{\lambda}}}/(M^{\Lambda\setminus\underline{\sigma}}_{Y_{\underline{\lambda}}})^{\mathrm{gp}}) = \varepsilon(\underline{\sigma})$.

3.4. We first describe the local case for the later use. So we assume the following:

(3.4.1) $Y = \{x_1 x_2 \cdots x_r = 0\}$ in the polydisc Δ^n with coordinate functions $x_1, x_2, \dots, x_n$.

(3.4.2) $\Lambda = \{1, 2, \dots, r\}$ and $Y_\lambda = \{x_\lambda = 0\}$ for $\lambda \in \Lambda$.

(3.4.3) $t = x_1 x_2 \cdots x_r = \prod_{i=1}^r x_i$.

Let $\underline{\sigma} = \{\sigma_1, \sigma_2, \dots, \sigma_m\}$ be an element of $S_m(\Lambda)$. For a local section ω of $\omega^{p-m}_{Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}\cup\underline{\sigma}}}$, the morphism (3.3.1) sends

$$e_{\sigma_1} \wedge e_{\sigma_2} \wedge \cdots \wedge e_{\sigma_m} \otimes \omega$$

to the local section

$$\operatorname{dlog} x_{\sigma_1} \wedge \operatorname{dlog} x_{\sigma_2} \wedge \cdots \wedge \operatorname{dlog} x_{\sigma_m} \wedge \tilde{\omega}$$

where $\tilde{\omega}$ is a local section of $\omega^{p-m}_{Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}}}$ whose restriction to $Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}\cup\underline{\sigma}}$ coincides with ω.

Proposition 3.5. *The morphism* (3.3.1) *induces an isomorphism of complexes*

$$\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \omega_{Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}\cup\underline{\sigma}}}[-m] \longrightarrow \operatorname{Gr}^{W(\underline{\sigma})}_m \omega_{Y_{\underline{\lambda}}} \tag{3.5.1}$$

for every $\underline{\sigma} \in S_m(\Lambda)$.

Proof. Same as [4, Proposition 3.6]. □

Definition 3.6. For the case of $|\underline{\sigma}| = m$, the morphism

$$\operatorname{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}} : \omega_{Y_{\underline{\lambda}}} \longrightarrow \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \omega_{Y_{\underline{\lambda}\cup\underline{\sigma}}}[-m]$$

is defined as the composite of the three morphisms, the projection

$$\omega_{Y_{\underline{\lambda}}} = W(\underline{\sigma})_m \omega_{Y_{\underline{\lambda}}} \longrightarrow \operatorname{Gr}^{W(\underline{\sigma})}_m \omega_{Y_{\underline{\lambda}}},$$

the inverse of the isomorphism (3.5.1), and the morphism

$$\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \omega_{Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}\cup\underline{\sigma}}}[-m] \longrightarrow \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \omega_{Y_{\underline{\lambda}\cup\underline{\sigma}}}[-m]$$

induced from the inclusion $\omega_{Y^{\Lambda\setminus\underline{\sigma}}_{\underline{\lambda}\cup\underline{\sigma}}} \subset \omega_{Y_{\underline{\lambda}\cup\underline{\sigma}}}$. Note that $\operatorname{Res}^{\emptyset}_{Y_{\underline{\lambda}}} = \mathrm{id}$.

Moreover we set

$$\operatorname{Res}^m_{Y_{\underline{\lambda}}} = \sum_{\underline{\sigma}\in S_m(\Lambda)} \operatorname{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}} : \omega_{Y_{\underline{\lambda}}} \longrightarrow \bigoplus_{\underline{\sigma}\in S_m(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \omega_{Y_{\underline{\lambda}\cup\underline{\sigma}}}[-m]$$

for every non-negative integer m. Here we remark that the definition of the residue morphisms above is different from that by Deligne in [5, (3.1.5.2)].

Lemma 3.7. *In the situation above, the morphism* $\operatorname{Res}^m_{Y_{\underline{\lambda}}}$ *induces an isomorphism*

$$\operatorname{Gr}^W_m \omega_{Y_{\underline{\lambda}}} \xrightarrow{\simeq} \bigoplus_{\underline{\sigma}\in S_m(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda}\cup\underline{\sigma}}}[-m] \tag{3.7.1}$$

for every m.

Proof. Same as [4, Proposition 3.6]. □

3.8. A global section $\operatorname{dlog} t$ of $\omega^1_{Y_{\underline{\lambda}}}$ is obtained from $t \in \Gamma(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})$. A morphism of complexes

$$\operatorname{dlog} t \wedge : \omega_{Y_{\underline{\lambda}}} \longrightarrow \omega_{Y_{\underline{\lambda}}}[1] \tag{3.8.1}$$

is defined by sending a local section $\omega \in \omega^p_{Y_{\underline{\lambda}}}$ to $\operatorname{dlog} t \wedge \omega \in \omega^{p+1}_{Y_{\underline{\lambda}}}$. We can easily see the property $(\operatorname{dlog} t \wedge)(W_m \omega_{Y_{\underline{\lambda}}}) \subset W_{m+1} \omega_{Y_{\underline{\lambda}}}[1]$ for every m. Therefore, for all m, the morphism $\operatorname{dlog} t\wedge$ above induces a morphism of complexes

$$\operatorname{dlog} t \wedge : \operatorname{Gr}^W_m \omega_{Y_{\underline{\lambda}}} \longrightarrow \operatorname{Gr}^W_{m+1} \omega_{Y_{\underline{\lambda}}}[1]. \tag{3.8.2}$$

Lemma 3.9. *We have*

$$\begin{aligned}
&\mathrm{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}}[1](\mathrm{dlog}\, t\wedge) \\
&\quad= (-1)^m(\mathrm{id}\otimes(\mathrm{dlog}\, t\wedge)[-m])\,\mathrm{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}} + \sum_{\lambda\in\underline{\sigma}}((e_\lambda\wedge)\otimes a^*_{\underline{\lambda}\cup(\underline{\sigma}\setminus\{\lambda\}),\underline{\lambda}\cup\underline{\sigma}})\,\mathrm{Res}^{\underline{\sigma}\setminus\{\lambda\}}_{Y_{\underline{\lambda}}} \\
&\qquad : \omega_{Y_{\underline{\lambda}}} \longrightarrow \varepsilon(\underline{\sigma})\otimes_{\mathbb{Z}}\omega_{Y_{\underline{\lambda}\cup\underline{\sigma}}}[1-m]
\end{aligned}$$

for $\underline{\sigma}\in S_m(\Lambda)$.

Proof. We may assume the conditions (3.4.1)–(3.4.3) in 3.4. Now we take a subset $\underline{\sigma}=\{\sigma_1,\sigma_2,\dots,\sigma_m\}$ of Λ. For a local section ω of $\omega_{Y_{\underline{\lambda}}}$, we write

$$\begin{aligned}
\omega &= \mathrm{dlog}\, x_{\sigma_1}\wedge \mathrm{dlog}\, x_{\sigma_2}\wedge\cdots\wedge \mathrm{dlog}\, x_{\sigma_m}\wedge\eta \\
&\qquad + \sum_{i=1}^m \mathrm{dlog}\, x_{\sigma_1}\wedge\cdots\wedge \widehat{\mathrm{dlog}\, x_{\sigma_i}}\wedge\cdots\wedge \mathrm{dlog}\, x_{\sigma_m}\wedge\eta_i+\eta'
\end{aligned}$$

where $\widehat{\mathrm{dlog}\, x_{\sigma_i}}$ means to omit $\mathrm{dlog}\, x_{\sigma_i}$ and where $\eta,\eta_i\in W(\underline{\sigma})_0\omega_{Y_{\underline{\lambda}}}, \eta'\in W(\underline{\sigma})_{m-2}\omega_{Y_{\underline{\lambda}}}$. Then we have

$$\begin{aligned}
\mathrm{dlog}\, t\wedge\omega = (-1)^m &\sum_{\lambda\in\Lambda\setminus\underline{\sigma}} \mathrm{dlog}\, x_{\sigma_1}\wedge\cdots\wedge \mathrm{dlog}\, x_{\sigma_m}\wedge \mathrm{dlog}\, x_\lambda\wedge\eta \\
&+\sum_{i=1}^m \mathrm{dlog}\, x_{\sigma_i}\wedge \mathrm{dlog}\, x_{\sigma_1}\wedge\cdots\wedge \widehat{\mathrm{dlog}\, x_{\sigma_i}}\wedge\cdots\wedge \mathrm{dlog}\, x_{\sigma_m}\wedge\eta_i \\
&+\sum_{\lambda\in\Lambda\setminus\underline{\sigma}} \mathrm{dlog}\, x_\lambda\wedge \mathrm{dlog}\, x_{\sigma_0}\wedge\cdots\wedge \widehat{\mathrm{dlog}\, x_{\sigma_i}}\wedge\cdots\wedge \mathrm{dlog}\, x_{\sigma_m}\wedge\eta_i \\
&+\mathrm{dlog}\, t\wedge\eta',
\end{aligned}$$

and then

$$\begin{aligned}
\mathrm{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}}(\mathrm{dlog}\, t\wedge\omega) &= (-1)^m e_{\sigma_1}\wedge\cdots\wedge e_{\sigma_m}\otimes\sum_{\lambda\in\Lambda\setminus\underline{\sigma}}\mathrm{dlog}\, x_\lambda\wedge\eta \\
&\qquad+\sum_{i=1}^m e_{\sigma_i}\wedge e_{\sigma_1}\wedge\cdots\wedge \widehat{e_{\sigma_i}}\wedge\cdots\wedge e_{\sigma_m}\otimes\eta_i, \\
\mathrm{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}}(\omega) &= e_{\sigma_1}\wedge\cdots\wedge e_{\sigma_m}\otimes\eta
\end{aligned}$$

by definition. On the other hand,

$$\mathrm{Res}^{\underline{\sigma}\setminus\{\sigma_i\}}_{Y_{\underline{\lambda}}}(\omega) = e_{\sigma_1}\wedge\cdots\wedge\widehat{e_{\sigma_i}}\wedge\cdots\wedge e_{\sigma_m}\otimes((-1)^{m-i}\,\mathrm{dlog}\, x_{\sigma_i}\wedge\eta+\eta_i)$$

holds for $i=1,2,\dots,m$. Therefore

$$\begin{aligned}
&\mathrm{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}}(\mathrm{dlog}\, t\wedge\omega) \\
&\quad= (-1)^m e_{\sigma_1}\wedge\cdots\wedge e_{\sigma_m}\otimes\sum_{\lambda\in\Lambda\setminus\underline{\sigma}}\mathrm{dlog}\, x_\lambda\wedge\eta \\
&\qquad+\sum_{i=1}^m(e_{\sigma_i}\wedge\otimes\,\mathrm{id})\,\mathrm{Res}^{\underline{\sigma}\setminus\{\sigma_i\}}_{Y_{\underline{\lambda}}}(\omega)+(-1)^m e_{\sigma_1}\wedge\cdots\wedge e_{\sigma_m}\otimes\sum_{i=1}^m\mathrm{dlog}\, x_{\sigma_i}\wedge\eta \\
&\quad= (-1)^m(\mathrm{id}\otimes(\mathrm{dlog}\, t\wedge)[-m])\,\mathrm{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}}(\omega) \\
&\qquad\qquad+\sum_{\nu\in\underline{\sigma}}((e_\nu\wedge)\otimes a^*_{\underline{\lambda}\cup(\underline{\sigma}\setminus\{\nu\}),\underline{\lambda}\cup\underline{\sigma}}\cdot\mathrm{Res}^{\underline{\sigma}\setminus\{\nu\}}_{Y_{\underline{\lambda}}}(\omega)
\end{aligned}$$

is obtained. □

Corollary 3.10. *The morphism* (3.8.2) *is identified with the morphism*

$$\bigoplus_{\underline{\sigma}\in S_m(\Lambda)} \sum_{\mu\in\Lambda\setminus\underline{\sigma}} (e_\mu\wedge)\otimes(a_{\underline{\lambda}\cup\underline{\sigma},\underline{\lambda}\cup\underline{\sigma}\cup\{\mu\}})^*$$

$$: \bigoplus_{\underline{\sigma}\in S_m(\Lambda)} \varepsilon(\underline{\sigma})\otimes_{\mathbb{Z}}\Omega_{Y_{\underline{\lambda}\cup\underline{\sigma}}}[-m] \longrightarrow \bigoplus_{\underline{\tau}\in S_{m+1}(\Lambda)} \varepsilon(\underline{\tau})\otimes_{\mathbb{Z}}\Omega_{Y_{\underline{\lambda}\cup\underline{\tau}}}[-m]$$

under the identification (3.7.1).

3.11. We recall the definition of the Koszul complexes. Moreover, we will define the residue morphism of the Koszul complexes of a log deformation, and prove several results on it. The main reference is [6, Section 1].

For $\underline{\sigma}\in S(\Lambda)$, a $\mathbb{Q}$-sheaf $\mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}};n)^p$ is defined as in [6, (2.3)] by

$$\mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}};n)^p = \Gamma_{n-p}(\mathcal{O}_{Y_{\underline{\lambda}}})\otimes_{\mathbb{Z}} \bigwedge^p (M^{\sigma}_{Y_{\underline{\lambda}}})^{\mathrm{gp}} \tag{3.11.1}$$

for non-negative integers n,p with $p\le n$, where the divided power envelope $\Gamma_{n-p}(\mathcal{O}_{Y_{\underline{\lambda}}})$ is taken over the base field $\mathbb{Q}$. By setting

$$d(f_1^{[n_1]}f_2^{[n_2]}\cdots f_k^{[n_k]}\otimes m) = \sum_{i=1}^k f_1^{[n_1]}\cdots f_i^{[n_i-1]}\cdots f_k^{[n_k]}\otimes \exp\left(2\pi\sqrt{-1}f_i\right)\wedge m$$

for integers $n_1,n_2,\dots,n_k$ with $n_1+n_2+\cdots+n_k=n-p$, the differential

$$d:\mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}};n)^p \longrightarrow \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}};n)^{p+1}$$

is defined. The global section 1 of $\mathcal{O}_{Y_{\underline{\lambda}}}$ gives us the morphism

$$\mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}};n) \longrightarrow \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}};n+1) \tag{3.11.2}$$

by sending $f\otimes m$ to $1^{[1]}f\otimes m$ for $f\in\Gamma_{n-p}(\mathcal{O}_{Y_{\underline{\lambda}}})$ and for $m\in\bigwedge^p(M^{\sigma}_{Y_{\underline{\lambda}}})^{\mathrm{gp}}$. Thus we obtain an inductive system

$$\cdots \longrightarrow \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}};n) \longrightarrow \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}};n+1) \longrightarrow \cdots,$$

and its limit

$$\mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}}) = \varinjlim_n \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}};n)$$

as in [6, Definition 1.8]. By setting

$$\psi_{(Y_{\underline{\lambda}},M_{Y_{\underline{\lambda}}})}(f_1^{[n_1]}\cdots f_k^{[n_k]}\otimes m) = \left(2\pi\sqrt{-1}\right)^{-p}(n_1!\cdots n_k!)^{-1}f_1^{n_1}\cdots f_k^{n_k}\left(\bigwedge^p \mathrm{dlog}\right)(m),$$

a morphism of complexes

$$\psi_{(Y_{\underline{\lambda}},M^{\sigma}_{Y_{\underline{\lambda}}})}:\mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}};n)\longrightarrow \omega_{Y^{\sigma}_{\underline{\lambda}}}$$

is defined for every n. These data $\psi_{(Y_{\underline{\lambda}},M^{\sigma}_{Y_{\underline{\lambda}}})}$ for all n induce a morphism of complexes

$$\psi_{(Y_{\underline{\lambda}},M^{\sigma}_{Y_{\underline{\lambda}}})}:\mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\sigma}_{Y_{\underline{\lambda}}})\longrightarrow \omega_{Y^{\sigma}_{\underline{\lambda}}} \tag{3.11.3}$$

as in [6, (2.4)].

For the case of $\underline{\sigma}=\emptyset$, the log structure $M^{\emptyset}_{Y_{\underline{\lambda}}}$ is nothing but the trivial log structure $\mathcal{O}^*_{Y_{\underline{\lambda}}}$. Then the morphism $\mathbb{Q}_{Y_{\underline{\lambda}}}\longrightarrow\Gamma_n(\mathcal{O}_{Y_{\underline{\lambda}}})=\mathrm{Kos}_{Y_{\underline{\lambda}}}(\mathcal{O}^*_{Y_{\underline{\lambda}}};n)^0$ which sends $f\in\mathbb{Q}$ to $(n!f)1^{[n]}\in\Gamma_n(\mathcal{O}_{Y_{\underline{\lambda}}})$

induces a morphism of complexes $\mathbb{Q}_{Y_{\underline{\lambda}}} \longrightarrow \mathrm{Kos}_{Y_{\underline{\lambda}}}(\mathcal{O}^*_{Y_{\underline{\lambda}}})$. Moreover these morphisms for all n are compatible with the morphisms (3.11.2). Therefore a morphism of complexes

$$\mathbb{Q}_{Y_{\underline{\lambda}}} \longrightarrow \mathrm{Kos}_{Y_{\underline{\lambda}}}(\mathcal{O}^*_{Y_{\underline{\lambda}}}) \tag{3.11.4}$$

is obtained.

Lemma 3.12. *The morphism* (3.11.4) *is a quasi-isomorphism, which fits in the commutative diagram*

$$\begin{array}{ccc} \mathbb{Q}_{Y_{\underline{\lambda}}} & \longrightarrow & \mathrm{Kos}_{Y_{\underline{\lambda}}}(\mathcal{O}^*_{Y_{\underline{\lambda}}}) \\ \big\downarrow & & \big\downarrow \psi_{(Y_{\underline{\lambda}}, \mathcal{O}^*_{Y_{\underline{\lambda}}})} \\ \mathbb{C}_{Y_{\underline{\lambda}}} & \longrightarrow & \Omega_{Y_{\underline{\lambda}}}, \end{array}$$

where the left vertical arrow is the canonical inclusion $\mathbb{Q}_{Y_{\underline{\lambda}}} \longrightarrow \mathbb{C}_{Y_{\underline{\lambda}}}$ and the bottom horizontal arrow is the usual morphism induced by the canonical inclusion $\mathbb{C}_{Y_{\underline{\lambda}}} \longrightarrow \mathcal{O}_{Y_{\underline{\lambda}}}$.

Proof. Easy by definition. □

3.13. For the case of $\underline{\sigma} = \Lambda$, the global section t of $M_{Y_{\underline{\lambda}}}$ defines a morphism of complexes

$$\mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}; n) \longrightarrow \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}; n+1)[1]$$

by sending $f \otimes m$ to $f \otimes t \wedge m$. This induces a morphism of complexes

$$t\wedge : \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \longrightarrow \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})[1] \tag{3.13.1}$$

as in [6, (1.11)]. Direct computation shows that the diagram

$$\begin{array}{ccc} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) & \xrightarrow{t\wedge} & \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})[1] \\ \psi_{(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})} \big\downarrow & & \big\downarrow \left(2\pi\sqrt{-1}\right)\psi_{(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})} \\ \omega_{Y_{\underline{\lambda}}} & \xrightarrow[\mathrm{dlog}\, t\wedge]{} & \omega_{Y_{\underline{\lambda}}}[1] \end{array} \tag{3.13.2}$$

is commutative.

3.14. For $\underline{\lambda}, \underline{\mu} \in S(\Lambda)$ with $\underline{\lambda} \subset \underline{\mu}$, the inclusion $a_{\underline{\lambda},\underline{\mu}} : Y^{\underline{\sigma}}_{\underline{\mu}} \longrightarrow Y^{\underline{\sigma}}_{\underline{\lambda}}$ induces morphisms of complexes

$$a^*_{\underline{\lambda},\underline{\mu}} : a^{-1}_{\underline{\lambda},\underline{\mu}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}}) \longrightarrow \mathrm{Kos}_{Y_{\underline{\mu}}}(M^{\underline{\sigma}}_{Y_{\underline{\mu}}}) \tag{3.14.1}$$

and

$$\mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}}) \longrightarrow \mathrm{Kos}_{Y_{\underline{\mu}}}(M^{\underline{\sigma}}_{Y_{\underline{\mu}}}) = (a_{\underline{\lambda},\underline{\mu}})_* \mathrm{Kos}_{Y_{\underline{\mu}}}(M^{\underline{\sigma}}_{Y_{\underline{\mu}}})$$

in the trivial way. These two morphisms are denoted by the same letter $a^*_{\underline{\lambda},\underline{\mu}}$, by abuse of the notation.

3.15. For $\underline{\sigma} \in S(\Lambda)$, the subsheaf $(M^{\Lambda\setminus\underline{\sigma}}_{Y_{\underline{\lambda}}})^{\mathrm{gp}}$ of $M^{\mathrm{gp}}_{Y_{\underline{\lambda}}}$ yields the filtration $W((M^{\Lambda\setminus\underline{\sigma}}_{Y_{\underline{\lambda}}})^{\mathrm{gp}})$ on $\mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})$ as defined in [6, Definition 1.8]. This filtration on $\mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})$ is denoted by $W(\underline{\sigma})$ in this article. The filtration $W(\Lambda)$ is denoted by W. The morphism $\psi_{(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})}$ above preserves the filtration $W(\underline{\sigma})$ for any subset $\underline{\sigma}$ of Λ. As proved in [6, Proposition 1.10], we have an isomorphism of complexes

$$\mathrm{Gr}^{W(\underline{\sigma})}_m \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \simeq \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} a^{-1}_{\underline{\lambda},\underline{\lambda}\cup\underline{\sigma}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\Lambda\setminus\underline{\sigma}}_{Y_{\underline{\lambda}}})[-m] \tag{3.15.1}$$

for every integer m.

We have the inclusion

$$a^{-1}_{\underline{\lambda},\underline{\lambda}\cup\underline{\sigma}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\Lambda\setminus\underline{\sigma}}_{Y_{\underline{\lambda}}}) \longrightarrow a^{-1}_{\underline{\lambda},\underline{\lambda}\cup\underline{\sigma}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})$$

induced by the inclusion $M^{\Lambda\setminus\underline{\sigma}}_{Y_{\underline{\lambda}}} \subset M_{Y_{\underline{\lambda}}}$. Therefore we obtain the morphism

$$\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} a^{-1}_{\underline{\lambda},\underline{\lambda}\cup\underline{\sigma}} \operatorname{Kos}_{Y_{\underline{\lambda}}}(M^{\Lambda\setminus\underline{\sigma}}_{Y_{\underline{\lambda}}})[-m] \longrightarrow \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} a^{-1}_{\underline{\lambda},\underline{\lambda}\cup\underline{\sigma}} \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})[-m] \tag{3.15.2}$$

by tensoring the identity and by shifting.

On the other hand, we have the canonical morphism (3.14.1)

$$a^*_{\underline{\lambda},\underline{\lambda}\cup\underline{\sigma}} : a^{-1}_{\underline{\lambda},\underline{\lambda}\cup\underline{\sigma}} \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \longrightarrow \operatorname{Kos}_{Y_{\underline{\lambda}\cup\underline{\sigma}}}(M_{Y_{\underline{\lambda}\cup\underline{\sigma}}})$$

for $\underline{\lambda}, \underline{\sigma} \in S(\Lambda)$. Thus the morphism

$$\begin{aligned} \operatorname{id} \otimes a^*_{\underline{\lambda},\underline{\lambda}\cup\underline{\sigma}}[-m] : \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} a^{-1}_{\underline{\lambda},\underline{\lambda}\cup\underline{\sigma}} \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})[-m] \\ \longrightarrow \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \operatorname{Kos}_{Y_{\underline{\lambda}\cup\underline{\sigma}}}(M_{Y_{\underline{\lambda}\cup\underline{\sigma}}})[-m] \end{aligned} \tag{3.15.3}$$

is obtained.

Definition 3.16. For $\underline{\sigma} \in S_m(\Lambda)$, the equality

$$W(\underline{\sigma})_m \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) = \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})$$

can be easily seen. Then the composite of four morphisms, the projection

$$\operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) = W(\underline{\sigma})_m \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \longrightarrow \operatorname{Gr}^{W(\underline{\sigma})}_m \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}),$$

the isomorphism (3.15.1), the morphisms (3.15.2) and (3.15.3), is denoted by

$$\operatorname{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}} : \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \longrightarrow \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \operatorname{Kos}_{Y_{\underline{\lambda}\cup\underline{\sigma}}}(M_{Y_{\underline{\lambda}\cup\underline{\sigma}}})[-m]$$

by abuse of the language. Moreover we set

$$\operatorname{Res}^m_{Y_{\underline{\lambda}}} = \sum_{\underline{\sigma}\in S_m(\Lambda)} \operatorname{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}} : \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \longrightarrow \bigoplus_{\underline{\sigma}\in S_m(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \operatorname{Kos}_{Y_{\underline{\lambda}\cup\underline{\sigma}}}(M_{Y_{\underline{\lambda}\cup\underline{\sigma}}})[-m]$$

as in Definition 3.6 again.

Lemma 3.17. *The morphism* $\operatorname{Res}^m_{Y_{\underline{\lambda}}}$ *induces a quasi-isomorphism*

$$\operatorname{Res}^m_{Y_{\underline{\lambda}}} : \operatorname{Gr}^W_m \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \longrightarrow \bigoplus_{\underline{\sigma}\in S_m(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \operatorname{Kos}_{Y_{\underline{\lambda}\cup\underline{\sigma}}}(\mathcal{O}^*_{Y_{\underline{\lambda}\cup\underline{\sigma}}})[-m] \tag{3.17.1}$$

for every m. *In particular, we have an isomorphism*

$$\operatorname{Gr}^W_m \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \longrightarrow \bigoplus_{\underline{\sigma}\in S_m(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathbb{Q}_{Y_{\underline{\lambda}\cup\underline{\sigma}}}[-m] \tag{3.17.2}$$

in the derived category for every m.

Proof. We have an isomorphism

$$\operatorname{Gr}^W_m \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \longrightarrow \bigoplus_{\underline{\sigma}\in S_m(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} a^{-1}_{\underline{\lambda},\underline{\lambda}\cup\underline{\sigma}} \operatorname{Kos}_{Y_{\underline{\lambda}}}(\mathcal{O}^*_{Y_{\underline{\lambda}}})[-m]$$

as in the case of $\omega_{Y_{\underline{\lambda}}}$. We can check that the canonical morphism

$$a^{-1}_{\underline{\lambda},\underline{\lambda}\cup\underline{\sigma}} \operatorname{Kos}_{Y_{\underline{\lambda}}}(\mathcal{O}^*_{Y_{\underline{\lambda}}}) \longrightarrow \operatorname{Kos}_{Y_{\underline{\lambda}\cup\underline{\sigma}}}(\mathcal{O}^*_{Y_{\underline{\lambda}\cup\underline{\sigma}}})$$

is a quasi-isomorphism. Thus the morphism (3.17.1) is a quasi-isomorphism. Combining with the quasi-isomorphism (3.11.4) for $\underline{\lambda} \cup \underline{\sigma}$, we obtain the isomorphism (3.17.2). See [6, Section 1] for the detail. □

Lemma 3.18. *For a non-negative integer m, the diagram*

$$\begin{array}{ccc} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) & \xrightarrow{\mathrm{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}}} & \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathrm{Kos}_{Y_{\underline{\lambda}\cup\underline{\sigma}}}(M_{Y_{\underline{\lambda}\cup\underline{\sigma}}})[-m] \\ {\scriptstyle \psi_{(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes(2\pi\sqrt{-1})^{-m}\psi_{(Y_{\underline{\lambda}\cup\underline{\sigma}}, M_{Y_{\underline{\lambda}\cup\underline{\sigma}}})[-m]}} \\ \omega_{Y_{\underline{\lambda}}} & \xrightarrow[\mathrm{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}}]{} & \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \omega_{Y_{\underline{\lambda}\cup\underline{\sigma}}}[-m] \end{array}$$

is commutative for every $\underline{\sigma} \in S_m(\Lambda)$.

Proof. Easy by definition. □

Lemma 3.19. *We have*

$$\begin{aligned} &\mathrm{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}}[1](t\wedge) \\ &\quad= (-1)^m(\mathrm{id}\otimes(t\wedge)[-m])\,\mathrm{Res}^{\underline{\sigma}}_{Y_{\underline{\lambda}}} + \sum_{\nu\in\underline{\sigma}}((e_\nu\wedge)\otimes a^*_{\underline{\lambda}\cup(\underline{\sigma}\setminus\{\nu\}),\underline{\lambda}\cup\underline{\sigma}})\,\mathrm{Res}^{\underline{\sigma}\setminus\{\nu\}}_{Y_{\underline{\lambda}}} \\ &\quad: \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \longrightarrow \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathrm{Kos}_{Y_{\underline{\lambda}\cup\underline{\sigma}}}(M_{Y_{\underline{\lambda}\cup\underline{\sigma}}})[1-m] \end{aligned}$$

for $\underline{\sigma} \in S_m(\Lambda)$.

Proof. Similar to the case of the log de Rham complex $\omega_{Y_{\underline{\lambda}}}$ in Lemma 3.9. □

4. Gysin morphisms

In this section, we fix the notation on the so called "Gysin map". Because the signs of objects in the cohomology groups are crucial for our computation, we start with the well-known objects and fix the signs explicitly.

4.1. Let X be a complex analytic space equipped with a log structure M_X. For the log de Rham complex ω_X of a log complex analytic space X, we set an increasing filtration W by

$$W_m\omega^p_X = \mathrm{Image}(\omega^m_X \otimes_{\mathcal{O}_X} \Omega^{p-m}_X \longrightarrow \omega^p_X)$$

as in 3.3. Then we have a morphism

$$\gamma_m(\omega_X, W) : \mathrm{Gr}^W_m \omega_X \longrightarrow \mathrm{Gr}^W_{m-1} \omega_X[1]$$

in the derived category as in 1.5. We use the symbol $\gamma_m(X, M_X)$ instead of $\gamma_m(\omega_X, W)$. We sometimes drop the subscript m, if there is no danger of confusion.

4.2. First, we recall the simplest example. Let X be a complex manifold and D a smooth hypersurface in X. The log structure $M_X(D)$ associated to the divisor D is equipped to X. In this case, the log de Rham complex ω_X is nothing but $\Omega_X(\log D)$, and the increasing filtration W coincides with the usual weight filtration on $\Omega_X(\log D)$ in [4]. Then we have

$$\mathrm{Gr}^W_0 \Omega_X(\log D) = W_0\Omega_X(\log D) = \Omega_X$$

by definition. Moreover we have $W_1\Omega_X(\log D) = \Omega_X(\log D)$ because D is smooth. We have the residue isomorphism of complexes

$$\mathrm{Res}^D_X : \mathrm{Gr}^W_1 \Omega_X(\log D) \xrightarrow{\simeq} \Omega_D[-1],$$

by which we identify $\mathrm{Gr}^W_1 \Omega_X(\log D)$ and $\Omega_D[-1]$. Thus we obtain the morphism

$$\gamma(X, D) = \gamma(X, M_X(D)) : \Omega_D[-1] \longrightarrow \Omega_X[1]$$

in the derived category.

Proposition 4.3. *In addition to the situation above, we assume that X is compact. Then we have the equality*

$$\int_X \mathrm{H}^{p+1}(X, \gamma(X,D))(a) \cup b = -\big(2\pi\sqrt{-1}\big)\int_D a \cup (b|_D)$$

for any $a \in \mathrm{H}^p(D, \Omega_D)$ and $b \in \mathrm{H}^{2\dim X - 2 - p}(X, \Omega_X)$.

Proof. See Griffiths-Schmid [10, §2 (b)]. □

4.4. Let $Y \longrightarrow *$ be a log deformation. As defined in 4.1, we have the morphism

$$\gamma_m(Y_{\underline{\lambda}}, M^{\underline{\sigma}}_{Y_{\underline{\lambda}}}) : \mathrm{Gr}^W_m \omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}} \longrightarrow \mathrm{Gr}^W_{m-1} \omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}}[1]$$

in the derived category for $\underline{\lambda}, \underline{\sigma} \in S(\Lambda)$. In particular, the morphism

$$\gamma(Y_{\underline{\lambda}}, M^{\lambda}_{Y_{\underline{\lambda}}}) : \mathrm{Gr}^W_1 \omega_{Y^{\lambda}_{\underline{\lambda}}} \longrightarrow \mathrm{Gr}^W_0 \omega_{Y^{\lambda}_{\underline{\lambda}}}[1] = \Omega_{Y_{\underline{\lambda}}}[1]$$

is obtained for $\lambda \in \Lambda$. By the identification

$$\Omega_{Y_{\underline{\lambda} \cup \{\lambda\}}}[-1] \simeq \varepsilon(\lambda) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda} \cup \{\lambda\}}}[-1] \simeq \mathrm{Gr}^W_1 \omega_{Y^{\lambda}_{\underline{\lambda}}}$$

we obtain a morphism

$$\gamma_{Y^{\lambda}_{\underline{\lambda}}} : \Omega_{Y_{\underline{\lambda} \cup \{\lambda\}}}[-1] \longrightarrow \Omega_{Y_{\underline{\lambda}}}[1]$$

in the derived category. We have

$$\gamma_{Y^{\lambda}_{\underline{\lambda}}} = \gamma(Y_{\underline{\lambda}}, Y_{\underline{\lambda} \cup \{\lambda\}}) \tag{4.4.1}$$

for $\lambda \notin \underline{\lambda}$.

The following proposition is very similar to [16, Proposition 4.3, Proposition 4.5]. However, we restate it for the completeness, because our definition of the residue isomorphism are different from Nakkajima's. Here we only give a sketch of the proof because it is almost the same as Proposition 4.5 in [16].

Proposition 4.5. *For a positive integer m, the morphism*

$$\gamma_m(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}}) : \mathrm{Gr}^W_m \omega_{Y_{\underline{\lambda}}} \longrightarrow \mathrm{Gr}^W_{m-1} \omega_{Y_{\underline{\lambda}}}[1]$$

is identified with

$$\bigoplus_{\underline{\sigma} \in S_m(\Lambda)} \sum_{\nu \in \underline{\sigma}} (e_\nu \wedge)^{-1} \otimes \gamma_{Y^{\nu}_{\underline{\lambda} \cup (\underline{\sigma} \setminus \{\nu\})}}[1-m]$$

$$: \bigoplus_{\underline{\sigma} \in S_m(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda} \cup \underline{\sigma}}}[-m] \longrightarrow \bigoplus_{\underline{\tau} \in S_{m-1}(\Lambda)} \varepsilon(\underline{\tau}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda} \cup \underline{\tau}}}[2-m]$$

under the isomorphism (3.7.1), where $e_\nu \wedge$ denotes the isomorphism $\varepsilon(\underline{\sigma} \setminus \{\nu\}) \longrightarrow \varepsilon(\underline{\sigma})$ in (2.2.3).

Proof. The canonical inclusion $\omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}} \hookrightarrow \omega_{Y_{\underline{\lambda}}}$ induces the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Gr}^W_{m-1} \omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}} & \longrightarrow & W_m\omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}}/W_{m-2}\omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}} & \longrightarrow & \mathrm{Gr}^W_m \omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathrm{Gr}^W_{m-1} \omega_{Y_{\underline{\lambda}}} & \longrightarrow & W_m\omega_{Y_{\underline{\lambda}}}/W_{m-2}\omega_{Y_{\underline{\lambda}}} & \longrightarrow & \mathrm{Gr}^W_m \omega_{Y_{\underline{\lambda}}} & \longrightarrow & 0
\end{array}$$

with exact rows for $\underline{\sigma} \in S_m(\Lambda)$. Thus the restriction of $\gamma_m(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})$ on the direct summand $\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda} \cup \underline{\sigma}}}[-m]$ under the identification (3.7.1) coincides with $\gamma_m(Y_{\underline{\lambda}}, M^{\underline{\sigma}}_{Y_{\underline{\lambda}}})$ under the identification $\mathrm{Gr}^W_m \omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}} \simeq \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda} \cup \underline{\sigma}}}[-m]$. For $\underline{\tau} \in S_{m-1}(\underline{\sigma})$, $\mathrm{Res}^{\underline{\tau}}_{Y_{\underline{\lambda}}}$ induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Gr}^W_{m-1} \omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}} & \longrightarrow & W_m\omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}}/W_{m-2}\omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}} & \longrightarrow & \mathrm{Gr}^W_m \omega_{Y^{\underline{\sigma}}_{\underline{\lambda}}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varepsilon(\underline{\tau}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda} \cup \underline{\sigma}}}[1-m] & \longrightarrow & \varepsilon(\underline{\tau}) \otimes_{\mathbb{Z}} \omega_{Y^{\nu}_{\underline{\lambda} \cup \underline{\tau}}}[1-m] & \longrightarrow & \varepsilon(\underline{\tau}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda} \cup \underline{\sigma}}}[-m] & \longrightarrow & 0 \end{array}$$

with exact rows, where $\underline{\sigma} \setminus \underline{\tau} = \{\nu\}$. Then we have

$$\begin{aligned} \mathrm{Res}^{\underline{\tau}}_{Y_{\underline{\lambda}}}[1]\gamma_m(Y_{\underline{\lambda}}, M^{\underline{\sigma}}_{Y_{\underline{\lambda}}}) =& (-1)^{m-1}\chi(\underline{\tau}, \{\nu\})^{-1} \otimes \gamma_{Y^{\nu}_{\underline{\lambda} \cup \underline{\tau}}} \\ & : \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda} \cup \underline{\sigma}}}[-m] \longrightarrow \varepsilon(\underline{\tau}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda} \cup \underline{\tau}}}[2-m] \end{aligned}$$

by using (1.5.1). Because of $\chi(\underline{\tau}, \{\nu\}) = (-1)^{m-1} e_\nu \wedge$ under the identification $\varepsilon(\nu) \simeq \mathbb{Z}$, the conclusion is obtained. □

5. Comparison between A and K

In this section, we first recall results in [6] and adjust them to the case of a log deformation. The definition and the notation are slightly changed from that in [6]. In addition to this change, the method in Section 2 is used instead of the simplicial method in [6] because it can work without fixing the total order on the index set. After recalling the results in Steenbrink [22], Fujisawa-Nakayama [8] briefly, we construct a morphism from Steenbrink's cohomological mixed Hodge complex to the complex in [6].

5.1. Let $Y \longrightarrow *$ be a log deformation. We assume that

(5.1.1) Y has finitely many irreducible components

in the remainder of this article. Then the index set Λ of the irreducible components of Y is a finite set. This assumption does not affect to our main results because we are interested only in the case where Y is compact.

5.2. Fix an element $\underline{\lambda} \in S(\Lambda)$. A morphism

$$\nabla : \mathbb{C}[u] \otimes_{\mathbb{C}} \omega^p_{Y_{\underline{\lambda}}} \longrightarrow \mathbb{C}[u] \otimes_{\mathbb{C}} \omega^{p+1}_{Y_{\underline{\lambda}}}$$

is defined by

$$\nabla = \mathrm{id} \otimes d + \left(2\pi\sqrt{-1}\right)^{-1} \frac{d}{du} \otimes \mathrm{dlog}\, t \wedge$$

for a non-negative integer p, where $\mathrm{dlog}\, t\wedge$ is the morphism (3.8.1). We can easily see the equality $\nabla^2 = 0$. Thus we obtain a complex of $\mathbb{C}$-sheaves on $Y_{\underline{\lambda}}$, which is denoted by $(\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}, \nabla)$ or simply by $\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}$. A morphism of complexes

$$\omega_{Y_{\underline{\lambda}}} \longrightarrow \mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}} \tag{5.2.1}$$

is induced by the natural inclusion $\mathbb{C} \longrightarrow \mathbb{C}[u]$. We consider $\omega_{Y_{\underline{\lambda}}}$ as a subcomplex of $\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}$ by the inclusion above.

By using the identity

$$\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}} = \bigoplus_{r \geq 0} \mathbb{C}u^r \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}},$$

the weight filtration W and the Hodge filtration F on $\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}$ are defined by

$$W_m(\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}) = \bigoplus_{r \geq 0} \mathbb{C}u^r \otimes_{\mathbb{C}} W_{m-2r}\omega_{Y_{\underline{\lambda}}}$$
$$F^p(\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}) = \bigoplus_{r \geq 0} \mathbb{C}u^r \otimes_{\mathbb{C}} F^{p-r}\omega_{Y_{\underline{\lambda}}}$$

for every m, p, where F on $\omega_{Y_{\underline{\lambda}}}$ denotes the stupid filtration as in [5, (1.4.6)]. It can be easily seen that the filtrations W and F are preserved by ∇. Thus we obtain a bifiltered complex $(\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}, W, F)$.

By setting

$$\pi_{\mathbb{C},\underline{\lambda},r}(P(u) \otimes \omega) = \frac{d^r P}{du^r}(0) \otimes \omega,$$

a morphism

$$\pi_{\mathbb{C},\underline{\lambda},r} : \mathbb{C}[u] \otimes_{\mathbb{C}} \omega^p_{Y_{\underline{\lambda}}} \longrightarrow \omega^p_{Y_{\underline{\lambda}}}$$

is obtained for every non-negative integer r. Note that $\pi_{\mathbb{C},\underline{\lambda},r}$ does not define a morphism of complexes. We have

$$\pi_{\mathbb{C},\underline{\lambda},r}(W_m(\mathbb{C}[u] \otimes_{\mathbb{C}} \omega^p_{Y_{\underline{\lambda}}})) \subset W_{m-2r}\omega^p_{Y_{\underline{\lambda}}}$$
$$\pi_{\mathbb{C},\underline{\lambda},r}(F^q(\mathbb{C}[u] \otimes_{\mathbb{C}} \omega^p_{Y_{\underline{\lambda}}})) \subset F^{q-r}\omega^p_{Y_{\underline{\lambda}}}$$

for every m, q. It is easy to see that

$$\mathrm{Gr}^W_m \pi_{\mathbb{C},\underline{\lambda},r} : (\mathrm{Gr}^W_m(\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}), F) \longrightarrow (\mathrm{Gr}^W_{m-2r} \omega_{Y_{\underline{\lambda}}}, F[-r])$$

defines a morphism of filtered complexes, although the morphism $\pi_{\mathbb{C},\underline{\lambda},r}$ is not a morphism of complexes. Moreover, the morphism of filtered complexes

$$\pi_{\underline{\lambda}/*} : (\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}, F) \longrightarrow (\omega_{Y_{\underline{\lambda}}/*}, F)$$

is given by composing the morphism $\pi_{\mathbb{C},\underline{\lambda},0}$ and the canonical projection $\omega^p_{Y_{\underline{\lambda}}} \longrightarrow \omega^p_{Y_{\underline{\lambda}}/*}$.

5.3. We have a complex

$$\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})$$

with the differential

$$\nabla = \mathrm{id} \otimes d + \frac{d}{du} \otimes t\wedge,$$

where $t\wedge$ is the morphism defined in (3.13.1). Moreover, an increasing filtration W on $\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})$ is defined by

$$W_m(\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})) = \bigoplus_{r \geq 0} \mathbb{Q}u^r \otimes_{\mathbb{Q}} W_{m-2r} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})$$

for every m.

We define a morphism

$$\pi_{\mathbb{Q},\underline{\lambda}} : \mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})^p \longrightarrow \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})^p$$

by substituting 0 for the variable u as in the case of $\pi_{\mathbb{C},\underline{\lambda},0}$. Note that $\pi_{\mathbb{Q},\underline{\lambda}}$ is not a morphism of complexes. It is clear that $\pi_{\mathbb{Q},\underline{\lambda}}$ preserves the filtration W and induces a morphism of complexes

$$\mathrm{Gr}^W_m \pi_{\mathbb{Q},\underline{\lambda}} : \mathrm{Gr}^W_m(\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})) \longrightarrow \mathrm{Gr}^W_m \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})$$

for every m.

5.4. We have the morphism of complexes

$$\psi_{\underline{\lambda},0} = \psi_{(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})} : \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \longrightarrow \omega_{Y_{\underline{\lambda}}}$$

defined in (3.11.3).

Tensoring with the canonical inclusion $\mathbb{Q}[u] \longrightarrow \mathbb{C}[u]$ with the morphism $\psi_{(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})}$, we obtain a morphism

$$\psi_{\underline{\lambda}} : \mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \longrightarrow \mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}$$

for every $\underline{\lambda}$. The commutative diagram (3.13.2) tells us that the morphism $\psi_{\underline{\lambda}}$ is a morphism of complexes.

5.5. The construction in 5.1–5.4 is functorial with respect to the morphisms induced from the canonical inclusion $Y_{\underline{\mu}} \subset Y_{\underline{\lambda}}$ for $\underline{\lambda} \subset \underline{\mu}$. Thus we obtain the corresponding co-cubical objects over the cubical log complex manifold $Y_\bullet$.

Then we obtain the filtered co-cubical complexes of $\mathbb{Q}$-sheaves

$$(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}), W), \quad (\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}), W),$$

the (bi)filtered co-cubical complexes of $\mathbb{C}$-sheaves

$$(\omega_{Y_\bullet/*}, F), \quad (\omega_{Y_\bullet}, W, F), \quad (\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_\bullet}, W, F)$$

and the morphisms

$$\begin{aligned}
&\psi_{\bullet,0} : (\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}), W) \longrightarrow (\omega_{Y_\bullet}, W) \\
&\psi_\bullet : (\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}), W) \longrightarrow (\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_\bullet}, W) \\
&\pi_{\mathbb{Q},\bullet} : (\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})^p, W) \longrightarrow (\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})^p, W) \\
&\pi_{\mathbb{C},\bullet,r} : (\mathbb{C}[u] \otimes_{\mathbb{C}} \omega^p_{Y_\bullet}, W, F) \longrightarrow (\omega^p_{Y_\bullet}, W[2r], F[-r]) \\
&\pi_{\bullet/*} : (\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_\bullet}, F) \longrightarrow (\omega_{Y_\bullet/*}, F)
\end{aligned}$$

on $Y_\bullet$. So we obtain the filtered complexes of $\mathbb{Q}$-sheaves

$$(\mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})), \delta W), \quad (\mathcal{C}(\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})), \delta W)$$

and the (bi)filtered complexes of $\mathbb{C}$-sheaves

$$(\mathcal{C}(\omega_{Y_\bullet/*}), F), \quad (\mathcal{C}(\omega_{Y_\bullet}), \delta W, F), \quad (\mathcal{C}(\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_\bullet}), \delta W, F),$$

on Y as in 2.5 and 2.6. We set

$$\begin{aligned}
(K_{\mathbb{Q}}, W) &= (\mathcal{C}(\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})), \delta W) \\
(K_{\mathbb{C}}, W, F) &= (\mathcal{C}(\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_\bullet}), \delta W, F)
\end{aligned}$$

for short. Moreover the morphisms of filtered complexes

$$\begin{aligned}
&\psi_0 = \mathcal{C}(\psi_{\bullet,0}) : (\mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})), \delta W) \longrightarrow (\mathcal{C}(\omega_{Y_\bullet}), \delta W) \\
&\psi = \mathcal{C}(\psi_\bullet) : (K_{\mathbb{Q}}, W) \longrightarrow (K_{\mathbb{C}}, W) \\
&\pi_{/*} = \mathcal{C}(\pi_{\bullet/*}) : (K_{\mathbb{C}}, F) \longrightarrow (\mathcal{C}(\omega_{Y_\bullet/*}), F)
\end{aligned}$$

are obtained. Moreover, $\mathcal{C}(\omega_{Y_\bullet})$ is considered as a subcomplex of $K_{\mathbb{C}}$ by the inclusion (5.2.1).

5.6. The morphisms $\pi_{\mathbb{Q},\bullet}$ and $\pi_{\mathbb{C},\bullet,r}$ induce morphisms

$$\begin{aligned}
\pi_{\mathbb{Q}} &= \mathcal{C}(\pi_{\mathbb{Q},\bullet}) : (K^p_{\mathbb{Q}}, W) \longrightarrow (\mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}))^p, \delta W) \\
\pi_{\mathbb{C},r} &= \mathcal{C}(\pi_{\mathbb{C},\bullet,r}) : (K^p_{\mathbb{C}}, W, F) \longrightarrow (\mathcal{C}(\omega_{Y_\bullet})^p, \delta W[2r], F[-r])
\end{aligned}$$

for every p, which induce a morphism of complexes

$$\mathrm{Gr}^W_m \pi_{\mathbb{Q}} : \mathrm{Gr}^W_m K_{\mathbb{Q}} \longrightarrow \mathrm{Gr}^{\delta W}_m \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}))$$

and a morphism of filtered complexes

$$\mathrm{Gr}_m^W \pi_{\mathbb{C},r} : (\mathrm{Gr}_m^W K_{\mathbb{C}}, F) \longrightarrow (\mathrm{Gr}_{m-2r}^{\delta W} \mathcal{C}(\omega_{Y_\bullet}), F[-r])$$

for every m, r. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_m^W K_{\mathbb{Q}} & \xrightarrow{\mathrm{Gr}_m^W \psi} & \mathrm{Gr}_m^W K_{\mathbb{C}} \\ {\scriptstyle \mathrm{Gr}_m^W \pi_{\mathbb{Q},0}} \downarrow & & \downarrow {\scriptstyle \mathrm{Gr}_m^W \pi_{\mathbb{C},0}} \\ \mathrm{Gr}_m^{\delta W} \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) & \xrightarrow[\mathrm{Gr}_m^{\delta W} \psi_0]{} & \mathrm{Gr}_m^{\delta W} \mathcal{C}(\omega_{Y_\bullet}) \end{array} \tag{5.6.1}$$

for every m.

5.7. For an element $\lambda \in \Lambda$, the morphism

$$a_\lambda^* : \omega_Y^p \longrightarrow \omega_{Y_\lambda}^p$$

can be regarded as a morphism $\omega_Y^p \longrightarrow \mathcal{C}(\omega_{Y_\bullet})^{0,p} \subset \mathcal{C}(\omega_{Y_\bullet})^p$ for every p. We set

$$a_0^* = \sum_{\lambda \in \Lambda} a_\lambda^* : \omega_Y^p \longrightarrow \mathcal{C}(\omega_{Y_\bullet})^p,$$

which induces a morphism

$$a_0^* : (\omega_Y, W, F) \longrightarrow (\mathcal{C}(\omega_{Y_\bullet}), \delta W, F)$$

of bifiltered complexes. The composite of a_0^* and the canonical inclusion $\mathcal{C}(\omega_{Y_\bullet}) \longrightarrow K_{\mathbb{C}}$ is denoted by a^*. Thus a morphism of bifiltered complexes

$$a^* : (\omega_Y, W, F) \longrightarrow (K_{\mathbb{C}}, W, F)$$

is obtained. Then the equality $a_0^* = \pi_{\mathbb{C},0} a^*$ holds by definition. A morphism of filtered complexes

$$a_{/*}^* : (\omega_{Y/*}, F) \longrightarrow (\mathcal{C}(\omega_{Y/*}), F) \tag{5.7.1}$$

is defined by the same way.

Morphisms of filtered complexes

$$a_0^* : (\mathrm{Kos}_Y(M_Y), W) \longrightarrow (\mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})), \delta W)$$
$$a^* : (\mathrm{Kos}_Y(M_Y), W) \longrightarrow (K_{\mathbb{Q}}, W)$$

are defined similarly. Then the diagrams

$$\begin{array}{ccc} \mathrm{Kos}_Y(M_Y) & \xrightarrow{\psi_{(Y,M_Y)}} & \omega_Y \\ {\scriptstyle a_0^*} \downarrow & & \downarrow {\scriptstyle a_0^*} \\ \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) & \xrightarrow[\psi_0]{} & \mathcal{C}(\omega_{Y_\bullet}) \end{array} \qquad \begin{array}{ccc} \mathrm{Kos}_Y(M_Y) & \xrightarrow{\psi_{(Y,M_Y)}} & \omega_Y \\ {\scriptstyle a^*} \downarrow & & \downarrow {\scriptstyle a^*} \\ K_{\mathbb{Q}} & \xrightarrow[\psi]{} & K_{\mathbb{C}} \end{array} \tag{5.7.2}$$

are commutative.

Definition 5.8. We set

$$(K, W, F) = ((K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F), \psi)$$

and

$$(\mathrm{H}^q(Y, K), W, F) = ((\mathrm{H}^q(Y, K_{\mathbb{Q}}), W), (\mathrm{H}^q(Y, K_{\mathbb{C}}), W, F), \mathrm{H}^q(Y, \psi)),$$

for an integer q.

Theorem 5.9. *For a log deformation $Y \longrightarrow *$, the morphism $a^*_{/*}$ in (5.7.1) is a filtered quasi-isomorphism with respect to the filtrations F on both sides. Therefore the morphism $a^*_{/*}$ induces an isomorphism*

$$\mathrm{H}^q(Y, a^*_{/*}) : \mathrm{H}^q(Y, \omega_{Y/*}) \longrightarrow \mathrm{H}^q(Y, \mathcal{C}(\omega_{Y_\bullet/*}))$$

for every integer q, under which the filtrations F on both sides coincide.

If we assume the following conditions

(5.9.1) *$Y \longrightarrow *$ is proper, that is, Y is compact,*

(5.9.2) *all the irreducible components Y_λ are Kähler complex manifolds*

in addition, then we have the following:

(5.9.3) *The morphism $\pi_{/*}$ induces an isomorphism $\mathrm{H}^q(Y, K_{\mathbb{C}}) \longrightarrow \mathrm{H}^q(Y, \mathcal{C}(\omega_{Y_\bullet/*}))$ for every integer q, under which the filtrations F on both sides coincide.*

(5.9.4) *The data $(\mathrm{H}^q(Y, K), W[q], F)$ is a mixed Hodge structure for every integer q.*

(5.9.5) *The spectral sequence $E_r^{p,q}(K_{\mathbb{C}}, F)$ degenerates at E_1-terms.*

(5.9.6) *The spectral sequences $E_r^{p,q}(K_{\mathbb{Q}}, W)$ and $E_r^{p,q}(K_{\mathbb{C}}, W)$ degenerate at E_2-terms.*

Proof. We can deduce the conclusion from [6] by fixing a total order on Λ. □

Remark 5.10. Here, we recall several isomorphisms for the later use. We note that we can use $\bigoplus_{\lambda \in \Lambda^{k+1,\circ}}$ instead of $\prod_{\lambda \in \Lambda^{k+1,\circ}}$ by the assumption (5.1.1).

For the complex $\mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}))$ we have the isomorphism in the derived category

$$\begin{aligned}\mathrm{Gr}_m^{\delta W} \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) &= \bigoplus_{\lambda \in \prod^\circ \Lambda} \mathrm{Gr}_{m+d(\lambda)}^W \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})[-d(\lambda)] \\ &\simeq \bigoplus_{\lambda \in \prod^\circ \Lambda} \bigoplus_{\underline{\sigma} \in S_{m+d(\lambda)}(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathbb{Q}_{Y_{\underline{\lambda} \cup \underline{\sigma}}}[-m - 2d(\lambda)]\end{aligned} \tag{5.10.1}$$

by (2.6.1) and by (3.17.2). For $\mathcal{C}(\omega_{Y_\bullet})$, we have the isomorphism of complexes

$$\begin{aligned}\mathrm{Gr}_m^{\delta W} \mathcal{C}(\omega_{Y_\bullet}) &= \bigoplus_{\lambda \in \prod^\circ \Lambda} \mathrm{Gr}_{m+d(\lambda)}^W \omega_{Y_{\underline{\lambda}}}[-d(\lambda)] \\ &\simeq \bigoplus_{\lambda \in \prod^\circ \Lambda} \bigoplus_{\underline{\sigma} \in S_{m+d(\lambda)}(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda} \cup \underline{\sigma}}}[-m - 2d(\lambda)]\end{aligned} \tag{5.10.2}$$

by (2.6.1) and by the residue isomorphism (3.7.1). Under the identifications (5.10.1) and (5.10.2), the morphism $\mathrm{Gr}_m^W \psi_0$ coincides with the morphism induced by the inclusion

$$\left(2\pi\sqrt{-1}\right)^{-m-d(\lambda)} \iota : \mathbb{Q} \longrightarrow \mathbb{C}$$

on the direct summand $\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathbb{Q}_{Y_{\underline{\lambda} \cup \underline{\sigma}}}[-m - 2d(\lambda)]$ by Lemma 3.12 and by Lemma 3.18.

Similarly, we have the isomorphism in the derived category

$$\begin{aligned}\mathrm{Gr}_m^W K_{\mathbb{Q}} &= \bigoplus_{r \geq 0} \bigoplus_{\lambda \in \prod^\circ \Lambda} \mathbb{Q}u^r \otimes_{\mathbb{Q}} \mathrm{Gr}_{m+d(\lambda)-2r}^W \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})[-d(\lambda)] \\ &\simeq \bigoplus_{r \geq 0} \bigoplus_{\lambda \in \prod^\circ \Lambda} \bigoplus_{\underline{\sigma} \in S_{m+d(\lambda)-2r}(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathbb{Q}_{Y_{\underline{\lambda} \cup \underline{\sigma}}}[-m - 2d(\lambda) + 2r]\end{aligned}$$

and the isomorphism of complexes

$$\begin{aligned}\operatorname{Gr}_m^W K_{\mathbb{C}} &= \bigoplus_{r\geq 0}\bigoplus_{\lambda\in\prod^\circ\Lambda} \mathbb{C}u^r \otimes_{\mathbb{C}} \operatorname{Gr}_{m+d(\lambda)-2r}^W \omega_{Y_{\underline{\lambda}}}[-d(\lambda)] \\ &\simeq \bigoplus_{r\geq 0}\bigoplus_{\lambda\in\prod^\circ\Lambda}\bigoplus_{\underline{\sigma}\in S_{m+d(\lambda)-2r}(\Lambda)} \varepsilon(\underline{\sigma})\otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda}\cup\underline{\sigma}}}[-m-2d(\lambda)+2r]\end{aligned}$$

as above. Lemma 3.18 tells us that the morphism $\operatorname{Gr}_m^W \psi : \operatorname{Gr}_m^W K_{\mathbb{Q}} \longrightarrow \operatorname{Gr}_m^W K_{\mathbb{C}}$ is identified with the morphism induced by the inclusion

$$\left(2\pi\sqrt{-1}\right)^{2r-d(\lambda)-m}\iota : \mathbb{Q} \longrightarrow \mathbb{C}$$

on the direct summand $\varepsilon(\underline{\sigma})\otimes_{\mathbb{Z}} \mathbb{Q}_{Y_{\underline{\lambda}\cup\underline{\sigma}}}[-m-2d(\lambda)+2r]$.

5.11. Now we compare the Gysin morphisms of $K_{\mathbb{C}}$ and of $\mathcal{C}(\omega_{Y_\bullet})$ for the later use. For this purpose, we introduce a new complex.

The morphism

$$\operatorname{id}\otimes d : \mathbb{C}[u]\otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}^p \longrightarrow \mathbb{C}[u]\otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}^{p+1}$$

yields a complex $(\mathbb{C}[u]\otimes_{\mathbb{C}}\omega_{Y_{\underline{\lambda}}}, \operatorname{id}\otimes d)$ for every $\underline{\lambda}$. Thus we obtain a co-cubical complex $(\mathbb{C}[u]\otimes_{\mathbb{C}} \omega_{Y_\bullet}, \operatorname{id}\otimes d)$. We set $L=\mathcal{C}(\mathbb{C}[u]\otimes_{\mathbb{C}}\omega_{Y_\bullet}, \operatorname{id}\otimes d)$ for a while. The complex L carries the filtration W and F by the same definition for $K_{\mathbb{C}}$. Then we trivially have $(\operatorname{Gr}_m^W L, F) = (\operatorname{Gr}_m^W K_{\mathbb{C}}, F)$ for every m. Moreover, the morphism $\pi_{\mathbb{C},r}$ defines a morphism of bifiltered complexes

$$\pi_{\mathbb{C},r} : (L, W, F) \longrightarrow (\mathcal{C}(\omega_{Y_\bullet}), \delta W[2r], F[-r])$$

for every $r\geq 0$.

The morphism

$$(-1)^k \frac{d}{du}\otimes \operatorname{dlog} t\wedge : \mathbb{C}[u]\otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}^{p-k} \longrightarrow \mathbb{C}[u]\otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}^{p+1-k}$$

for $\lambda\in\Lambda^{k+1,\circ}$ induces morphisms of bifiltered complexes

$$(L, W, F)\longrightarrow (L[1], W[1], F), \quad (K_{\mathbb{C}}, W, F)\longrightarrow (K_{\mathbb{C}}[1], W[1], F)$$

which are denoted by $\mathcal{C}(d/du\otimes \operatorname{dlog} t\wedge)$. Similarly, morphisms of bifiltered complexes

$$\begin{aligned}\mathcal{C}(\operatorname{id}\otimes \operatorname{dlog} t\wedge) &: (K_{\mathbb{C}}, W, F)\longrightarrow (K_{\mathbb{C}}[1], W[-1], F[1]) \\ \mathcal{C}(\operatorname{dlog} t\wedge) &: (\mathcal{C}(\omega_{Y_\bullet}), \delta W, F) \longrightarrow (\mathcal{C}(\omega_{Y_\bullet})[1], \delta W[-1], F[1])\end{aligned}$$

are induced by the morphisms

$$\begin{aligned}&(-1)^k \operatorname{id}\otimes \operatorname{dlog} t\wedge : \mathbb{C}[u]\otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}^{p-k}\longrightarrow \mathbb{C}[u]\otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}^{p+1-k} \\ &(-1)^k \operatorname{dlog} t\wedge : \omega_{Y_{\underline{\lambda}}}^{p-k}\longrightarrow \omega_{Y_{\underline{\lambda}}}^{p+1-k}\end{aligned}$$

for every $\lambda\in\Lambda^{k+1,\circ}$ and for every p.

We have the equality

$$\pi_{\mathbb{C},r}[1]\,\mathcal{C}(\frac{d}{du}\otimes \operatorname{dlog} t\wedge) = \mathcal{C}(\operatorname{dlog} t\wedge)\pi_{\mathbb{C},r+1} \tag{5.11.1}$$

by definition.

Lemma 5.12. *We have*

$$\gamma_m(K_{\mathbb{C}}, W) = \gamma_m(L, W) + \left(2\pi\sqrt{-1}\right)^{-1}\operatorname{Gr}_m^W \mathcal{C}\left(\frac{d}{du}\otimes \operatorname{dlog} t\wedge\right)$$

for every m.

Proof. By Proposition 1.7. □

Proposition 5.13. *We have*

$$\begin{aligned}&\mathrm{Gr}^W_{m-1}\pi_{\mathbb{C},r}[1]\gamma_m(K_{\mathbb{C}},W)\\&\quad=\gamma_{m-2r}(\mathcal{C}(\omega_{Y_\bullet}),\delta W)\,\mathrm{Gr}^W_m\pi_{\mathbb{C},r}+\left(2\pi\sqrt{-1}\right)^{-1}\mathrm{Gr}^{\delta W}_{m-2r-2}\mathcal{C}(\operatorname{dlog}t\wedge)\,\mathrm{Gr}^W_m\pi_{\mathbb{C},r+1}\end{aligned}$$

for every m, r.

Proof. We obtain

$$\mathrm{Gr}^W_{m-1}\pi_{\mathbb{C},r}[1]\gamma_m(L,W)=\gamma_{m-2r}(\mathcal{C}(\omega_{Y_\bullet}),\delta W)\,\mathrm{Gr}^W_m\pi_{\mathbb{C},r}$$

by the functoriality of the Gysin morphism. Then we obtain the conclusion by (5.11.1) and by the lemma above. □

Definition 5.14. For $\lambda\in\Lambda^{k+1,\circ}$ and for $\underline{\sigma}\in S_{m+k}(\Lambda)$, a morphism

$$\Pi_0(\lambda,\underline{\sigma}):\mathrm{Gr}^{\delta W}_m\mathcal{C}(\omega_{Y_\bullet})\longrightarrow\varepsilon(\underline{\sigma})\otimes_{\mathbb{Z}}\Omega_{Y_{\underline{\lambda}\cup\underline{\sigma}}}[-m-2k]$$

is defined as the projection onto a direct summand under the identification (5.10.2). In particular, we have a morphism

$$\Pi_0(\lambda)=\Pi_0(\lambda,\underline{\lambda}):\mathrm{Gr}^{\delta W}_1\mathcal{C}(\omega_{Y_\bullet})\longrightarrow\varepsilon(\underline{\lambda})\otimes_{\mathbb{Z}}\Omega_{Y_{\underline{\lambda}}}[-1-2k]$$

for $\lambda\in\Lambda^{k+1,\circ}$. We set

$$\Theta_{\mathbb{C},0}(\lambda)=((e_\lambda\wedge)^{-1}\otimes\mathrm{id})\Pi_0(\lambda):\mathrm{Gr}^{\delta W}_1\mathcal{C}(\omega_{Y_\bullet})\longrightarrow\Omega_{Y_{\underline{\lambda}}}[-1-2k]\qquad(5.14.1)$$

for every $\lambda\in\Lambda^{k+1,\circ}$, where $e_\lambda\wedge$ is the isomorphism (2.2.3). Moreover a morphism

$$\Theta_{\mathbb{C}}(\lambda):\mathrm{Gr}^W_1K_{\mathbb{C}}\longrightarrow\Omega_{Y_{\underline{\lambda}}}[-1-2k]$$

is defined by $\Theta_{\mathbb{C}}(\lambda)=\Theta_{\mathbb{C},0}(\lambda)\,\mathrm{Gr}^W_1\pi_{\mathbb{C},0}$, for $\lambda\in\Lambda^{k+1,\circ}$.

Lemma 5.15. *In the situation above, we have*

$$\begin{aligned}&\Theta_{\mathbb{C}}(\lambda)[2]\,\mathrm{Gr}^W_0\mathcal{C}(\mathrm{id}\otimes\operatorname{dlog}t\wedge)[1]\gamma_1(K_{\mathbb{C}},W)\\&\quad=\Theta_{\mathbb{C},0}(\lambda)[2]\,\mathrm{Gr}^{\delta W}_0\mathcal{C}(\operatorname{dlog}t\wedge)[1]\gamma_1(\mathcal{C}(\omega_{Y_\bullet}),\delta W)\,\mathrm{Gr}^W_1\pi_{\mathbb{C},0}\end{aligned}$$

for $\lambda\in\Lambda^{k+1,\circ}$.

Proof. Easy by Proposition 5.13. □

5.16. The morphism

$$\frac{d}{du}\otimes\mathrm{id}:\mathbb{C}[u]\otimes_{\mathbb{C}}\omega_{Y_{\underline{\lambda}}}\longrightarrow\mathbb{C}[u]\otimes_{\mathbb{C}}\omega_{Y_{\underline{\lambda}}}$$

induces a morphism of complexes

$$\frac{d}{du}\otimes\mathrm{id}:K_{\mathbb{C}}\longrightarrow K_{\mathbb{C}},$$

which satisfies the conditions

$$\begin{aligned}&(\frac{d}{du}\otimes\mathrm{id})(W_mK_{\mathbb{C}})\subset W_{m-2}K_{\mathbb{C}}\\&(\frac{d}{du}\otimes\mathrm{id})(F^pK_{\mathbb{C}})\subset F^{p-1}K_{\mathbb{C}}\end{aligned}$$

for every m, p. Similarly, the morphism

$$\frac{d}{du}\otimes\mathrm{id}:\mathbb{Q}[u]\otimes_{\mathbb{Q}}\mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})\longrightarrow\mathbb{Q}[u]\otimes_{\mathbb{Q}}\mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})$$

induces a morphism of complexes

$$\frac{d}{du} \otimes \mathrm{id} : K_{\mathbb{Q}} \longrightarrow K_{\mathbb{Q}},$$

which satisfies the condition

$$(\frac{d}{du} \otimes \mathrm{id})(W_m K_{\mathbb{Q}}) \subset W_{m-2} K_{\mathbb{Q}}$$

for every m as above. Trivially, these morphisms are compatible with $\psi : K_{\mathbb{Q}} \longrightarrow K_{\mathbb{C}}$. Thus the morphism $d/du \otimes \mathrm{id}$ induces a morphism

$$\mathrm{H}^q(Y, \frac{d}{du} \otimes \mathrm{id}) : (\mathrm{H}^q(Y, K), W[q], F) \longrightarrow (\mathrm{H}^q(Y, K), W[q+2], F[-1]) \tag{5.16.1}$$

for every q, denoted by N_K for short.

5.17. We recall results in Steenbrink [22] and in Fujisawa-Nakayama [8], which are analogues of results in Steenbrink [21] from the viewpoint of log geometry.

We set

$$A^p_{\mathbb{C}} = \bigoplus_{r \geq 0} \omega^{p+1}_Y / W_r \omega^{p+1}_Y$$

for every p. The morphism $\mathrm{dlog}\, t\wedge$ in (3.8.1) induces a morphism

$$\mathrm{dlog}\, t\wedge : \omega^{p+1}_Y / W_r \omega^{p+1}_Y \longrightarrow \omega^{p+2}_Y / W_{r+1} \omega^{p+2}_Y \subset A^{p+1}_{\mathbb{C}}$$

for every p, r. By setting

$$d = \bigoplus_{r \geq 0} (-d - \mathrm{dlog}\, t\wedge) : A^p_{\mathbb{C}} \longrightarrow A^{p+1}_{\mathbb{C}},$$

we obtain a complex of $\mathbb{C}$-sheaves $A_{\mathbb{C}}$ on Y. The weight filtration W on $A_{\mathbb{C}}$ is given by

$$W_m A^p_{\mathbb{C}} = \bigoplus_{r \geq 0} W_{m+2r+1} \omega^{p+1}_Y / W_r \omega^{p+1}_Y$$

for every m, and the Hodge filtration F by

$$F^n A^p_{\mathbb{C}} = \bigoplus_{0 \leq r \leq p-n} \omega^{p+1}_Y / W_r \omega^{p+1}_Y$$

for every n.

We set

$$A^p_{\mathbb{Q}} = \bigoplus_{r \geq 0} \mathrm{Kos}_Y(M_Y)^{p+1} / W_r \,\mathrm{Kos}_Y(M_Y)^{p+1}$$

for every p. The morphism $t\wedge$ in (3.13.1) induces a morphism

$$t\wedge : \mathrm{Kos}_Y(M_Y)^{p+1} / W_r \,\mathrm{Kos}_Y(M_Y)^{p+1} \longrightarrow \mathrm{Kos}_Y(M_Y)^{p+2} / W_{r+1} \,\mathrm{Kos}_Y(M_Y)^{p+2} \subset A^{p+1}_{\mathbb{Q}}$$

for every p, r. By setting

$$d = \bigoplus_{r \geq 0} (-d - t\wedge) : A^p_{\mathbb{Q}} \longrightarrow A^{p+1}_{\mathbb{Q}},$$

we obtain a complex $A_{\mathbb{Q}}$. The weight filtration W on $A_{\mathbb{Q}}$ is defined by

$$W_m A^p_{\mathbb{Q}} = \bigoplus_{r \geq 0} W_{m+2r+1} \,\mathrm{Kos}_Y(M_Y)^{p+1} / W_r \,\mathrm{Kos}_Y(M_Y)^{p+1}$$

for every m.

We can easily check that the morphism $(2\pi\sqrt{-1})^{r+1}\psi_Y : \mathrm{Kos}_Y(M_Y) \longrightarrow \omega_Y$ induces a morphism of filtered complexes

$$\alpha = \bigoplus_{r \geq 0} (2\pi\sqrt{-1})^{r+1}\psi_Y : (A_{\mathbb{Q}}, W) \longrightarrow (A_{\mathbb{C}}, W)$$

by using the commutative diagram (3.13.2).

Definition 5.18. We set

$$(A, W, F) = ((A_{\mathbb{Q}}, W), (A_{\mathbb{C}}, W, F), \alpha)$$

and

$$(\mathrm{H}^q(Y, A), W, F) = ((\mathrm{H}^q(Y, A_{\mathbb{Q}}), W), (\mathrm{H}^q(Y, A_{\mathbb{C}}), W, F), \mathrm{H}^q(Y, \alpha))$$

for every q.

Remark 5.19. The signs in the definition of the differentials above are different from that in [22], [8]. Moreover, the $\mathbb{Q}$-structure $A_{\mathbb{Q}}$ above is slightly different from the ones in [22], [8]. We can prove that the above $\mathbb{Q}$-structure induces the same $\mathbb{Q}$-structure as the ones in [22], [8] on the cohomology groups. However, we will not give the proof here because we do not need this fact in this article. What we need in this article is Theorem 5.21 below.

5.20. The composite of the morphism

$$\mathrm{dlog}\, t\wedge : \omega_Y^p \longrightarrow \omega_Y^{p+1}$$

and the projection $\omega_Y^{p+1} \longrightarrow \omega_Y^{p+1}/W_0\omega_Y^{p+1}$ can be regarded as a morphism

$$\theta : \omega_Y^p \longrightarrow \omega_Y^{p+1}/W_0\omega_Y^{p+1} \subset A_{\mathbb{C}}^p$$

for every p. It is easy to check that this morphism defines a morphism of complexes, which is denoted by $\theta : \omega_Y \longrightarrow A_{\mathbb{C}}$. The morphism θ factors through the surjection $\omega_Y \longrightarrow \omega_{Y/*}$. Thus a morphism of complexes $\theta_{/*} : \omega_{Y/*} \longrightarrow A_{\mathbb{C}}$ is obtained.

Theorem 5.21. *In the situation above, the morphism $\theta_{/*} : (\omega_{Y/*}, F) \longrightarrow (A_{\mathbb{C}}, F)$ is a filtered quasi-isomorphism. Therefore the morphism*

$$\mathrm{H}^q(Y, \theta_{/*}) : \mathrm{H}^q(Y, \omega_{Y/*}) \longrightarrow \mathrm{H}^q(Y, A_{\mathbb{C}})$$

is an isomorphism for every q, under which the filtrations F on both sides coincide.

If we assume the conditions (5.9.1) and (5.9.2), the data A is a cohomological mixed Hodge complex on Y. In particular, $(\mathrm{H}^q(Y, A), W[q], F)$ is a mixed Hodge structure for every q.

Proof. By Lemmas 3.7, 3.17, and 3.18, the same proof as in [22], [8] can work. □

Remark 5.22. We have an isomorphism in the derived category

$$\begin{aligned} \mathrm{Gr}_m^W A_{\mathbb{Q}} &= \bigoplus_{r \geq \max(0,-m)} \mathrm{Gr}_{m+2r+1}^W \mathrm{Kos}_Y(M_Y)[1] \\ &\simeq \bigoplus_{r \geq \max(0,-m)} \bigoplus_{\underline{\sigma} \in S_{m+2r+1}(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathbb{Q}_{Y_{\underline{\sigma}}}[-m-2r] \end{aligned} \tag{5.22.1}$$

and the isomorphism of complexes

$$\begin{aligned} \mathrm{Gr}_m^W A_{\mathbb{C}} &= \bigoplus_{r \geq \max(0,-m)} \mathrm{Gr}_{m+2r+1}^W \omega_Y[1] \\ &\simeq \bigoplus_{r \geq \max(0,-m)} \bigoplus_{\underline{\sigma} \in S_{m+2r+1}(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\sigma}}}[-m-2r] \end{aligned} \tag{5.22.2}$$

as in Remark 5.10. Under these identification, the morphism

$$\mathrm{Gr}_m^W \alpha : \mathrm{Gr}_m^W A_{\mathbb{Q}} \longrightarrow \mathrm{Gr}_m^W A_{\mathbb{C}}$$

is identified with the morphism whose restriction on $\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathbb{Q}_{Y_{\underline{\sigma}}}[-m-2r]$ coincides with the morphism $\left(2\pi\sqrt{-1}\right)^{-m-r}\iota : \mathbb{Q} \longrightarrow \mathbb{C}$.

5.23. As in Steenbrink [22, (5.6)] a morphism of bifiltered complexes

$$\nu_{\mathbb{C}} : (A_{\mathbb{C}}, W, F) \longrightarrow (A_{\mathbb{C}}, W[2], F[-1])$$

is induced from the projection $\omega_Y^{p+1}/W_r\omega_Y^{p+1} \longrightarrow \omega_Y^{p+1}/W_{r+1}\omega_Y^{p+1}$ for $r \geq 0$.

Similarly, the projection

$$\mathrm{Kos}_Y(M_Y)^{p+1}/W_r \,\mathrm{Kos}_Y(M_Y)^{p+1} \longrightarrow \mathrm{Kos}_Y(M_Y)^{p+1}/W_{r+1} \,\mathrm{Kos}_Y(M_Y)^{p+1}$$

induces a morphism of filtered complexes $\nu_{\mathbb{Q}} : (A_{\mathbb{Q}}, W) \longrightarrow (A_{\mathbb{Q}}, W[2])$ as above. Since the diagram

$$\begin{array}{ccc} A_{\mathbb{Q}} & \xrightarrow{\nu_{\mathbb{Q}}} & A_{\mathbb{Q}} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ A_{\mathbb{C}} & \xrightarrow[(2\pi\sqrt{-1})\nu_{\mathbb{C}}]{} & A_{\mathbb{C}} \end{array}$$

is commutative, the pair of the morphism

$$\nu = (\nu_{\mathbb{Q}}, \left(2\pi\sqrt{-1}\right)\nu_{\mathbb{C}}) : (A, W, F) \longrightarrow (A, W[2], F[-1])$$

induces a morphism

$$N_A = \mathrm{H}^q(Y, \nu) : (\mathrm{H}^q(Y, A), W[q], F) \longrightarrow (\mathrm{H}^q(Y, A), W[q+2], F[-1])$$

for every q.

Next, we will define morphisms of complexes $A_{\mathbb{Q}} \longrightarrow K_{\mathbb{Q}}$ and $A_{\mathbb{C}} \longrightarrow K_{\mathbb{C}}$ which play an important role in the remainder of this article.

Definition 5.24. We set

$$\mathrm{Res}_Y^{\lambda} = ((e_{\lambda}\wedge)^{-1} \otimes \mathrm{id})\,\mathrm{Res}_Y^{\underline{\lambda}} : \omega_Y \longrightarrow \omega_{Y_{\underline{\lambda}}}[-|\underline{\lambda}|]$$

for an element $\lambda \in \prod^{\circ} \Lambda$.

Because the morphism Res_Y^{λ} sends $W_r\omega_Y$ to zero for $r \leq d(\lambda)$, a morphism

$$u^{[d(\lambda)-r]} \otimes \mathrm{Res}_Y^{\lambda} : \omega_Y^{p+1}/W_r\omega_Y^{p+1} \longrightarrow \mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}^{p-d(\lambda)} \subset K_{\mathbb{C}}^p$$

is induced, where we set $u^{[n]} = u^n/n!$ for a non-negative integer n. Then we set

$$\varphi_{\mathbb{C}} = \bigoplus_{r \geq 0} \sum_{\substack{\lambda \in \prod^{\circ} \Lambda \\ d(\lambda) \geq r}} (-1)^{d(\lambda)} \left(2\pi\sqrt{-1}\right)^{d(\lambda)-r} u^{[d(\lambda)-r]} \otimes \mathrm{Res}_Y^{\lambda} : A_{\mathbb{C}}^p \longrightarrow K_{\mathbb{C}}^p$$

for every p.

Lemma 5.25. *The morphism $\varphi_{\mathbb{C}}$ above defines a morphism of complexes preserving the filtrations W and F.*

Proof. Since it is easy to check that $\varphi_{\mathbb{C}}$ preserves the filtrations W and F, it suffices to prove that $\varphi_{\mathbb{C}}$ is a morphism of complexes.

For any $\lambda \in \prod^{\circ} \Lambda$, the equality

$$\operatorname{Res}_Y^{\lambda} d = (-1)^{d(\lambda)+1} d \operatorname{Res}_Y^{\lambda} \tag{5.25.1}$$

can be easily checked by definition. Moreover, we can easily see the equality

$$\operatorname{Res}_Y^{\lambda} \operatorname{dlog} t\wedge = (-1)^{d(\lambda)+1} (\operatorname{dlog} t\wedge) \operatorname{Res}_Y^{\lambda} + \sum_{i=0}^{d(\lambda)} (-1)^i a^*_{\underline{\lambda_i},\underline{\lambda}} \operatorname{Res}_Y^{\lambda_i} \tag{5.25.2}$$

as morphisms from ω_Y^p to $\omega_{Y_{\underline{\lambda}}}^{p-d(\lambda)}$ by Lemma 3.9.

We write the differentials of the complexes $K_{\mathbb{C}}$ and $A_{\mathbb{C}}$ by d_K and d_A and the projection

$$K^p_{\mathbb{C}} = \bigoplus_{\lambda \in \prod^{\circ} \Lambda} \mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}^{p-d(\lambda)} \longrightarrow \mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}^{p-d(\lambda)}$$

by $\operatorname{pr}_\lambda$ for a while. In order to prove that $\varphi_{\mathbb{C}}$ is a morphism of complexes, it suffices to prove the equality

$$\operatorname{pr}_\lambda d_K \varphi_{\mathbb{C}} = \operatorname{pr}_\lambda \varphi_{\mathbb{C}} d_A : A^p_{\mathbb{C}} \longrightarrow \mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}^{p+1-d(\lambda)} \tag{5.25.3}$$

for any $\lambda \in \prod^{\circ} \Lambda$. By the definition of the differential of the Čech complex in 2.5, we have

$$\begin{aligned}\operatorname{pr}_\lambda d_K = \sum_{i=0}^{d(\lambda)} (-1)^i a^*_{\underline{\lambda_i},\underline{\lambda}} \operatorname{pr}_{\lambda_i} + (-1)^{d(\lambda)} (\operatorname{id} \otimes d) \operatorname{pr}_\lambda& \\ + (-1)^{d(\lambda)} \big(\big(2\pi\sqrt{-1}\big)^{-1} \frac{d}{du} \otimes \operatorname{dlog} t\wedge \big) \operatorname{pr}_\lambda&,\end{aligned}$$

where λ_i is the element $\prod \Lambda$ defined in (2.1.1). On the other hand, we have

$$\begin{aligned}&\operatorname{pr}_\lambda \varphi_{\mathbb{C}}|_{\omega_Y^{p+1}/W_r\omega_Y^{p+1}} \\ &\quad = \begin{cases} (-1)^{d(\lambda)} \big(2\pi\sqrt{-1}\big)^{d(\lambda)-r} u^{[d(\lambda)-r]} \otimes \operatorname{Res}_Y^{\lambda} & d(\lambda) \geq r \\ 0 & d(\lambda) < r \end{cases}\end{aligned}$$

for a non-negative integer r. Take $r \geq 0$ and $\lambda \in \prod^{\circ} \Lambda$ with $d(\lambda) = k$. If $k \geq r+1$, we have

$$\begin{aligned}\operatorname{pr}_\lambda d_K \varphi_{\mathbb{C}}|_{\omega_Y^{p+1}/W_r\omega_Y^{p+1}} = \sum_{i=0}^{k} &(-1)^{k+i+1} \big(2\pi\sqrt{-1}\big)^{k-r-1} u^{[k-r-1]} \otimes (a^*_{\underline{\lambda_i},\underline{\lambda}} \operatorname{Res}_Y^{\lambda_i}) \\ &+ \big(2\pi\sqrt{-1}\big)^{k-r} u^{[k-r]} \otimes (d \operatorname{Res}_Y^{\lambda}) \\ &+ \big(2\pi\sqrt{-1}\big)^{k-r-1} u^{[k-r-1]} \otimes (\operatorname{dlog} t \wedge \operatorname{Res}_Y^{\lambda})\end{aligned}$$

by using $d(\lambda_i) = d(\lambda) - 1 = k - 1$, and

$$\begin{aligned}\operatorname{pr}_\lambda \varphi_{\mathbb{C}} d_A|_{\omega_Y^{p+1}/W_r\omega_Y^{p+1}} &= (-1)^k \big(2\pi\sqrt{-1}\big)^{k-r} u^{[k-r]} \otimes (\operatorname{Res}_Y^{\lambda}(-d)) \\ &\quad + (-1)^k \big(2\pi\sqrt{-1}\big)^{k-r-1} u^{[k-r-1]} \otimes (\operatorname{Res}_Y^{\lambda}(-\operatorname{dlog} t\wedge)) \\ &= (-1)^{k+1} \big(2\pi\sqrt{-1}\big)^{k-r} u^{[k-r]} \otimes (\operatorname{Res}_Y^{\lambda} d) \\ &\quad + (-1)^{k+1} \big(2\pi\sqrt{-1}\big)^{k-r-1} u^{[k-r-1]} \otimes (\operatorname{Res}_Y^{\lambda} \operatorname{dlog} t\wedge)\end{aligned}$$

because $(\operatorname{dlog} t\wedge)(\omega_Y^{p+1}/W_r\omega_Y^{p+1}) \subset \omega_Y^{p+2}/W_{r+1}\omega_Y^{p+2}$ Then we obtain (5.25.3) by (5.25.1) and by (5.25.2). If $k = r$, we have

$$\operatorname{pr}_\lambda d_K \varphi_{\mathbb{C}}|_{\omega_Y^{p+1}/W_r\omega_Y^{p+1}} = u^{[0]} \otimes (d \operatorname{Res}_Y^\lambda)$$

by $d(\lambda_i) = k - 1 < r$ and by $(d/du)u^{[0]} = 0$. On the other hand,

$$\operatorname{pr}_\lambda \varphi_{\mathbb{C}} d_A = (-1)^{k+1} u^{[0]} \otimes (\operatorname{Res}_Y^\lambda d)$$

because $(\operatorname{dlog} t\wedge)(\omega_Y^{p+1}/W_r\omega_Y^{p+1}) \subset \omega_Y^{p+2}/W_{r+1}\omega_Y^{p+2}$ again. Thus we obtain (5.25.3) by (5.25.1). If $k < r$, we have

$$\operatorname{pr}_\lambda d_K \varphi_{\mathbb{C}} = \operatorname{pr}_\lambda \varphi_{\mathbb{C}} d_A = 0$$

by definition. Thus we obtain (5.25.3) for any $\lambda \in \prod^\circ \Lambda$. □

Definition 5.26. We set

$$\operatorname{Res}_Y^\lambda = ((e_\lambda\wedge)^{-1} \otimes \operatorname{id}) \operatorname{Res}_Y^{\underline{\lambda}} : \operatorname{Kos}_Y(M_Y) \longrightarrow \operatorname{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}})[-|\underline{\lambda}|]$$

for an element $\lambda \in \prod^\circ \Lambda$, and

$$\varphi_{\mathbb{Q}} = \bigoplus_{r \geq 0} \sum_{\substack{\lambda \in \prod^\circ \Lambda \\ d(\lambda) \geq r}} (-1)^{d(\lambda)} u^{[d(\lambda)-r]} \otimes \operatorname{Res}_Y^\lambda : A_{\mathbb{Q}}^p \longrightarrow K_{\mathbb{Q}}^p$$

for every p. By Lemma 3.19, $\varphi_{\mathbb{Q}}$ is a morphism of complexes as in the case of $\varphi_{\mathbb{C}}$, which also preserves the filtration W.

Lemma 5.27. *The diagram*

$$\begin{array}{ccc} A_{\mathbb{Q}} & \xrightarrow{\varphi_{\mathbb{Q}}} & K_{\mathbb{Q}} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\psi} \\ A_{\mathbb{C}} & \xrightarrow[\varphi_{\mathbb{C}}]{} & K_{\mathbb{C}} \end{array}$$

is commutative.

Proof. Lemma 3.18 implies the conclusion. □

Definition 5.28. The pair $(\varphi_{\mathbb{Q}}, \varphi_{\mathbb{C}})$ is abbreviated as

$$\varphi : A \longrightarrow K$$

and the pair $(\mathrm{H}^q(Y, \varphi_{\mathbb{Q}}), \mathrm{H}^q(Y, \varphi_{\mathbb{C}}))$ as

$$\mathrm{H}^q(Y, \varphi) : \mathrm{H}^q(Y, A) \longrightarrow \mathrm{H}^q(Y, K)$$

for simplicity.

Theorem 5.29. *If a log deformation $Y \longrightarrow *$ satisfies the conditions (5.9.1) and (5.9.2), then*

$$\mathrm{H}^q(Y, \varphi) : \mathrm{H}^q(Y, A) \longrightarrow \mathrm{H}^q(Y, K)$$

is an isomorphism of mixed Hodge structures for every q.

Proof. It is clear that $\mathrm{H}^q(Y,\varphi)$ is a morphism of mixed Hodge structures. Therefore it is sufficient to prove that the morphism $\mathrm{H}^q(Y,\varphi_{\mathbb{C}}) : \mathrm{H}^q(Y,A_{\mathbb{C}}) \longrightarrow \mathrm{H}^q(Y,K_{\mathbb{C}})$ is an isomorphism. Lemma 3.9 implies that the diagram

$$\begin{array}{ccc} \omega_{Y/*} & \xrightarrow{a^*_{/*}} & \mathcal{C}(\omega_{Y_\bullet/*}) \\ {\scriptstyle \theta_{/*}}\big\downarrow & & \big\uparrow{\scriptstyle \pi_{/*}} \\ A_{\mathbb{C}} & \xrightarrow[\varphi_{\mathbb{C}}]{} & K_{\mathbb{C}} \end{array}$$

is commutative. The morphisms $\mathrm{H}^q(Y,a^*_{/*})$, $\mathrm{H}^q(Y,\pi_{/*})$ are isomorphisms for every q by Theorem 5.9 and the morphism $\mathrm{H}^q(Y,\theta_{/*})$ is an isomorphism for every q by Theorem 5.21. Therefore the morphism $\mathrm{H}^q(Y,\varphi_{\mathbb{C}})$ is an isomorphism for every q. □

Corollary 5.30. *In the situation above, the morphism $\varphi_{\mathbb{C}}$ induces filtered isomorphism*

$$E_2^{p,q}(\varphi_{\mathbb{C}}) : (E_2^{p,q}(A_{\mathbb{C}},W),F) \xrightarrow{\simeq} (E_2^{p,q}(K_{\mathbb{C}},W),F)$$

for every p,q.

Proposition 5.31. *The diagrams*

$$\begin{array}{ccc} A_{\mathbb{C}} & \xrightarrow{\varphi_{\mathbb{C}}} & K_{\mathbb{C}} \\ {\scriptstyle (2\pi\sqrt{-1})\nu_{\mathbb{C}}}\big\downarrow & & \big\downarrow{\scriptstyle \frac{d}{du}\otimes\mathrm{id}} \\ A_{\mathbb{C}} & \xrightarrow[\varphi_{\mathbb{C}}]{} & K_{\mathbb{C}} \end{array} \qquad \begin{array}{ccc} A_{\mathbb{Q}} & \xrightarrow{\varphi_{\mathbb{Q}}} & K_{\mathbb{Q}} \\ {\scriptstyle \nu_{\mathbb{Q}}}\big\downarrow & & \big\downarrow{\scriptstyle \frac{d}{du}\otimes\mathrm{id}} \\ A_{\mathbb{Q}} & \xrightarrow[\varphi_{\mathbb{Q}}]{} & K_{\mathbb{Q}} \end{array}$$

are commutative. Therefore the morphism N_A and N_K are identified under the isomorphism $\mathrm{H}^q(Y,\varphi)$ *in Theorem 5.29.*

Definition 5.32. We set

$$\varphi_{\mathbb{C},0} = \pi_{\mathbb{C},0}\varphi_{\mathbb{C}} : (A^p_{\mathbb{C}},W,F) \longrightarrow (\mathcal{C}(\omega_{Y_\bullet})^p,\delta W,F)$$

for every p. It does not define a morphism of complexes. However it induces a morphism of filtered complexes

$$\mathrm{Gr}^W_m \varphi_{\mathbb{C},0} : (\mathrm{Gr}^W_m A_{\mathbb{C}},F) \longrightarrow (\mathrm{Gr}^{\delta W}_m \mathcal{C}(\omega_{Y_\bullet}),F)$$

for every m. Explicitly, we have

$$\varphi_{\mathbb{C},0} = (-1)^r \sum_{\lambda\in\Lambda^{r+1,\circ}} ((e_\lambda\wedge)^{-1}\otimes\mathrm{id})\,\mathrm{Res}^\lambda_Y : \omega^{p+1}_Y/W_r\omega^{p+1}_Y \longrightarrow \bigoplus_{\lambda\in\prod\Lambda} \omega^{p-d(\lambda)}_{Y_{\underline{\lambda}}} = \mathcal{C}(\omega_{Y_\bullet})^p$$

on the direct summand $\omega^{p+1}_Y/W_r\omega^{p+1}_Y$ for every $r\geq 0$.

6. Product

6.1. Let $Y \longrightarrow *$ be a log deformation satisfying condition (5.1.1). By sending

$$(\mathbb{C}[u]\otimes_{\mathbb{C}}\omega^p_{Y_{\underline{\lambda}}})\otimes_{\mathbb{C}}(\mathbb{C}[u]\otimes_{\mathbb{C}}\omega^q_{Y_{\underline{\lambda}}}) \ni (P(u)\otimes\omega)\otimes(Q(u)\otimes\eta)$$

to

$$P(u)Q(u)\otimes\omega\wedge\eta \in \mathbb{C}[u]\otimes_{\mathbb{C}}\omega^{p+q}_{Y_{\underline{\lambda}}}$$

we obtain a morphism

$$(\mathbb{C}[u]\otimes_{\mathbb{C}}\omega_{Y_{\underline{\lambda}}})\otimes_{\mathbb{C}}(\mathbb{C}[u]\otimes_{\mathbb{C}}\omega_{Y_{\underline{\lambda}}}) \longrightarrow \mathbb{C}[u]\otimes_{\mathbb{C}}\omega_{Y_{\underline{\lambda}}},$$

denoted by the same letter $\wedge$. In fact, it is easy to check that the morphism $\wedge$ is a morphism of complexes. Thus the morphisms of complexes

$$\begin{aligned}
&\wedge : \omega_{Y_{\underline{\lambda}}} \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}} \longrightarrow \omega_{Y_{\underline{\lambda}}} \\
&\wedge : \omega_{Y_{\underline{\lambda}}/*} \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}/*} \longrightarrow \omega_{Y_{\underline{\lambda}}/*} \\
&\wedge : (\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}) \otimes_{\mathbb{C}} (\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}) \longrightarrow \mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}
\end{aligned}$$

are obtained. For these three morphisms, the images of $W_a \otimes W_b$ (resp. $F^a \otimes F^b$) are contained in W_{a+b} (resp. F^{a+b}) by definition. These morphisms are compatible with the morphisms $\pi_{\mathbb{C},\underline{\lambda},0}, \pi_{\underline{\lambda}/*}$ and with the morphism induced from the inclusion $Y_{\underline{\mu}} \subset Y_{\underline{\lambda}}$ for $\underline{\lambda} \subset \underline{\mu}$.

6.2. Now we consider the case of the Koszul complex. We take local sections

$$f_1^{[i_1]} f_2^{[i_2]} \cdots f_k^{[i_k]} \otimes x \in \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}}; n)^p, \quad g_1^{[j_1]} g_2^{[j_2]} \cdots g_l^{[j_l]} \otimes y \in \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}}; m)^q$$

respectively, where

$$f_1, f_2, \ldots, f_k, g_1, g_2, \ldots g_l \in \mathcal{O}_{Y_{\underline{\lambda}}}, \quad x \in \bigwedge^p (M^{\underline{\sigma}}_{Y_{\underline{\lambda}}})^{\mathrm{gp}}, \quad y \in \bigwedge^q (M^{\underline{\sigma}}_{Y_{\underline{\lambda}}})^{\mathrm{gp}}$$

and $i_1, i_2, \ldots i_k, j_1, j_2, \ldots j_l$ are positive integers with the conditions $i_1 + i_2 + \cdots + i_k = n - p$ and $j_1 + j_2 + \cdots + j_l = m - q$. Then

$$f_1^{[i_1]} f_2^{[i_2]} \cdots f_k^{[i_k]} g_1^{[j_1]} g_2^{[j_2]} \cdots g_l^{[j_l]} \otimes x \wedge y$$

is a local section of

$$\Gamma_{n+m-p-q}(\mathcal{O}_{Y_{\underline{\lambda}}}) \otimes \bigwedge^{p+q} (M^{\underline{\sigma}}_{Y_{\underline{\lambda}}})^{\mathrm{gp}} = \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}}; n+m)^{p+q}$$

by the definition (3.11.1). We can check that this correspondence induces a morphism of complexes

$$\wedge : \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}}) \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}}) \longrightarrow \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}}), \tag{6.2.1}$$

which sends $W_a \otimes W_b$ to W_{a+b}. Similarly to the case of $\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}}$, we define a morphism

$$(\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}})) \otimes_{\mathbb{Q}} (\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}})) \longrightarrow \mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}})$$

by using the product of $\mathbb{Q}[u]$ and the product $\wedge$ of $\mathrm{Kos}_{Y_{\underline{\lambda}}}(M^{\underline{\sigma}}_{Y_{\underline{\lambda}}})$. This morphism is also denoted by $\wedge$ by abuse of the language.

For the case of $\underline{\sigma} = \emptyset$, we have the commutative diagram

$$\begin{array}{ccc}
\mathbb{Q}_{Y_{\underline{\lambda}}} \otimes_{\mathbb{Q}} \mathbb{Q}_{Y_{\underline{\lambda}}} & \longrightarrow & \mathrm{Kos}_{Y_{\underline{\lambda}}}(\mathcal{O}^*_{Y_{\underline{\lambda}}}) \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(\mathcal{O}^*_{Y_{\underline{\lambda}}}) \\
\downarrow & & \downarrow \wedge \\
\mathbb{Q}_{Y_{\underline{\lambda}}} & \longrightarrow & \mathrm{Kos}_{Y_{\underline{\lambda}}}(\mathcal{O}^*_{Y_{\underline{\lambda}}})
\end{array}$$

for every $\underline{\lambda} \in S(\Lambda)$, where the top horizontal arrow is the tensor product of the morphism (3.11.4), the bottom horizontal arrow is the morphism (3.11.4) itself and the left vertical arrow is the canonical morphism which sends $a \otimes b \in \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ to $ab \in \mathbb{Q}$. Because of this compatibility, the canonical morphism on the left is denoted by

$$\wedge : \mathbb{Q}_{Y_{\underline{\lambda}}} \otimes_{\mathbb{Q}} \mathbb{Q}_{Y_{\underline{\lambda}}} \longrightarrow \mathbb{Q}_{Y_{\underline{\lambda}}} \tag{6.2.2}$$

in the remainder of this article.

For every $\underline{\lambda}$, we can check the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) & \xrightarrow{\wedge} & \mathrm{Kos}_{Y_{\underline{\lambda}}}(M_{Y_{\underline{\lambda}}}) \\ {\scriptstyle \psi_{(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})} \otimes \psi_{(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})}} \downarrow & & \downarrow {\scriptstyle \psi_{(Y_{\underline{\lambda}}, M_{Y_{\underline{\lambda}}})}} \\ \omega_{Y_{\underline{\lambda}}} \otimes_{\mathbb{C}} \omega_{Y_{\underline{\lambda}}} & \xrightarrow[\wedge]{} & \omega_{Y_{\underline{\lambda}}} \end{array} \tag{6.2.3}$$

by direct computation.

6.3. The morphisms $\wedge$ in 6.1 and 6.2 define the morphisms

$$\begin{aligned} &\omega_{Y_\bullet} \otimes_{\mathbb{C}} \omega_{Y_\bullet} \longrightarrow \omega_{Y_\bullet} \\ &\omega_{Y_\bullet/*} \otimes_{\mathbb{C}} \omega_{Y_\bullet/*} \longrightarrow \omega_{Y_\bullet/*} \\ &(\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_\bullet}) \otimes_{\mathbb{C}} (\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_\bullet}) \longrightarrow \mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_\bullet} \\ &\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}) \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}) \longrightarrow \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}) \\ &(\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \otimes_{\mathbb{Q}} (\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \longrightarrow \mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}) \end{aligned}$$

of co-cubical complexes over $Y_\bullet$, where the left hand sides denote the co-cubical complexes defined in 2.8. Thus the morphisms of complexes

$$\begin{aligned} &\mathcal{C}(\omega_{Y_\bullet} \otimes \omega_{Y_\bullet}) \longrightarrow \mathcal{C}(\omega_{Y_\bullet}) \\ &\mathcal{C}(\omega_{Y_\bullet/*} \otimes \omega_{Y_\bullet/*}) \longrightarrow \mathcal{C}(\omega_{Y_\bullet/*}) \\ &\mathcal{C}((\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_\bullet}) \otimes_{\mathbb{C}} (\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_\bullet})) \longrightarrow K_{\mathbb{C}} = \mathcal{C}(\mathbb{C}[u] \otimes_{\mathbb{C}} \omega_{Y_\bullet}) \\ &\mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}) \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \longrightarrow \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \\ &\mathcal{C}((\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \otimes_{\mathbb{Q}} (\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}))) \\ &\qquad\qquad\qquad \longrightarrow K_{\mathbb{Q}} = \mathcal{C}(\mathbb{Q}[u] \otimes_{\mathbb{Q}} \mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \end{aligned} \tag{6.3.1}$$

are induced.

Definition 6.4. We define the morphisms of complexes

$$\begin{aligned} &\Phi_{\mathbb{C},0} : \mathcal{C}(\omega_{Y_\bullet}) \otimes_{\mathbb{C}} \mathcal{C}(\omega_{Y_\bullet}) \longrightarrow \mathcal{C}(\omega_{Y_\bullet}) \\ &\Phi_{/*} : \mathcal{C}(\omega_{Y_\bullet/*}) \otimes_{\mathbb{C}} \mathcal{C}(\omega_{Y_\bullet/*}) \longrightarrow \mathcal{C}(\omega_{Y_\bullet/*}) \\ &\Phi_{\mathbb{C}} : K_{\mathbb{C}} \otimes_{\mathbb{C}} K_{\mathbb{C}} \longrightarrow K_{\mathbb{C}} \\ &\Phi_{\mathbb{Q},0} : \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \otimes_{\mathbb{Q}} \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \longrightarrow \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \\ &\Phi_{\mathbb{Q}} : K_{\mathbb{Q}} \otimes_{\mathbb{Q}} K_{\mathbb{Q}} \longrightarrow K_{\mathbb{Q}} \end{aligned}$$

by composing the morphisms in (6.3.1) with the morphisms τ in Definition 2.9.

6.5. We have the commutative diagrams

$$\begin{array}{ccc} \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \otimes_{\mathbb{Q}} \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) & \xrightarrow{\Phi_{\mathbb{Q},0}} & \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \\ {\scriptstyle \psi_0 \otimes \psi_0} \downarrow & & \downarrow {\scriptstyle \psi_0} \\ \mathcal{C}(\omega_{Y_\bullet}) \otimes_{\mathbb{C}} \mathcal{C}(\omega_{Y_\bullet}) & \xrightarrow[\Phi_{\mathbb{C},0}]{} & \mathcal{C}(\omega_{Y_\bullet}) \end{array}$$

and

$$\begin{array}{ccc} K_{\mathbb{Q}} \otimes_{\mathbb{Q}} K_{\mathbb{Q}} & \xrightarrow{\Phi_{\mathbb{Q}}} & K_{\mathbb{Q}} \\ {\scriptstyle \psi \otimes \psi} \downarrow & & \downarrow {\scriptstyle \psi} \\ K_{\mathbb{C}} \otimes_{\mathbb{C}} K_{\mathbb{C}} & \xrightarrow[\Phi_{\mathbb{C}}]{} & K_{\mathbb{C}} \end{array} \tag{6.5.1}$$

from the commutativity of the diagram (6.2.3). Moreover, we also have the commutative diagram

$$\begin{array}{ccc} K_{\mathbb{C}} \otimes_{\mathbb{C}} K_{\mathbb{C}} & \xrightarrow{\Phi_{\mathbb{C}}} & K_{\mathbb{C}} \\ {\scriptstyle \pi_{/*} \otimes \pi_{/*}} \downarrow & & \downarrow {\scriptstyle \pi_{/*}} \\ \mathcal{C}(\omega_{Y_\bullet/*}) \otimes_{\mathbb{C}} \mathcal{C}(\omega_{Y_\bullet/*}) & \xrightarrow[\Phi_{/*}]{} & \mathcal{C}(\omega_{Y_\bullet/*}) \end{array}$$

from the compatibility of $\wedge$ with the morphism $\pi_{\underline{\lambda}/*}$.

Lemma 6.6. *The diagram*

$$\begin{array}{ccc} \omega_{Y/*} \otimes_{\mathbb{C}} \omega_{Y/*} & \xrightarrow{\wedge} & \omega_{Y/*} \\ {\scriptstyle a^*_{/*} \otimes a^*_{/*}} \downarrow & & \downarrow {\scriptstyle a^*_{/*}} \\ \mathcal{C}(\omega_{Y_\bullet/*}) \otimes_{\mathbb{C}} \mathcal{C}(\omega_{Y_\bullet/*}) & \xrightarrow[\Phi_{/*}]{} & \mathcal{C}(\omega_{Y_\bullet/*}) \end{array}$$

is commutative.

Lemma 6.7. *The equalities*

$$\Phi_{\mathbb{C}}\left(\left(\frac{d}{du} \otimes \mathrm{id}\right) \otimes \mathrm{id} + \mathrm{id} \otimes \left(\frac{d}{du} \otimes \mathrm{id}\right)\right) = \left(\frac{d}{du} \otimes \mathrm{id}\right)\Phi_{\mathbb{C}}$$
$$\Phi_{\mathbb{Q}}\left(\left(\frac{d}{du} \otimes \mathrm{id}\right) \otimes \mathrm{id} + \mathrm{id} \otimes \left(\frac{d}{du} \otimes \mathrm{id}\right)\right) = \left(\frac{d}{du} \otimes \mathrm{id}\right)\Phi_{\mathbb{Q}}$$

hold.

6.8. From Corollary 2.12, we obtain

$$\begin{aligned} &\Phi_{\mathbb{C},0}(\delta W_a\, \mathcal{C}(\omega_{Y_\bullet}) \otimes \delta W_b\, \mathcal{C}(\omega_{Y_\bullet})) \subset \delta W_{a+b}\, \mathcal{C}(\omega_{Y_\bullet}) \\ &\Phi_{\mathbb{C},0}(F^a\, \mathcal{C}(\omega_{Y_\bullet}) \otimes F^b\, \mathcal{C}(\omega_{Y_\bullet})) \subset F^{a+b}\, \mathcal{C}(\omega_{Y_\bullet}) \\ &\Phi_{/*}(F^a\, \mathcal{C}(\omega_{Y_\bullet/*}) \otimes F^b\, \mathcal{C}(\omega_{Y_\bullet/*})) \subset F^{a+b}\, \mathcal{C}(\omega_{Y_\bullet/*}) \\ &\Phi_{\mathbb{C}}(W_a K_{\mathbb{C}} \otimes W_b K_{\mathbb{C}}) \subset W_{a+b} K_{\mathbb{C}} \\ &\Phi_{\mathbb{C}}(F^a K_{\mathbb{C}} \otimes F^b K_{\mathbb{C}}) \subset W_{a+b} K_{\mathbb{C}} \\ &\Phi_{\mathbb{Q},0}(\delta W_a\, \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \otimes \delta W_b\, \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet}))) \subset \delta W_{a+b}\, \mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})) \\ &\Phi_{\mathbb{Q}}(W_a K_{\mathbb{Q}} \otimes W_b K_{\mathbb{Q}}) \subset W_{a+b} K_{\mathbb{Q}} \end{aligned} \tag{6.8.1}$$

for every a, b. Therefore we obtain morphisms

$$\begin{aligned} &\mathrm{Gr}^W_{a,b}\, \Phi_{\mathbb{C}} : \mathrm{Gr}^W_a\, K_{\mathbb{C}} \otimes \mathrm{Gr}^W_b\, K_{\mathbb{C}} \longrightarrow \mathrm{Gr}^W_{a+b}\, K_{\mathbb{C}} \\ &\mathrm{Gr}^{\delta W}_{a,b}\, \Phi_{\mathbb{C},0} : \mathrm{Gr}^{\delta W}_a\, \mathcal{C}(\omega_{Y_\bullet}) \otimes \mathrm{Gr}^{\delta W}_b\, \mathcal{C}(\omega_{Y_\bullet}) \longrightarrow \mathrm{Gr}^{\delta W}_{a+b}\, \mathcal{C}(\omega_{Y_\bullet}) \end{aligned}$$

and so on, for every a, b.

Definition 6.9. We set

$$\Psi_{\mathbb{C}} = \Phi_{\mathbb{C}}(\varphi_{\mathbb{C}} \otimes \varphi_{\mathbb{C}}) : A_{\mathbb{C}} \otimes_{\mathbb{C}} A_{\mathbb{C}} \longrightarrow K_{\mathbb{C}},$$

which is a morphism of complexes. Moreover, we set

$$\Psi_{\mathbb{C},0} = \Phi_{\mathbb{C},0}(\varphi_{\mathbb{C},0} \otimes \varphi_{\mathbb{C},0}) : A^p_{\mathbb{C}} \otimes_{\mathbb{C}} A^q_{\mathbb{C}} \longrightarrow \mathcal{C}(\omega_{Y_\bullet})^{p+q}$$

for every p, q. Although $\Psi_{\mathbb{C},0}$ dose not define a morphism of complexes, it induces a morphism of complexes

$$\mathrm{Gr}^W_{a,b}\, \Psi_{\mathbb{C},0} : \mathrm{Gr}^W_a\, A_{\mathbb{C}} \otimes_{\mathbb{C}} \mathrm{Gr}^W_b\, A_{\mathbb{C}} \longrightarrow \mathrm{Gr}^{\delta W}_{a+b}\, \mathcal{C}(\omega_{Y_\bullet})$$

for every a, b as before.

6.10. We can easily see the equality

$$\Psi_{\mathbb{C},0} = \pi_{\mathbb{C},0}\Psi_{\mathbb{C}} \tag{6.10.1}$$

by $\pi_{\mathbb{C},0}\Phi_{\mathbb{C}} = \Phi_{\mathbb{C},0}(\pi_{\mathbb{C},0} \otimes \pi_{\mathbb{C},0})$.

Proposition 6.11. *The equality*

$$\Psi_{\mathbb{C}}(\nu_{\mathbb{C}} \otimes \mathrm{id} + \mathrm{id} \otimes \nu_{\mathbb{C}}) = (\frac{d}{du} \otimes \mathrm{id})\Psi_{\mathbb{C}}$$

holds

Proof. Proposition 5.31 and Lemma 6.7 yield the conclusion. □

6.12. For the later use, we describe the morphism

$$\Theta_{\mathbb{C},0}(\lambda)[1] \operatorname{Gr}_0^{\delta W} \mathcal{C}(\operatorname{dlog} t\wedge) \operatorname{Gr}_{m,-m}^W \Psi_{\mathbb{C},0} : \operatorname{Gr}_m^W A_{\mathbb{C}} \otimes_{\mathbb{C}} \operatorname{Gr}_{-m}^W A_{\mathbb{C}} \longrightarrow \Omega_{Y_{\underline{\lambda}}}[-2k] \tag{6.12.1}$$

for $\lambda \in \Lambda^{k+1,\circ}$ and for a non-negative integer m, where $\Theta_{\mathbb{C},0}(\lambda)$ is the morphism (5.14.1).

Lemma 6.13. *Under the identification (5.22.2), the restriction of the morphism (6.12.1) on the direct summand*

$$(\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\sigma}}}[-m-2r]) \otimes_{\mathbb{C}} (\varepsilon(\underline{\tau}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\tau}}}[m-2s])$$

is the zero morphism except for the case of $\underline{\sigma} = \underline{\tau} = \underline{\lambda}, s = r + m$.

For the case of $\underline{\sigma} = \underline{\tau} = \underline{\lambda}, s = r + m$, the restriction of the morphism (6.12.1) on the direct summand

$$(\varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda}}}[-m-2r]) \otimes_{\mathbb{C}} (\varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda}}}[-m-2r])$$

coincides with the composite of the exchange isomorphism

$$\begin{aligned}(\varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda}}}[-m-2r]) \otimes_{\mathbb{C}} (\varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda}}}[-m-2r])& \\ \simeq \varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda}}}[-m-2r] \otimes_{\mathbb{C}} \Omega_{Y_{\underline{\lambda}}}[-m-2r]&\end{aligned}$$

and the morphism

$$\begin{aligned}(-1)^r\vartheta(\underline{\lambda}) \otimes \wedge [-m-2r, -m-2r]& \\ : \varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda}}}[-m-2r] \otimes_{\mathbb{C}} \Omega_{Y_{\underline{\lambda}}}[-m-2r] \longrightarrow \Omega_{Y_{\underline{\lambda}}}[-2m-4r]&\end{aligned}$$

where $\vartheta(\underline{\lambda})$ is the morphism (2.2.1), and where $\wedge[-m-2r, -m-2r]$ denotes the morphism induced from $\wedge$ as in (1.4.2).

Proof. On the direct summand

$$(\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\sigma}}}[-m-2r]) \otimes_{\mathbb{C}} (\varepsilon(\underline{\tau}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\tau}}}[m-2s])$$

of $\operatorname{Gr}_m^W A_{\mathbb{C}} \otimes_{\mathbb{C}} \operatorname{Gr}_{-m}^W A_{\mathbb{C}}$, the morphism $\operatorname{Gr}_m^W \varphi_{\mathbb{C},0} \otimes \operatorname{Gr}_{-m}^W \varphi_{\mathbb{C},0}$ coincides with the morphism

$$(-1)^{r+s} \sum_{\mu,\nu} ((e_\mu\wedge)^{-1} \otimes \mathrm{id}) \otimes ((e_\nu\wedge)^{-1} \otimes \mathrm{id}), \tag{6.13.1}$$

where $\mu \in \Lambda^{r+1,\circ}, \nu \in \Lambda^{s+1,\circ}$ with $\underline{\mu} \subset \underline{\sigma}, \underline{\nu} \subset \underline{\tau}$. Therefore the image of the direct summand

$$(\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\sigma}}}[-m-2r]) \otimes_{\mathbb{C}} (\varepsilon(\underline{\tau}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\tau}}}[m-2s])$$

by the morphism $\operatorname{Gr}_m^W \varphi_{\mathbb{C},0} \otimes \operatorname{Gr}_{-m}^W \varphi_{\mathbb{C},0}$ is contained in

$$\bigoplus_{\mu,\nu} (\varepsilon(\underline{\sigma} \setminus \underline{\mu}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\sigma}}}[-m-2r]) \otimes_{\mathbb{C}} (\varepsilon(\underline{\tau} \setminus \underline{\nu}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\tau}}}[m-2s])$$

for $\mu \in \Lambda^{r+1,\circ}$, $\nu \in \Lambda^{s+1,\circ}$ with $\underline{\mu} \subset \underline{\sigma}, \underline{\nu} \subset \underline{\tau}$.

On the other hand, Corollary 3.10 implies that the morphism

$$\Theta_{\mathbb{C},0}(\lambda)\operatorname{Gr}_0^{\delta W}\mathcal{C}(\operatorname{dlog} t\wedge)\operatorname{Gr}_{m,-m}^{\delta W}\Phi_{\mathbb{C},0} \tag{6.13.2}$$

is equal to zero on the direct summand

$$(\varepsilon(\underline{\sigma}\setminus\underline{\mu})\otimes_{\mathbb{Z}}\Omega_{Y_{\underline{\sigma}}}[-m-2r])\otimes_{\mathbb{C}}(\varepsilon(\underline{\tau}\setminus\underline{\nu})\otimes_{\mathbb{Z}}\Omega_{Y_{\underline{\tau}}}[m-2s])$$

of $\operatorname{Gr}_{m+r}^W\omega_{Y_{\underline{\mu}}}[-r]\otimes_{\mathbb{C}}\operatorname{Gr}_{-m+s}^W\omega_{Y_{\underline{\nu}}}[-s]$, unless the following two conditions are satisfied:

(6.13.3) $\mu=h_r(\lambda)$ and $\nu=t_r(\lambda)$

(6.13.4) $(\underline{\sigma}\setminus\underline{\mu})\cup(\underline{\tau}\setminus\underline{\nu})\subset\underline{\lambda}$.

By condition (6.13.3) we have $k=r+s$. Moreover, condition (6.13.3) implies $\underline{\lambda}=\underline{\mu}\cup\underline{\nu}\subset\underline{\sigma}\cup\underline{\tau}$ because of the conditions $\underline{\mu}\subset\underline{\sigma},\underline{\nu}\subset\underline{\tau}$. By (6.13.3) and (6.13.4) we have $\underline{\sigma}\subset\underline{\lambda},\underline{\tau}\subset\underline{\lambda}$. Therefore $\underline{\lambda}=\underline{\sigma}\cup\underline{\tau}$. Now we have the equalities

$$|\underline{\lambda}|=|\underline{\sigma}|+|\underline{\tau}|-|\underline{\sigma}\cap\underline{\tau}|=2(k+1)-|\underline{\sigma}\cap\underline{\tau}|$$

from the equality $k=r+s$, which imply $|\underline{\sigma}\cap\underline{\tau}|=k+1$. Then $\underline{\lambda}=\underline{\sigma}\cap\underline{\tau}=\underline{\sigma}\cup\underline{\tau}$. Thus we conclude that $\underline{\lambda}=\underline{\sigma}=\underline{\tau}$, $s=r+m$ and $k=m+2r$.

On the direct summand

$$(\varepsilon(\underline{\lambda}\setminus\underline{\mu})\otimes_{\mathbb{Z}}\Omega_{Y_{\underline{\lambda}}}^{p-m-2r})\otimes_{\mathbb{C}}(\varepsilon(\underline{\lambda}\setminus\underline{\nu})\otimes_{\mathbb{Z}}\Omega_{Y_{\underline{\lambda}}}^{q-m-2r}) \tag{6.13.5}$$

of $\operatorname{Gr}_{m+r}^W\omega_{Y_{\underline{\mu}}}^{p-r}\otimes_{\mathbb{C}}\operatorname{Gr}_r^W\omega_{Y_{\underline{\nu}}}^{q-m-r}$ with the conditions $s=r+m$, $k=m+2r$ and (6.13.3), the morphism $\operatorname{Gr}_{m,-m}^{\delta W}\Phi_{\mathbb{C},0}$ coincides with the composite of the exchange isomorphism

$$\begin{aligned}(\varepsilon(\underline{\lambda}\setminus\underline{\mu})\otimes_{\mathbb{Z}}\Omega_{Y_{\underline{\lambda}}}^{p-m-2r})&\otimes_{\mathbb{C}}(\varepsilon(\underline{\lambda}\setminus\underline{\nu})\otimes_{\mathbb{Z}}\Omega_{Y_{\underline{\lambda}}}^{q-m-2r})\\&\simeq\varepsilon(\underline{\lambda}\setminus\underline{\mu})\otimes_{\mathbb{Z}}\varepsilon(\underline{\lambda}\setminus\underline{\nu})\otimes_{\mathbb{Z}}\Omega_{Y_{\underline{\lambda}}}^{p-m-2r}\otimes_{\mathbb{C}}\Omega_{Y_{\underline{\lambda}}}^{q-m-2r}\end{aligned} \tag{6.13.6}$$

and the morphism

$$(-1)^{(p-r)(r+m)+(p-m)|\underline{\lambda}\setminus\underline{\nu}|}\chi(\underline{\lambda}\setminus\underline{\mu},\underline{\lambda}\setminus\underline{\nu})\otimes\wedge=(-1)^{r+pm}\chi(\underline{\lambda}\setminus\underline{\mu},\underline{\lambda}\setminus\underline{\nu})\otimes\wedge$$

by using $|\underline{\lambda}\setminus\underline{\nu}|=r$, where $\chi(\underline{\lambda}\setminus\underline{\mu},\underline{\lambda}\setminus\underline{\nu})$ is the morphism (2.2.2). Because $\mathcal{C}(\operatorname{dlog} t\wedge)$ on the direct summand $\omega_{Y_{\underline{\lambda}}}[-k]$ of $\mathcal{C}(\omega_{Y_\bullet})$ is the morphism $(-1)^k(\operatorname{dlog} t\wedge)$, the morphism (6.13.2) is equal to the composite of the isomorphism (6.13.6) and the morphism

$$(-1)^{m+r+pm}(e_\lambda\wedge)^{-1}(e_{\lambda(r)}\wedge)\chi(\underline{\lambda}\setminus\underline{\mu},\underline{\lambda}\setminus\underline{\nu})\otimes\wedge$$

on the direct summand (6.13.5) by Corollary 3.10, by $k=m+2r$ and by $(\underline{\lambda}\setminus\underline{\mu})\cup(\underline{\lambda}\setminus\underline{\nu})=\underline{\lambda_r}$. Here, the equality

$$\vartheta(\underline{\lambda})=(e_\lambda\wedge)^{-1}(e_{\lambda(r)}\wedge)\chi(\underline{\lambda}\setminus\underline{\mu},\underline{\lambda}\setminus\underline{\nu})((e_\mu\wedge)^{-1}\otimes(e_\nu\wedge)^{-1}):\varepsilon(\underline{\lambda})\otimes_{\mathbb{Z}}\varepsilon(\underline{\lambda})\longrightarrow\mathbb{Z}$$

can be easily checked. Thus we obtain the conclusion by considering (6.13.1) with $r+s=m+2r$ and by the sign convention (1.4.1). □

Definition 6.14. From the commutativity of the diagram (6.5.1), a pair of the morphisms $(\mathrm{H}^{p,q}(Y,\Phi_{\mathbb{Q}}),\mathrm{H}^{p,q}(Y,\Phi_{\mathbb{C}}))$ is denoted by

$$\mathrm{H}^{p,q}(Y,\Phi):\mathrm{H}^p(Y,K)\otimes\mathrm{H}^q(Y,K)\longrightarrow\mathrm{H}^{p+q}(Y,K)$$

for every p,q. Sometimes, for $x\in\mathrm{H}^p(Y,K)$ and $y\in\mathrm{H}^q(Y,K)$, $\mathrm{H}^{p,q}(Y,\Phi)(x\otimes y)$ is simply denoted by $x\cup y$ if there is no danger of confusion.

Lemma 6.15. *We have*

$$(x\cup y)\cup z=x\cup(y\cup z)$$

for every $x\in\mathrm{H}^p(Y,K)$, $y\in\mathrm{H}^q(Y,K)$ *and* $z\in\mathrm{H}^r(Y,K)$.

Proof. By Lemma 2.11. □

Lemma 6.16. *As for the filtration, we have*

$$W_a \mathrm{H}^p(Y, K) \cup W_b \mathrm{H}^q(Y, K) \subset W_{a+b} \mathrm{H}^{p+q}(Y, K) \tag{6.16.1}$$

$$F^a \mathrm{H}^p(Y, K) \cup F^b \mathrm{H}^q(Y, K) \subset F^{a+b} \mathrm{H}^{p+q}(Y, K) \tag{6.16.2}$$

for every a, b. *In particular, we obtain the morphism of mixed Hodge structures*

$$\cup : (\mathrm{H}^p(Y, K), W[p], F) \otimes (\mathrm{H}^q(Y, K), W[q], F) \longrightarrow (\mathrm{H}^{p+q}(Y, K), W[p+q], F)$$

for every p, q *if we assume the conditions* (5.9.1) *and* (5.9.2).

Proof. Easy by (6.8.1). □

Lemma 6.17. *Under the assumptions* (5.9.1) *and* (5.9.2)*, we have*

$$y \cup x = (-1)^{pq} x \cup y$$

for $x \in \mathrm{H}^p(Y, K)$ *and for* $y \in \mathrm{H}^q(Y, K)$.

Proof. The commutativity of the diagrams

$$\begin{array}{ccc} \mathrm{H}^p(Y, K) \otimes \mathrm{H}^q(Y, K) & \xrightarrow{\mathrm{H}^{p,q}(Y, \Phi)} & \mathrm{H}^{p+q}(Y, K) \\ {\scriptstyle \mathrm{H}^p(Y, \pi_{/*}) \otimes \mathrm{H}^q(Y, \pi_{/*})} \downarrow & & \downarrow {\scriptstyle \mathrm{H}^{p+q}(Y, \pi_{/*})} \\ \mathrm{H}^p(Y, \mathcal{C}(\omega_{Y_\bullet/*})) \otimes \mathrm{H}^q(Y, \mathcal{C}(\omega_{Y_\bullet/*})) & \xrightarrow[\mathrm{H}^{p,q}(Y, \Phi_{/*})]{} & \mathrm{H}^{p+q}(Y, \mathcal{C}(\omega_{Y_\bullet/*})) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{H}^p(Y, \omega_{Y/*}) \otimes \mathrm{H}^q(Y, \omega_{Y/*}) & \xrightarrow{\wedge} & \mathrm{H}^{p+q}(Y, \omega_{Y/*}) \\ {\scriptstyle \mathrm{H}^p(Y, a^*_{/*}) \otimes \mathrm{H}^q(Y, a^*_{/*})} \downarrow & & \downarrow {\scriptstyle \mathrm{H}^{p+q}(Y, a^*_{/*})} \\ \mathrm{H}^p(Y, \mathcal{C}(\omega_{Y_\bullet/*})) \otimes \mathrm{H}^q(Y, \mathcal{C}(\omega_{Y_\bullet/*})) & \xrightarrow[\mathrm{H}^{p,q}(Y, \Phi_{/*})]{} & \mathrm{H}^{p+q}(Y, \mathcal{C}(\omega_{Y_\bullet/*})) \end{array}$$

implies the conclusion, by using the fact that $\mathrm{H}^p(Y, \pi_{/*})$ and $\mathrm{H}^p(Y, a^*_{/*})$ are isomorphisms for all p. □

Definition 6.18. The wedge product on ω_Y induces a morphism

$$\omega_Y^{p+1}/W_r \omega_Y^{p+1} \otimes_{\mathbb{C}} W_0 \omega_Y^q \longrightarrow \omega_Y^{p+q+1}/W_r \omega_Y^{p+q+1}$$

because of the inclusion $W_r \omega_Y^{p+1} \wedge W_0 \omega_Y^q \subset W_r \omega_Y^{p+q+1}$. We define a morphism

$$\overline{\Psi}_{\mathbb{C}} : A_{\mathbb{C}}^p \otimes_{\mathbb{C}} W_0 \omega_Y^q \longrightarrow A_{\mathbb{C}}^{p+q}$$

by the direct sum of the morphism above. It is easy to see that $\overline{\Psi}_{\mathbb{C}}$ defines a morphism of complexes $A_{\mathbb{C}} \otimes_{\mathbb{C}} W_0 \omega_Y \longrightarrow A_{\mathbb{C}}$.

On the other hand, the wedge product (6.2.1) on $\mathrm{Kos}_Y(M_Y)$ induces a morphism

$$\mathrm{Kos}_Y(M_Y)^{p+1}/W_r \mathrm{Kos}_Y(M_Y)^{p+1} \otimes_{\mathbb{Q}} W_0 \mathrm{Kos}_Y(M_Y)^q \longrightarrow \mathrm{Kos}_Y(M_Y)^{p+q+1}/W_r \mathrm{Kos}_Y(M_Y)^{p+q+1}$$

for every p, q, r. These morphisms define a morphism of complexes

$$\overline{\Psi}_{\mathbb{Q}} : A_{\mathbb{Q}} \otimes_{\mathbb{Q}} W_0 \mathrm{Kos}_Y(M_Y) \longrightarrow A_{\mathbb{Q}}$$

as above.

6.19. We can easily see that the diagram

$$\begin{array}{ccc} A_{\mathbb{Q}} \otimes_{\mathbb{Q}} W_0 \operatorname{Kos}_Y(M_Y) & \xrightarrow{\overline{\Psi}_{\mathbb{Q}}} & A_{\mathbb{Q}} \\ {\scriptstyle \alpha\otimes\psi}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A_{\mathbb{C}} \otimes_{\mathbb{C}} W_0\omega_Y & \xrightarrow[\overline{\Psi}_{\mathbb{C}}]{} & A_{\mathbb{C}} \end{array}$$

is commutative. As for the filtration, we easily see

$$\overline{\Psi}_{\mathbb{C}}(W_m A_{\mathbb{C}} \otimes_{\mathbb{C}} W_0\omega_Y) \subset W_m A_{\mathbb{C}}$$
$$\overline{\Psi}_{\mathbb{C}}(F^p A_{\mathbb{C}} \otimes F^q W_0\omega_Y) \subset F^{p+q} A_{\mathbb{C}}$$
$$\overline{\Psi}_{\mathbb{Q}}(W_m A_{\mathbb{Q}} \otimes_{\mathbb{Q}} W_0 \operatorname{Kos}_Y(M_Y)) \subset W_m A_{\mathbb{Q}}$$

for every m, p, q. Therefore the morphisms

$$\operatorname{Gr}_m^W \overline{\Psi}_{\mathbb{C}} : \operatorname{Gr}_m^W A_{\mathbb{C}} \otimes_{\mathbb{C}} W_0\omega_Y \longrightarrow \operatorname{Gr}_m^W A_{\mathbb{C}}$$
$$\operatorname{Gr}_m^W \overline{\Psi}_{\mathbb{Q}} : \operatorname{Gr}_m^W A_{\mathbb{Q}} \otimes_{\mathbb{Q}} W_0 \operatorname{Kos}_Y(M_Y) \longrightarrow \operatorname{Gr}_m^W A_{\mathbb{Q}}$$

are induced for every m. The following two lemmas are easily proved. We omit the proofs here.

Lemma 6.20. *Under the identification (5.22.2), the morphism $\operatorname{Gr}_m^W \overline{\Psi}_{\mathbb{C}}$ coincides with the morphism*

$$(\mathrm{id} \otimes \wedge[-m-2r,0]) \cdot (\mathrm{id} \otimes a_{\underline{\sigma}}^*)$$
$$: \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\sigma}}}[-m-2r] \otimes_{\mathbb{C}} W_0\omega_Y \longrightarrow \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\sigma}}}[-m-2r]$$

on the direct summand $\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\sigma}}}[-m-2r] \otimes_{\mathbb{C}} W_0\omega_Y$, where $a_{\underline{\sigma}}^*$ denotes the morphism induced by the inclusion $a_{\underline{\sigma}} : Y_{\underline{\sigma}} \longrightarrow Y$.

Similarly, under the identification (5.22.1), the restriction of the morphism $\operatorname{Gr}_m^W \overline{\Psi}_{\mathbb{Q}}$ is identified with the morphism

$$(\mathrm{id} \otimes \wedge[-m-2r,0])(\mathrm{id} \otimes a_{\underline{\sigma}}^{-1})$$
$$: \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathbb{Q}_{Y_{\underline{\sigma}}}[-m-2r] \otimes_{\mathbb{Q}} \mathbb{Q}_Y \longrightarrow \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathbb{Q}_{Y_{\underline{\sigma}}}[-m-2r]$$

on the direct summand $\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathbb{Q}_{Y_{\underline{\sigma}}}[-m-2r] \otimes_{\mathbb{Q}} \mathbb{Q}_Y$ of $\operatorname{Gr}_m^W A_{\mathbb{Q}} \otimes_{\mathbb{Q}} W_0 \operatorname{Kos}_Y(M_Y)$, where $\wedge$ is the morphism (6.2.2).

Lemma 6.21. *The diagrams*

$$\begin{array}{ccc} A_{\mathbb{Q}} \otimes_{\mathbb{Q}} W_0 \operatorname{Kos}_Y(M_Y) & \xrightarrow{\overline{\Psi}_{\mathbb{Q}}} & A_{\mathbb{Q}} \\ {\scriptstyle \varphi_{\mathbb{Q}}\otimes a^*}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathbb{Q}}} \\ K_{\mathbb{Q}} \otimes_{\mathbb{Q}} K_{\mathbb{Q}} & \xrightarrow[\Phi_{\mathbb{Q}}]{} & K_{\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} A_{\mathbb{C}} \otimes_{\mathbb{C}} W_0\omega_Y & \xrightarrow{\overline{\Psi}_{\mathbb{C}}} & A_{\mathbb{C}} \\ {\scriptstyle \varphi_{\mathbb{C}}\otimes a^*}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathbb{C}}} \\ K_{\mathbb{C}} \otimes_{\mathbb{C}} K_{\mathbb{C}} & \xrightarrow[\Phi_{\mathbb{C}}]{} & K_{\mathbb{C}} \end{array}$$

are commutative.

7. Trace morphism

7.1. Let $Y \longrightarrow *$ be a log deformation satisfying conditions (5.9.1) and (5.9.2). In addition, we assume

(7.1.1) all the irreducible components Y_λ are of dimension n

in the remainder of this article.

Lemma 7.2. *The condition* $\operatorname{Gr}_m^W \mathrm{H}^q(Y, K_{\mathbb{C}}) \neq 0$ *implies the inequalities* $-q \leq m \leq q$ *and* $-2n + q \leq m \leq 2n - q$.

Proof. The condition $\operatorname{Gr}_m^W \mathrm{H}^q(Y, K_{\mathbb{C}}) \neq 0$ is equivalent to $\operatorname{Gr}_m^W \mathrm{H}^q(Y, A_{\mathbb{C}}) \neq 0$ by Theorem 5.29. The condition $\operatorname{Gr}_m^W \mathrm{H}^q(Y, A_{\mathbb{C}}) \neq 0$ implies $E_1^{-m,q+m}(A_{\mathbb{C}}, W) \neq 0$. If this condition is the case, then the identification

$$\begin{aligned} E_1^{-m,q+m}(A_{\mathbb{C}}, W) &= \mathrm{H}^q(Y, \operatorname{Gr}_m^W A_{\mathbb{C}}) \\ &\simeq \bigoplus_{r \geq \max(0,-m)} \bigoplus_{\underline{\sigma} \in S_{m+2r+1}(\Lambda)} \varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathrm{H}^{q-m-2r}(Y_{\underline{\sigma}}, \Omega_{Y_{\underline{\sigma}}}) \end{aligned}$$

induced by (5.22.2) gives us the inequalities $0 \leq q - m - 2r \leq 2 \dim Y_{\underline{\sigma}} = 2(n - m - 2r)$. The conclusion can be easily obtained from these inequalities. □

Corollary 7.3. *The condition* $\mathrm{H}^q(Y, K) \neq 0$ *implies* $0 \leq q \leq 2n$.

Lemma 7.4. *We have*

$$W_{-1}\mathrm{H}^{2n}(Y, K_{\mathbb{C}}) = 0, W_0\mathrm{H}^{2n}(Y, K_{\mathbb{C}}) = \mathrm{H}^{2n}(Y, K_{\mathbb{C}})$$

for the weight filtration W *on* $\mathrm{H}^{2n}(Y, K_{\mathbb{C}})$. *On the other hand, we have*

$$F^n\mathrm{H}^{2n}(Y, K_{\mathbb{C}}) = \mathrm{H}^{2n}(Y, K_{\mathbb{C}}), F^{n+1}\mathrm{H}^{2n}(Y, K_{\mathbb{C}}) = 0$$

for the Hodge filtration F.

Proof. Lemma 7.2 shows that $\operatorname{Gr}_m^W \mathrm{H}^{2n}(Y, K_{\mathbb{C}}) = 0$ for $m \neq 0$, Hence we obtain the conclusion for the filtration W.

We have

$$(E_1^{0,2n}(A_{\mathbb{C}}, W), F) \simeq \bigoplus_{r \geq 0} \bigoplus_{\underline{\sigma} \in S_{2r+1}(\Lambda)} (\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \mathrm{H}^{2n-2r}(Y_{\underline{\sigma}}, \Omega_{Y_{\underline{\sigma}}}), F[-r])$$

as in the proof of Lemma 7.2. Since $\dim Y_{\underline{\sigma}} = n - 2r$, $\mathrm{H}^{2n-2r}(Y_{\underline{\sigma}}, \Omega_{Y_{\underline{\sigma}}}) = 0$ for $r > 0$. Therefore

$$\operatorname{Gr}_F^p E_1^{0,2n}(A_{\mathbb{C}}, W) = \bigoplus_{\sigma \in \Lambda} \varepsilon(\sigma) \otimes_{\mathbb{Z}} \operatorname{Gr}_F^p \mathrm{H}^{2n}(Y_\sigma, \Omega_{Y_\sigma}) \neq 0$$

implies $p = n$. Thus we conclude that

$$\operatorname{Gr}_F^p E_2^{0,2n}(A_{\mathbb{C}}, W) \simeq \operatorname{Gr}_F^p E_2^{0,2n}(K_{\mathbb{C}}, W) \neq 0$$

implies $p = n$ as desired. □

Corollary 7.5. *We have an exact sequence*

$$E_1^{-1,2n}(K_{\mathbb{C}}, W) \longrightarrow Z_{\mathbb{C}} \longrightarrow \mathrm{H}^{2n}(Y, K_{\mathbb{C}}) \longrightarrow 0 \tag{7.5.1}$$

by setting $Z_{\mathbb{C}} = \operatorname{Ker}(d_1 : E_1^{0,2n}(K_{\mathbb{C}}, W) \longrightarrow E_1^{1,2n}(K_{\mathbb{C}}, W))$.

Proof. We have

$$\mathrm{H}^{2n}(Y, K_{\mathbb{C}}) \simeq \operatorname{Gr}_0^W \mathrm{H}^{2n}(Y, K_{\mathbb{C}}) \simeq E_2^{0,2n}(K_{\mathbb{C}}, W)$$

by Corollary 7.4 and by E_2-degeneration of the spectral sequence $E_r^{p,q}(K_{\mathbb{C}}, W)$. □

7.6. For $\lambda \in \prod^\circ \Lambda$, we have the morphism

$$\int_{Y_{\underline{\lambda}}} : \mathrm{H}^{2n-2d(\lambda)}(Y_{\underline{\lambda}}, \Omega_{Y_{\underline{\lambda}}}) \longrightarrow \mathbb{C}$$

because $\dim Y_{\underline{\lambda}} = n - d(\lambda)$ for $\lambda \in \prod^\circ \Lambda$. On the other hand, we have the morphisms

$$\Theta_{\mathbb{C},0}(\lambda)[1] \operatorname{Gr}_0^{\delta W} \mathcal{C}(\operatorname{dlog} t\wedge) : \operatorname{Gr}_0^{\delta W} \mathcal{C}(\omega_{Y_\bullet}) \longrightarrow \Omega_{Y_{\underline{\lambda}}}[-2d(\lambda)]$$
$$\Theta_{\mathbb{C}}(\lambda)[1] \operatorname{Gr}_0^{W} \mathcal{C}(\mathrm{id} \otimes \operatorname{dlog} t\wedge) : \operatorname{Gr}_0^{W} K_{\mathbb{C}} \longrightarrow \Omega_{Y_{\underline{\lambda}}}[-2d(\lambda)]$$

for every $\lambda \in \prod^\circ \Lambda$.

Definition 7.7. The morphisms

$$\Theta_{\mathbb{C},0} : E_1^{0,2n}(\mathcal{C}(\omega_{Y_\bullet}), \delta W) = \mathrm{H}^{2n}(Y, \operatorname{Gr}_0^{\delta W} \mathcal{C}(\omega_{Y_\bullet})) \longrightarrow \mathbb{C}$$
$$\Theta_{\mathbb{C}} : E_1^{0,2n}(K_{\mathbb{C}}, W) = \mathrm{H}^{2n}(Y, \operatorname{Gr}_0^{W} K_{\mathbb{C}}) \longrightarrow \mathbb{C}$$

are defined by

$$\Theta_{\mathbb{C},0} = \sum_{\lambda \in \prod^\circ \Lambda} \epsilon(d(\lambda))(|\underline{\lambda}|!)^{-1}\left(2\pi\sqrt{-1}\right)^{d(\lambda)} \int_{Y_{\underline{\lambda}}} \mathrm{H}^{2n}(Y, \Theta_{\mathbb{C},0}(\lambda)[1] \operatorname{Gr}_0^{\delta W} \mathcal{C}(\operatorname{dlog} t\wedge))$$
$$\Theta_{\mathbb{C}} = \sum_{\lambda \in \prod^\circ \Lambda} \epsilon(d(\lambda))(|\underline{\lambda}|!)^{-1}\left(2\pi\sqrt{-1}\right)^{d(\lambda)} \int_{Y_{\underline{\lambda}}} \mathrm{H}^{2n}(Y, \Theta_{\mathbb{C}}(\underline{\lambda})[1] \operatorname{Gr}_0^{W} \mathcal{C}(\mathrm{id} \otimes \operatorname{dlog} t\wedge))$$

respectively, where $\epsilon(a) = (-1)^{a(a-1)/2}$ as in [11, (3.3)], [18, I-14].

7.8. We can easily check the equality

$$\Theta_{\mathbb{C}} = \Theta_{\mathbb{C},0}\mathrm{H}^{2n}(Y, \operatorname{Gr}_0^W \pi_{\mathbb{C}}) \tag{7.8.1}$$

by direct computation.

Proposition 7.9. *We have* $\Theta_{\mathbb{C}} d_1 = 0$.

Proof. The morphism $d_1 : E_1^{-1,2n}(K_{\mathbb{C}}, W) \longrightarrow E_1^{0,2n}(K_{\mathbb{C}}, W)$ is induced by the Gysin morphism $\gamma_1(K_{\mathbb{C}}, W) : \operatorname{Gr}_1^W K_{\mathbb{C}} \longrightarrow \operatorname{Gr}_0^W K_{\mathbb{C}}[1]$. Therefore we have

$$\begin{aligned}
\Theta_{\mathbb{C}} d_1 &= \sum_{\lambda \in \prod^\circ \Lambda} \epsilon(d(\lambda))(|\underline{\lambda}|!)^{-1}\left(2\pi\sqrt{-1}\right)^{d(\lambda)} \int_{Y_{\underline{\lambda}}} \\
&\qquad \mathrm{H}^{2n-1}(Y, \Theta_{\mathbb{C}}(\lambda)[2] \operatorname{Gr}_0^{W} \mathcal{C}(\mathrm{id} \otimes \operatorname{dlog} t\wedge)[1]\gamma_1(K_{\mathbb{C}}, W)) \\
&= \sum_{\lambda \in \prod^\circ \Lambda} \epsilon(d(\lambda))(|\underline{\lambda}|!)^{-1}\left(2\pi\sqrt{-1}\right)^{d(\lambda)} \int_{Y_{\underline{\lambda}}} \\
&\qquad \mathrm{H}^{2n-1}(Y, \Theta_{\mathbb{C},0}(\lambda)[2] \operatorname{Gr}_0^{\delta W} \mathcal{C}(\operatorname{dlog} t\wedge)[1]\gamma_1(\mathcal{C}(\omega_{Y_\bullet}), \delta W) \operatorname{Gr}_1^W \pi_{\mathbb{C},0}) \\
&= \Theta_{\mathbb{C},0} d_1 \mathrm{H}^{2n-1}(Y, \operatorname{Gr}_1^W \pi_{\mathbb{C},0})
\end{aligned}$$

by Lemma 5.15, where d_1 in the last equality stands for the morphism of E_1-terms of the spectral sequence $E_r^{p,q}(\mathcal{C}(\omega_{Y_\bullet}), \delta W)$. Therefore the following lemma implies the conclusion. □

Lemma 7.10. *For the morphism* $d_1 : E_1^{-1,2n}(\mathcal{C}(\omega_{Y_\bullet}), \delta W) \longrightarrow E_1^{0,2n}(\mathcal{C}(\omega_{Y_\bullet}), \delta W)$*, we have* $\Theta_{\mathbb{C},0} d_1 = 0$.

Proof. Because we have

$$\begin{aligned}
\operatorname{Gr}_0^{\delta W} \mathcal{C}(\operatorname{dlog} t\wedge)[1]\gamma_1(\mathcal{C}(\omega_{Y_\bullet}), \delta W) &= \gamma_2(\mathcal{C}(\omega_{Y_\bullet})[1], \delta W) \operatorname{Gr}_1^{\delta W} \mathcal{C}(\operatorname{dlog} t\wedge) \\
&= -\gamma_2(\mathcal{C}(\omega_{Y_\bullet}), \delta W)[1] \operatorname{Gr}_1^{\delta W} \mathcal{C}(\operatorname{dlog} t\wedge)
\end{aligned}$$

from the functoriality of the Gysin morphism and from the equality (1.5.1),

$$\Theta_{\mathbb{C},0} d_1 = - \sum_{\lambda \in \prod^\circ \Lambda} \epsilon(d(\lambda))(|\underline{\lambda}|!)^{-1} \bigl(2\pi\sqrt{-1}\bigr)^{d(\lambda)} \int_{Y_{\underline{\lambda}}} \mathrm{H}^{2n}(Y, \Theta_{\mathbb{C},0}(\lambda)[1]\gamma_2(\mathcal{C}(\omega_{Y_\bullet}), \delta W))\mathrm{H}^{2n-1}(Y, \mathrm{Gr}_1^{\delta W} \mathcal{C}(\mathrm{dlog}\, t\wedge))$$

is obtained. Then it suffices to prove that the morphism

$$\sum_{\lambda \in \prod^\circ \Lambda} \epsilon(d(\lambda))(|\underline{\lambda}|!)^{-1} \bigl(2\pi\sqrt{-1}\bigr)^{d(\lambda)} \int_{Y_{\underline{\lambda}}} \mathrm{H}^{2n}(Y, \Theta_{\mathbb{C},0}(\lambda)[1]\gamma_2(\mathcal{C}(\omega_{Y_\bullet}), \delta W))$$

from $\mathrm{H}^{2n}(Y, \mathrm{Gr}_2^{\delta W} \mathcal{C}(\omega_{Y_\bullet}))$ to $\mathbb{C}$ is the zero morphism.

For $\lambda \in \Lambda^{k+1,\circ}$, Lemma 2.7 and Proposition 4.5 imply that the restriction of the morphism

$$\Theta_{\mathbb{C},0}(\lambda)[1]\gamma_2(\mathcal{C}(\omega_{Y_\bullet}), \delta W) : \mathrm{Gr}_2^{\delta W} \mathcal{C}(\omega_{Y_\bullet}) \longrightarrow \Omega_{Y_{\underline{\lambda}}}[-2k] \tag{7.10.1}$$

is the zero morphism on the direct summand

$$\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\mu}\cup\underline{\sigma}}}[-2-2d(\mu)] \tag{7.10.2}$$

of $\mathrm{Gr}_2^{\delta W} \mathcal{C}(\omega_{Y_\bullet})$ for $\mu \in \prod^\circ \Lambda$ and for $\underline{\sigma} \in S_{2+d(\mu)}(\Lambda)$ under the identification (5.10.2), unless one of the following conditions is satisfied

(7.10.3) $\lambda = \mu$ and $\underline{\sigma} = \underline{\lambda} \cup \{\nu\}$ for some $\nu \in \Lambda \setminus \underline{\lambda}$

(7.10.4) $\mu = \lambda_i$ for some $i = 0, 1, \ldots, k$ and $\underline{\sigma} = \underline{\lambda}$.

For the case of (7.10.3), the restriction of the morphism (7.10.1) on the direct summand (7.10.2) coincides with the morphism

$$(-1)^k((e_\lambda\wedge)^{-1}(e_\nu\wedge)^{-1}) \otimes \gamma(Y_{\underline{\lambda}}, Y_{\underline{\lambda}\cup\{\nu\}})[-1-2k]$$

by (4.4.1), by Lemma 2.7 and by Proposition 4.5. For the case of (7.10.4), the restriction of the morphism (7.10.1) on the direct summand (7.10.2) coincides with the morphism

$$(-1)^i(e_\lambda\wedge)^{-1} \otimes \mathrm{id} = ((e_{\lambda_i}\wedge)^{-1}(e_{\lambda(i)}\wedge)^{-1}) \otimes \mathrm{id}$$

by Lemma 2.7.

Hence, on the direct summand

$$\varepsilon(\underline{\lambda} \cup \{\nu\}) \otimes_{\mathbb{Z}} \mathrm{H}^{2n-2-2k}(Y_{\underline{\lambda}\cup\{\nu\}}, \Omega_{Y_{\underline{\lambda}\cup\{\nu\}}})$$

for $\lambda \in \Lambda^{k+1,\circ}$ and for some $\nu \in \Lambda \setminus \underline{\lambda}$, the restriction of the morphism $\Theta_{\mathbb{C},0} d_1$ coincides with the morphism

$$\begin{aligned}
&\epsilon(k)((k+1)!)^{-1}\bigl(2\pi\sqrt{-1}\bigr)^k(-1)^k((e_\lambda\wedge)^{-1}(e_\nu\wedge)^{-1}) \otimes \int_{Y_{\underline{\lambda}}} \mathrm{H}^{2n-2-2k}(Y_{\underline{\lambda}}, \gamma(Y_{\underline{\lambda}}, Y_{\underline{\lambda}\cup\{\nu\}})) \\
&\qquad + (k+2)\epsilon(k+1)((k+2)!)^{-1}\bigl(2\pi\sqrt{-1}\bigr)^{k+1}((e_\lambda\wedge)^{-1}(e_\nu\wedge)^{-1}) \otimes \int_{Y_{\underline{\lambda}\cup\{\nu\}}} \\
&= \epsilon(k+1)((k+1)!)^{-1}\bigl(2\pi\sqrt{-1}\bigr)^k((e_\lambda\wedge)^{-1}(e_\nu\wedge)^{-1}) \\
&\qquad \otimes \Bigl(\int_{Y_{\underline{\lambda}}} \mathrm{H}^{2n-2-2k}(Y_{\underline{\lambda}}, \gamma(Y_{\underline{\lambda}}, Y_{\underline{\lambda}\cup\{\nu\}})) + \bigl(2\pi\sqrt{-1}\bigr) \int_{Y_{\underline{\lambda}\cup\{\nu\}}}\Bigr),
\end{aligned}$$

which turns out to be zero because of Proposition 4.3. ∎

Definition 7.11. By Proposition 7.9 and by Corollary 7.5, there exists the unique morphism

$$\mathrm{Tr} : \mathrm{H}^{2n}(Y, K_{\mathbb{C}}) \longrightarrow \mathbb{C},$$

such that the diagram

$$\begin{array}{ccc} Z_{\mathbb{C}} & \longrightarrow & \mathrm{H}^{2n}(Y, K_{\mathbb{C}}) \\ {\scriptstyle \Theta_{\mathbb{C}}|_{Z_{\mathbb{C}}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathbb{C} & = & \mathbb{C} \end{array}$$

commutes, where the top horizontal arrow stands for the morphism in the exact sequence (7.5.1). We call the morphism Tr the trace morphism for $Y \longrightarrow *$.

Proposition 7.12. *The morphism* Tr *is defined over* $\mathbb{Q}$*, that is, we have*

$$\mathrm{Tr}(\mathrm{H}^{2n}(Y, K_{\mathbb{Q}})) \subset \mathbb{Q}$$

under the identification $\mathrm{H}^{2n}(Y, K_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathrm{H}^{2n}(Y, K_{\mathbb{C}})$ *by* $\mathrm{H}^{2n}(Y, \psi)$*.*

Proof. It suffices to prove the inclusion

$$\Theta_{\mathbb{C},0}\mathrm{H}^{2n}(Y, \mathrm{Gr}_0^{\delta W} \psi_0)(E_1^{0,2n}(\mathcal{C}(\mathrm{Kos}_{Y_\bullet}(M_{Y_\bullet})), \delta W)) \subset \mathbb{Q} \tag{7.12.1}$$

by the commutative diagram (5.6.1) and by the equality (7.8.1). Since the morphism $\mathrm{Gr}_0^{\delta W} \psi_0$ is identified with the morphism induced by the inclusion

$$\left(2\pi\sqrt{-1}\right)^{-d(\lambda)}\iota : \mathbb{Q} \longrightarrow \mathbb{C}$$

on the direct summand $\varepsilon(\underline{\sigma}) \otimes_{\mathbb{Z}} \Omega_{Y_{\underline{\lambda} \cup \sigma}}[-2d(\lambda)]$ for $\lambda \in \prod^{\circ} \Lambda$ under the identifications (5.10.1) and (5.10.2), we can easily obtain the inclusion (7.12.1) as desired. □

Definition 7.13. We define a pairing

$$Q_K : \mathrm{H}^q(Y, K_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathrm{H}^{2n-q}(Y, K_{\mathbb{C}}) \longrightarrow \mathbb{C}$$

by setting

$$Q_K = \mathrm{Tr} \cdot \mathrm{H}^{q,2n-q}(Y, \Phi_{\mathbb{C}}),$$

that is, $Q_K(x \otimes y) = \mathrm{Tr}(x \cup y)$ for $x \in \mathrm{H}^q(Y, K)$ and $y \in \mathrm{H}^{2n-q}(Y, K)$.

Lemma 7.14. *We have the property, for all* q*,*

$$Q_K(\mathrm{H}^q(Y, K_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathrm{H}^{2n-q}(Y, K_{\mathbb{Q}})) \subset \mathbb{Q}.$$

Proof. Easy from Proposition 7.12. □

Lemma 7.15. *We have*

$$Q_K(y \otimes x) = (-1)^q Q_K(x \otimes y)$$

for $x \in \mathrm{H}^q(Y, K)$ *and for* $y \in \mathrm{H}^{2n-q}(Y, K)$*.*

Proof. Easy by Lemma 6.17. □

Lemma 7.16. *For the morphism of mixed Hodge structures* N_K *in* (5.16.1)*, the equality*

$$Q_K(N_K(x) \otimes y) + Q_K(x \otimes N_K(y)) = 0$$

holds for every $x \in \mathrm{H}^q(Y, K)$*,* $y \in \mathrm{H}^{2n-q}(Y, K)$*.*

Proof. We have

$$\mathrm{H}^{p,q}(Y, \Phi)(N_K \otimes \mathrm{id} + \mathrm{id} \otimes N_K) = N_K \cdot \mathrm{H}^{p,q}(Y, \Phi)$$

by Lemma 6.7. On the other hand, $N_K = 0$ on $\mathrm{H}^{2n}(Y, K)$ by Lemma 7.4. □

Lemma 7.17. *We have, for all p,*

$$Q_K(F^p\mathrm{H}^q(Y,K)\otimes_{\mathbb{C}} F^{n-p+1}\mathrm{H}^{2n-q}(Y,K))=0.$$

Proof. Easy by (6.16.2) and by the conclusion on F in Lemma 7.4. □

Lemma 7.18. *We have*

$$Q_K(W_a\mathrm{H}^q(Y,K)\otimes W_b\mathrm{H}^{2n-q}(Y,K))=0$$

if $a+b\le -1$.

Proof. We easily obtain the conclusion from the property (6.16.1) and from the fact

$$W_{-1}\mathrm{H}^{2n}(Y,K)=0$$

in Lemma 7.4. □

Definition 7.19. By the lemma above, the morphism Q_K induces the morphism

$$\mathrm{Gr}_m^W\mathrm{H}^q(Y,K)\otimes\mathrm{Gr}_{-m}^W\mathrm{H}^{2n-q}(Y,K)\longrightarrow\mathbb{C}$$

which is denoted by $\mathrm{Gr}_{m,-m}^W Q_K$ for m,q.

Lemma 7.20. *For $x\in\mathrm{Gr}_m^W\mathrm{H}^q(Y,K), y\in\mathrm{Gr}_{-m}^W\mathrm{H}^{2n-q}(Y,K)$, we have*

$$\mathrm{Gr}_{m,-m}^W Q_K(Cx\otimes Cy)=\mathrm{Gr}_{m,-m}^W Q_K(x\otimes y),$$

where C's denote the Weil operators on $\mathrm{Gr}_m^W\mathrm{H}^q(Y,K)$ and $\mathrm{Gr}_{-m}^W\mathrm{H}^{2n-q}(Y,K)$ which are the Hodge structures of weight $m+q$ and $2n-m-q$ respectively.

Proof. The Weil operator on the Hodge structure $\mathrm{Gr}_0^W\mathrm{H}^{2n}(Y,K)$ of weight n coincides with the identity by Lemma 7.4. Then we can easily see the conclusion from the fact that $\cup$ is a morphism of mixed Hodge structures. □

8. Main results

8.1. Let $Y\longrightarrow *$ be a log deformation. We assume

(8.1.1) Y is projective,

together with condition (7.1.1). We fix an ample invertible sheaf $\mathcal{L}$ on Y.

8.2. The morphism

$$\mathrm{dlog}:\mathcal{O}_Y^*\longrightarrow W_0\omega_Y[1]$$

is defined by sending a local section $f\in\mathcal{O}_Y^*$ to $df/f\in\Omega_Y^1=W_0\omega_Y^1$. We note that the image of the morphism dlog is contained in $F^1\omega_Y[1]$.

On the other hand, we have the morphism

$$\mathcal{O}_Y^*\longrightarrow\Gamma_{n-1}(\mathcal{O}_Y)\otimes_{\mathbb{Q}}\mathcal{O}_Y^*=\mathrm{Kos}_Y(\mathcal{O}_Y^*;n)^1$$

which sends a local section $f\in\mathcal{O}_Y^*$ to $(n-1)!1^{[n-1]}\otimes f\in\Gamma_{n-1}(\mathcal{O}_Y)\otimes_{\mathbb{Q}}\mathcal{O}_Y^*$. Then we obtain a morphism of complexes

$$\mathcal{O}_Y^*\longrightarrow\mathrm{Kos}_Y(\mathcal{O}_Y^*)[1]=W_0\,\mathrm{Kos}_Y(M_Y)[1]$$

denoted by $\mathrm{dlog}_{\mathbb{Q}}$. The diagram

$$\begin{array}{ccc}\mathcal{O}_Y^* & \xrightarrow{\mathrm{dlog}_{\mathbb{Q}}} & W_0\,\mathrm{Kos}_Y(M_Y)[1]\\ \Big\| & & \Big\downarrow{\scriptstyle (2\pi\sqrt{-1})\psi_{(Y,M_Y)}[1]}\\ \mathcal{O}_Y^* & \xrightarrow[\mathrm{dlog}]{} & W_0\omega_Y[1]\end{array}\qquad(8.2.1)$$

is commutative by definition.

Definition 8.3. We set

$$c_{\mathbb{Q}}(\mathcal{L}) = \mathrm{H}^1(Y, \mathrm{dlog}_{\mathbb{Q}})([\mathcal{L}]) \in \mathrm{H}^2(Y, W_0 \operatorname{Kos}_Y(M_Y))$$
$$c_{\mathbb{C}}(\mathcal{L}) = \mathrm{H}^1(Y, \mathrm{dlog})([\mathcal{L}]) \in \mathrm{H}^2(Y, W_0\omega_Y)$$

where $[\mathcal{L}]$ denotes the isomorphism class of $\mathcal{L}$ in $\mathrm{H}^1(Y, \mathcal{O}_Y^*)$. Moreover, we set

$$c_{K,\mathbb{Q}}(\mathcal{L}) = \mathrm{H}^2(Y, a^*)(c_{\mathbb{Q}}(\mathcal{L})) \in \mathrm{H}^2(Y, K_{\mathbb{Q}})$$
$$c_{K,\mathbb{C}}(\mathcal{L}) = \mathrm{H}^2(Y, a^*)(c_{\mathbb{C}}(\mathcal{L})) \in \mathrm{H}^2(Y, K_{\mathbb{C}}),$$

where a^* denotes the restriction of the morphism $a^* : \operatorname{Kos}_Y(M_Y) \longrightarrow K_{\mathbb{Q}}$ (resp. $a^* : \omega_Y \longrightarrow K_{\mathbb{C}}$) to the subcomplex $W_0 \operatorname{Kos}_Y(M_Y) \subset \operatorname{Kos}_Y(M_Y)$ (resp. $W_0\omega_Y \subset \omega_Y$).

Lemma 8.4. *We have the following*:

(8.4.1) $c_{K,\mathbb{Q}}(\mathcal{L}) \in W_0\mathrm{H}^2(Y, K_{\mathbb{Q}})$

(8.4.2) $c_{K,\mathbb{C}}(\mathcal{L}) \in W_0\mathrm{H}^2(Y, K_{\mathbb{C}}) \cap F^1\mathrm{H}^2(Y, K_{\mathbb{C}})$

(8.4.3) $c_{K,\mathbb{C}}(\mathcal{L}) = \big(2\pi\sqrt{-1}\big)\mathrm{H}^2(Y, \psi)(c_{K,\mathbb{Q}}(\mathcal{L}))$

Proof. The first two properties are easily seen by the definition of $c_{K,\mathbb{Q}}(\mathcal{L})$ and $c_{K,\mathbb{C}}(\mathcal{L})$. The equality

$$c_{\mathbb{C}}(\mathcal{L}) = \big(2\pi\sqrt{-1}\big)\mathrm{H}^2(Y, \psi_{(Y,M_Y)})(c_{\mathbb{Q}}(\mathcal{L})),$$

is obtained by the commutative diagram (8.2.1). Then the third equality follows the commutative diagram in (5.7.2). □

Definition 8.5. For every q, morphisms

$$l_{K,\mathbb{Q}} : \mathrm{H}^q(Y, K_{\mathbb{Q}}) \longrightarrow \mathrm{H}^{q+2}(Y, K_{\mathbb{Q}})$$
$$l_{K,\mathbb{C}} : \mathrm{H}^q(Y, K_{\mathbb{C}}) \longrightarrow \mathrm{H}^{q+2}(Y, K_{\mathbb{C}})$$

are defined by

$$l_{K,\mathbb{Q}}(x) = -c_{K,\mathbb{Q}}(\mathcal{L}) \cup x = -\mathrm{H}^{2,q}(Y, \Phi_{\mathbb{Q}})(c_{K,\mathbb{Q}}(\mathcal{L}) \otimes x)$$
$$l_{K,\mathbb{C}}(y) = -c_{K,\mathbb{C}}(\mathcal{L}) \cup y = -\mathrm{H}^{2,q}(Y, \Phi_{\mathbb{C}})(c_{K,\mathbb{C}}(\mathcal{L}) \otimes y)$$

for $x \in \mathrm{H}^q(Y, K_{\mathbb{Q}})$ and for $y \in \mathrm{H}^q(Y, K_{\mathbb{C}})$.

Lemma 8.6. *The diagram*

$$\begin{array}{ccc}
\mathrm{H}^q(Y, K_{\mathbb{Q}}) & \xrightarrow{l_{K,\mathbb{Q}}} & \mathrm{H}^{q+2}(Y, K_{\mathbb{Q}}) \\
{\scriptstyle \mathrm{H}^q(Y,\psi)}\downarrow & & \downarrow{\scriptstyle (2\pi\sqrt{-1})\mathrm{H}^{q+2}(Y,\psi)} \\
\mathrm{H}^q(Y, K_{\mathbb{C}}) & \xrightarrow[l_{K,\mathbb{C}}]{} & \mathrm{H}^{q+2}(Y, K_{\mathbb{C}})
\end{array}$$

is commutative. Moreover we have, for every a, m,

$$l_{K,\mathbb{Q}}(W_m\mathrm{H}^q(Y, K_{\mathbb{Q}})) \subset W_m\mathrm{H}^{q+2}(Y, K_{\mathbb{Q}})$$
$$l_{K,\mathbb{C}}(W_m\mathrm{H}^q(Y, K_{\mathbb{C}})) \subset W_m\mathrm{H}^{q+2}(Y, K_{\mathbb{C}})$$
$$l_{K,\mathbb{C}}(F^a\mathrm{H}^q(Y, K_{\mathbb{C}})) \subset F^{a+1}\mathrm{H}^{q+2}(Y, K_{\mathbb{C}}).$$

Proof. Easy from the properties of $\Phi_{\mathbb{Q}}$ and $\Phi_{\mathbb{C}}$. □

Definition 8.7. The lemma above implies that the pair of the morphism

$$l_K = (l_{K,\mathbb{Q}}, (2\pi\sqrt{-1})^{-1} l_{K,\mathbb{C}})$$

defines a morphism of mixed Hodge structures

$$l_K : (\mathrm{H}^q(Y,K), W[q], F) \longrightarrow (\mathrm{H}^{q+2}(Y,K), W[q], F[1])$$

for every q.

Lemma 8.8. *We have $x \cup l_K y = l_K x \cup y = l_K(x \cup y)$ for $x \in \mathrm{H}^p(Y,K), y \in \mathrm{H}^q(Y,K)$.*

Proof. Easy by Lemma 6.15, by Lemma 6.17 and by the fact $c_{K,\mathbb{C}}(\mathcal{L}) \in \mathrm{H}^2(Y,K)$. □

Lemma 8.9. *We have $l_K N_K = N_K l_K$ on $\mathrm{H}^q(Y,K)$ for all q.*

Proof. Lemma 6.7 tells us the equality

$$N_K(c_{K,\mathbb{C}}(\mathcal{L})) \cup x + c_{K,\mathbb{C}}(\mathcal{L}) \cup N_K(x) = N_K(c_{K,\mathbb{C}}(\mathcal{L}) \cup x)$$

for $x \in \mathrm{H}^q(Y,K)$. Because $a^* : \omega_Y \longrightarrow K_{\mathbb{C}}$ factors through the subcomplex $\mathcal{C}(\omega_{Y_\bullet})$ by definition, we have $N_K(c_{K,\mathbb{C}}(\mathcal{L})) = 0$. Thus we have

$$c_{K,\mathbb{C}}(\mathcal{L}) \cup N_K(x) = N_K(c_{K,\mathbb{C}}(\mathcal{L}) \cup x)$$

as desired. □

Definition 8.10. We set

$$L^{i,j}_{\mathbb{Q}} = \mathrm{Gr}^W_{-i} \mathrm{H}^{n+j}(Y, K_{\mathbb{Q}}), \qquad L^{i,j}_{\mathbb{C}} = \mathrm{Gr}^W_{-i} \mathrm{H}^{n+j}(Y, K_{\mathbb{C}})$$

and

$$L^{i,j} = (L^{i,j}_{\mathbb{Q}}, L^{i,j}_{\mathbb{C}})$$

for every i, j. Note that $L^{i,j}$ is a Hodge structure of weight $n+j-i$. Then

$$L_{\mathbb{Q}} = \bigoplus_{i,j} L^{i,j}_{\mathbb{Q}}, \qquad L_{\mathbb{C}} = \bigoplus_{i,j} L^{i,j}_{\mathbb{C}}$$

is a pair of a finite dimensional $\mathbb{Q}$-vector space and a $\mathbb{C}$-vector space such that $L_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \simeq L_{\mathbb{C}}$. Moreover, a morphism

$$\langle\ \rangle : L \otimes_{\mathbb{C}} L \longrightarrow \mathbb{C}$$

is defined by

$$\langle\, x \otimes y \,\rangle = \begin{cases} \epsilon(j-n)\, \mathrm{Gr}^W_{i,-i} Q_K(x \otimes y) & \text{if } x \in L^{-i,-j},\ y \in L^{i,j} \\ 0 & \text{otherwise,} \end{cases}$$

which turns out to be a morphism defined over $\mathbb{Q}$, that is, $\langle\, x \otimes y \,\rangle \in \mathbb{Q}$ if $x \otimes y \in L_{\mathbb{Q}} \otimes_{\mathbb{Q}} L_{\mathbb{Q}}$.

Theorem 8.11. *The data $(L, N_K, l_K, \langle\ \rangle)$ is a bigraded polarized Hodge-Lefschetz module in the sense of Guillén-Navarro Aznar* [11, (4.1)-(4.3)].

Proof. We have

$$\mathrm{Gr}^W_{-i} \mathrm{H}^{n+j}(Y, K_{\mathbb{C}}) = E_2^{i,n+j-i}(K_{\mathbb{C}}, W) \simeq E_2^{i,n+j-i}(A_{\mathbb{C}}, W)$$

by Corollary 5.30. Then L underlies a bigraded polarized Hodge-Lefschetz module by [11, (4.5)Théorème, (5.1)Théorème]. Therefore it is sufficient that our data $N_K, l_K, \langle\ \rangle$ coincide with the data $2\pi\sqrt{-1}N, (2\pi\sqrt{-1})^{-1} l, (2\pi\sqrt{-1})^n \psi$ used in [11, (5.1)Théorème]. (Our definition of the differential of A in 5.17 is different from that in [11, (2.4)]. However, we can apply the results in [11] because this difference only affects the sign of the morphism $d_1 : E_1^{p,q}(A,W) \longrightarrow E_1^{p+1,q}(A,W)$.)

The morphism

$$N_A : E_1^{p,q}(A_{\mathbb{C}}, W) \longrightarrow E_1^{p+2,q-2}(A_{\mathbb{C}}, W)$$

coincides with the morphism $\left(2\pi\sqrt{-1}\right)N$ because these two morphisms are induced by the same morphism $\left(2\pi\sqrt{-1}\right)\nu_{\mathbb{C}}$. Thus N_K on L coincides with $\left(2\pi\sqrt{-1}\right)N$ under the identification above. Lemma 6.20 and Lemma 6.21 tell us that cup product with $c_{K,\mathbb{C}}(\mathcal{L})$ on L is induced by the usual cup product on $Y_{\underline{\sigma}}$ with $a_{\underline{\sigma}}^*(c_{K,\mathbb{C}}(\mathcal{L}))$. Since $a_{\underline{\sigma}}^*(c_{K,\mathbb{C}}(\mathcal{L}))$ coincides with $c_1'(a_{\underline{\sigma}}^*\mathcal{L})$ in Deligne [5, (2.2.4.1)], the morphism l_K on L coincides with $\left(2\pi\sqrt{-1}\right)^{-1}l$.

We have

$$\Theta_{\mathbb{C}}\mathrm{H}^{n-j,n+j}(Y, \mathrm{Gr}_{i,-i}^W \Psi_{\mathbb{C}}) = \Theta_{\mathbb{C},0}\mathrm{H}^{n-j,n+j}(Y, \mathrm{Gr}_{i,-i}^W \Psi_{\mathbb{C},0})$$

by the equality (6.10.1). Therefore we have

$$\begin{aligned}
&\Theta_{\mathbb{C}}\mathrm{H}^{n-j,n+j}(Y, \mathrm{Gr}_{i,-i}^W \Psi_{\mathbb{C}}) \\
&\quad = \sum_{\lambda \in \prod^{\circ} \Lambda} \epsilon(d(\lambda))(|\underline{\lambda}|!)^{-1}\left(2\pi\sqrt{-1}\right)^{d(\lambda)} \int_{Y_{\underline{\lambda}}} \\
&\qquad\qquad \mathrm{H}^{n-j,n+j}(Y, \Theta_{\mathbb{C},0}(\lambda)[1]\,\mathrm{Gr}_0^{\delta W}\,\mathcal{C}(\mathrm{dlog}\, t\wedge)\,\mathrm{Gr}_{i,-i}^W \Psi_{\mathbb{C},0})
\end{aligned}$$

by definition. On the direct summand

$$\begin{aligned}
&(\varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \mathrm{H}^{n-j-i-2r}(Y_{\underline{\lambda}}, \Omega_{Y_{\underline{\lambda}}})) \otimes_{\mathbb{C}} (\varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \mathrm{H}^{n+j-i-2r}(Y_{\underline{\lambda}}, \Omega_{Y_{\underline{\lambda}}})) \\
&\qquad \simeq \varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \mathrm{H}^{n-j-i-2r}(Y_{\underline{\lambda}}, \Omega_{Y_{\underline{\lambda}}}) \otimes_{\mathbb{C}} \mathrm{H}^{n+j-i-2r}(Y_{\underline{\lambda}}, \Omega_{Y_{\underline{\lambda}}})
\end{aligned}$$

of $\mathrm{H}^{n-j}(Y, \mathrm{Gr}_i^W A_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathrm{H}^{n+j}(Y, \mathrm{Gr}_{-i}^W A_{\mathbb{C}})$ for $\underline{\lambda} \in S_{i+2r+1}(\Lambda)$, we have

$$\begin{aligned}
&\Theta_{\mathbb{C}}\mathrm{H}^{n-j,n+j}(Y, \mathrm{Gr}_{i,-i}^W \Psi_{\mathbb{C}}) \\
&\quad = (-1)^{r+(n-j)i}|\underline{\lambda}|!\epsilon(|\underline{\lambda}|-1)(|\underline{\lambda}|!)^{-1}\left(2\pi\sqrt{-1}\right)^{|\underline{\lambda}|-1} \int_{Y_{\underline{\lambda}}} (\vartheta(\underline{\lambda}) \otimes \cup) \\
&\quad = (-1)^{(n-j)i}\epsilon(i)\left(2\pi\sqrt{-1}\right)^{i+2r} \int_{Y_{\underline{\lambda}}} (\vartheta(\underline{\lambda}) \otimes \cup)
\end{aligned}$$

by Lemma 6.13, where $\cup$ in the equalities above denotes the product of the usual de Rham cohomology of $Y_{\underline{\lambda}}$. On the other direct summands, we have

$$\Theta_{\mathbb{C}}\mathrm{H}^{n-j,n+j}(Y, \mathrm{Gr}_{i,-i}^W \Psi_{\mathbb{C}}) = 0$$

by Lemma 6.13 again. Identifying $\varepsilon(\underline{\lambda}) \otimes_{\mathbb{Z}} \varepsilon(\underline{\lambda})$ and $\mathbb{Z}$ by the canonical isomorphism $\vartheta(\underline{\lambda})$, we conclude that the pairing $\langle\ \rangle$ coincides with $\left(2\pi\sqrt{-1}\right)^n\psi$ by using the equality

$$(-1)^{(n-j)i}\epsilon(j-n)\epsilon(i) = \epsilon(i+j-n).$$

□

Corollary 8.12. *For every $i \geq 0$ and for every q*

$$N_K^i : (\mathrm{Gr}_i^W \mathrm{H}^q(Y, K), F) \longrightarrow (\mathrm{Gr}_{-i}^W \mathrm{H}^q(Y, K), F[-i])$$

are isomorphisms of Hodge structures of weight $i+q$. Moreover,

$$l_K^j : (\mathrm{H}^{n-j}(Y, K), W[n-j], F) \longrightarrow (\mathrm{H}^{n+j}(Y, K), W[n-j], F[j])$$

is an isomorphism of mixed Hodge structures for every $j \geq 0$.

Definition 8.13. We set

$$\mathrm{H}^j(Y,K)_{\mathrm{prim}} = \mathrm{Ker}(l_K^{n-j+1} : \mathrm{H}^j(Y,K) \longrightarrow \mathrm{H}^{2n-j+2}(Y,K))$$

for $j \leq n$. Then a morphism $N_K : \mathrm{H}^j(Y,K)_{\mathrm{prim}} \longrightarrow \mathrm{H}^j(Y,K)_{\mathrm{prim}}$ is induced by Lemma 8.9. Moreover, we define a pairing

$$S_{j,\mathrm{prim}} : \mathrm{H}^j(Y,K)_{\mathrm{prim}} \otimes \mathrm{H}^j(Y,K)_{\mathrm{prim}} \longrightarrow \mathbb{C}$$

by

$$S_{j,\mathrm{prim}}(x \otimes y) = \epsilon(j) Q_K(x \otimes l_K^{n-j} y)$$

for $x, y \in \mathrm{H}^j(Y,K)_{\mathrm{prim}}$.

Theorem 8.14. *For $j \leq n$, the data*

$$(\mathrm{H}^j(Y,K)_{\mathrm{prim}}, W[j], F, N_K, S_{j,\mathrm{prim}})$$

is a polarized mixed Hodge structure over $\mathbb{Q}$ in the sense of Cattani-Kaplan-Schmid [2, Definition (2.26)].

Proof. Lemma 7.2 implies

$$W_{-j-1}\mathrm{H}^j(Y,K) = 0 \quad \text{and} \quad W_j \mathrm{H}^j(Y,K) = \mathrm{H}^j(Y,K).$$

Therefore $N_K^{j+1} = 0$. From (5.16.1), we have, for all p,

$$N_K(F^p \mathrm{H}^j(Y,K)_{\mathrm{prim}}) \subset F^{p-1}\mathrm{H}^j(Y,K)_{\mathrm{prim}}.$$

Since

$$l_K^{n-j+1} : (\mathrm{H}^j(Y,K), W[j], F) \longrightarrow (\mathrm{H}^{2n-j+2}(Y,K), W[j], F[n-j+1])$$

is a morphism of mixed Hodge structures, $(\mathrm{H}^j(Y,K)_{\mathrm{prim}}, W[j], F)$ is a mixed Hodge structure. Moreover, the sequences

$$0 \longrightarrow \mathrm{Gr}_m^W \mathrm{H}^j(Y,K)_{\mathrm{prim}} \longrightarrow \mathrm{Gr}_m^W \mathrm{H}^j(Y,K) \xrightarrow{l_K^{n-j+1}} \mathrm{Gr}_m^W \mathrm{H}^{2n-j+2}(Y,K)$$

are exact for all m. Therefore, we have

$$\mathrm{Gr}_m^W \mathrm{H}^j(Y,K)_{\mathrm{prim}} = \mathrm{Ker}(l_K^{n-j+1} : L^{-m,j-n} \longrightarrow L^{-m,n-j+2}) \tag{8.14.1}$$

for all m. The commutativity of N_K and l_K induces the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Gr}_i^W \mathrm{H}^j(Y,K)_{\mathrm{prim}} & \longrightarrow & \mathrm{Gr}_i^W \mathrm{H}^j(Y,K) & \xrightarrow{l_K^{n-j+1}} & \mathrm{Gr}_i^W \mathrm{H}^{2n-j+2}(Y,K) \\ & {\scriptstyle N_K^i}\downarrow & & \downarrow{\scriptstyle N_K^i} & & \downarrow{\scriptstyle N_K^i} \\ 0 \longrightarrow & \mathrm{Gr}_{-i}^W \mathrm{H}^j(Y,K)_{\mathrm{prim}} & \longrightarrow & \mathrm{Gr}_{-i}^W \mathrm{H}^j(Y,K) & \xrightarrow[l_K^{n-j+1}]{} & \mathrm{Gr}_{-i}^W \mathrm{H}^{2n-j+2}(Y,K), \end{array}$$

which shows that the morphism

$$N_K^i : \mathrm{Gr}_i^W \mathrm{H}^j(Y,K)_{\mathrm{prim}} \longrightarrow \mathrm{Gr}_{-i}^W \mathrm{H}^j(Y,K)_{\mathrm{prim}}$$

is isomorphism for $i \geq 0$. Therefore $W[j] = W(N_K)[j]$ as desired.

Take elements $x, y \in \mathrm{H}^j(Y,K)_{\mathrm{prim}}$. We have

$$\begin{aligned} S_{j,\mathrm{prim}}(y \otimes x) &= \epsilon(j) Q_K(y \otimes l_K^{n-j} x) = \epsilon(j) Q_K(l_K^{n-j} y \otimes x) \\ &= \epsilon(j)(-1)^j Q_K(x \otimes l_K^{n-j} y) = (-1)^j S_{j,\mathrm{prim}}(x \otimes y) \end{aligned}$$

by Lemma 7.15 and by Lemma 8.8. Moreover, we can easily check

$$S_{j,\mathrm{prim}}(N_K x \otimes y) + S_{j,\mathrm{prim}}(x \otimes N_K y) = 0$$

by the commutativity of N_K and l_K and by Lemma 7.16.

If $x \in F^p\mathrm{H}^j(Y,K), y \in F^{j-p+1}\mathrm{H}^j(Y,K)$, Lemma 7.17 implies $Q_K(x \otimes l_K^{n-j}y) = 0$ because $l_K^{n-j}y \in F^{n-p+1}\mathrm{H}^{2n-p}(Y,K)$. Thus we obtain, for all p,

$$S_{j,\mathrm{prim}}(F^p\mathrm{H}^j(Y,K)_{\mathrm{prim}} \otimes F^{j-p+1}\mathrm{H}^j(Y,K)_{\mathrm{prim}}) = 0.$$

Now we set

$$P_i = \mathrm{Ker}(N_K^{i+1} : \mathrm{Gr}_i^W \mathrm{H}^j(Y,K)_{\mathrm{prim}} \longrightarrow \mathrm{Gr}_{-i-2}^W \mathrm{H}^j(Y,K)_{\mathrm{prim}})$$

for every $i \geq 0$, which is a Hodge structure of weight $i+j$. Then we have

$$P_i = L^{-i,j-n} \cap \mathrm{Ker}(N_K^{i+1}) \cap \mathrm{Ker}(l_K^{n-j+1})$$

for every $i \geq 0$ by (8.14.1).

By the definition of polarization of bigraded Hodge-Lefschetz module in [11, (4.3)], we have

$$\langle\, x \otimes CN_K^i l_K^j \overline{x}\,\rangle > 0$$

for $x \in L^{-i,-j} \cap \mathrm{Ker}(N_K^{i+1}) \cap \mathrm{Ker}(l_K^{j+1})$ with $x \neq 0$, where C denotes the Weil operator on the Hodge structure $L^{i,j}$. For $x \in P_i \subset L^{-i,j-n}$ with $x \neq 0$, we have

$$\begin{aligned} S_{j,\mathrm{prim}}(Cx \otimes N_K^i\overline{x}) &= \epsilon(j)Q_K(Cx \otimes l_K^{n-j}N_K^i\overline{x}) \\ &= \epsilon(j)(-1)^{i+j}Q_K(x \otimes Cl_K^{n-j}N_K^i\overline{x}) \\ &= (-1)^j\epsilon(j)\epsilon(-j)\,\langle\, x \otimes CN_K^i l_K^{n-j}\overline{x}\,\rangle \\ &= \epsilon(j+1)\epsilon(j+1)\,\langle\, x \otimes CN_K^i l_K^{n-j}\overline{x}\,\rangle \\ &= \langle\, x \otimes CN_K^i l_K^{n-j}\overline{x}\,\rangle > 0 \end{aligned}$$

as desired. □

8.15. Now the standard procedure (e.g. [18, Example 2.10]) gives us a polarization of the mixed Hodge structure $(\mathrm{H}^q(Y,K), W[q], F)$ as follows.

We have a direct sum decomposition

$$(\mathrm{H}^q(Y,K), W[q], F) = \bigoplus_{j\geq 0} l_K^j(\mathrm{H}^{q-2j}(Y,K)_{\mathrm{prim}}, W[q], F[-j]) \quad \text{for } q \leq n$$

$$(\mathrm{H}^q(Y,K), W[q], F) = \bigoplus_{j\geq 0} l_K^{q-n+j}(\mathrm{H}^{2n-q-2j}(Y,K)_{\mathrm{prim}}, W[q], F[n-q-j]) \quad \text{for } q \geq n$$

by Corollary 8.12 and by the fact that l_K is a morphism of mixed Hodge structures.

For the case of $q \leq n$, we define

$$S_q(x \otimes y) = \sum_{j\geq 0} S_{q-2j,\mathrm{prim}}(x_j \otimes y_j),$$

where $x = \sum_{j\geq 0} l_K^j x_j$ and $y = \sum_{j\geq 0} l_K^j y_j$ for some $x_j, y_j \in \mathrm{H}^{q-2j}(Y,K)_{\mathrm{prim}}$.

For the case of $q \geq n$, we set

$$S_q(x \otimes y) = \sum_{j\geq 0} S_{2n-q-2j,\mathrm{prim}}(x_j, y_j),$$

where $x = \sum_{j\geq 0} l_K^{q-n+j}x_j$ and $y = \sum_{j\geq 0} l_K^{q-n+j}y_j$ for some $x_j, y_j \in \mathrm{H}^{2n-q-2j}(Y,K)_{\mathrm{prim}}$.

Theorem 8.16. *The data*

$$(\mathrm{H}^q(Y,K), W[q], F, N_K, S_q)$$

is a polarized mixed Hodge structure over $\mathbb{Q}$ *in the sense of Cattani-Kaplan-Schmid* [2, Definition (2.26)].

References

[1] E. Cattani and A. Kaplan, *Degenerating variations of Hodge structure*, Théorie de Hodge (Luminy, Juin 1987), Astérisque, vol. 179-180, Soc. Math. France, (1989), pp. 67–96.

[2] E. Cattani, A. Kaplan, and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. **123** (1986), 457–535. DOI: 10.2307/1971333

[3] B. Conrad, *Grothendieck duality and base change*, Lecture Notes in Math., vol. 1750, Springer-Verlag, (2000). DOI: 10.1007/b75857

[4] P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math., vol. 163, Springer-Verlag, (1970).

[5] ———, *Théorie de Hodge* II, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–58. DOI: 10.1007/BF02684692

[6] T. Fujisawa, *Mixed Hodge structures on log smooth degenerations*, Tôhoku Math. J. (2) **60** (2008), 71–100. DOI: 10.2748/tmj/1206734407

[7] ———, *Polarizations on limiting mixed Hodge structures* (*announcement*), Higher Dimensional Algebraic Geometry (Shigeru Mukai and Noboru Nakayama, eds.), RIMS Kôkyûroku Bessatsu, vol. B24, Research Institute for Mathematical Sciences, Kyoto University, (2011), pp. 47–66.

[8] T. Fujisawa and C. Nakayama, *Mixed Hodge Structures on Log Deformations*, Rend. Sem. Mat. Univ. Padova **110** (2003), 221–268.

[9] T. Fujisawa and C. Nakayama, *Geometric log Hodge structures on the standard log point*, preprint

[10] P. Griffiths and W. Schmid, *Recent developments in Hodge theory*: *A discussion of techniques and results*, Discrete subgroups of Lie Groups and Applications to Moduli, Oxford Univ. Press, (1975), pp. 31–127.

[11] F. Guillén and V. Navarro Aznar, *Sur le théorème local des cycles invariants*, Duke Math. J. **61** (1990), 133–155. DOI: 10.1215/S0012-7094-90-06107-1

[12] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math., vol. 20, Springer-Verlag, (1966).

[13] M. Kashiwara and T. Kawai, *The Poincaré lemma for variations of polarized Hodge structures*, Publ. RIMS **23** (1987), 345–407. DOI: http://dx.doi.org/10.2977/prims/1195176545

[14] M. Kashiwara and P. Schapira, *Categories and sheaves*, Grundlehren der mathematischen Wissenschaften, vol. 332, Springer-Verlag, (2005).

[15] K. Kato and S. Usui, *Classifying Spaces of Degenerating Polarized Hodge Structures*, Annales of Math. Stud., vol. 169, Princeton Univ. Press, (2009).

[16] Y. Nakkajima, *Sings in weight spectral sequences, monodromy weight conjectures, log Hodge symmetry and degenerations of surfaces*, Rend. Sem. Mat. Univ. Padova **116** (2006), 71–185.

[17] ———, *Weight filtration and slope filtration on the rigid cohomology of a variety in characteristic* $p > 0$, Mém. Soc. Math. France, **130-131** (2012).

[18] C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge Structures*, A Series of Modern Surveys in Mathematics, vol. 52, Springer-Verlag, (2008).

[19] M. Saito, *Modules de Hodge Polarisable*, Publ. RIMS. **24** (1988), 849–921. DOI: 10.2977/prims/1195173930

[20] W. Schmid, *Variation of Hodge structure* : *the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319. DOI: 10.1007/BF01389674

[21] J. Steenbrink, *Limits of Hodge Structures*, Invent. Math. **31** (1976), 229–257. DOI: 10.1007/BF01403146

[22] ———, *Logarithmic embeddings of varieties with normal crossings and mixed Hodge structures*, Math. Ann. **301** (1995), 105–118. DOI: 10.1007/BF01446621

[23] S. Usui, *Mixed Torelli problem for Todorov surfaces*, Osaka J. Math. **28** (1991), 697–735.

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