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DEGENERATIONS OF INVARIANT LAGRANGIAN MANIFOLDS

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Abstract. We consider a pair (H, I) where I is an involutive ideal of a Poisson algebra and $H \in I$. We show that if I defines a "$2n$-gon singularity" then, under arithmetical conditions on H, any deformation of H can be integrated as a deformation of (H, I).

In memoriam V.I. Arnold.

Introduction

Let us consider a real analytic Hamiltonian system in the neighbourhood of an elliptic critical point in $\mathbb{R}^{2n}$:

$$H = \sum_{i=1}^{n} \alpha_i \tau_i + o(\|\tau\|)$$

with $\tau_i := p_i^2 + q_i^2$, $\tau = (\tau_1, \dots, \tau_n)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

If the α_i are $\mathbb{Q}$-independent then, we may assume, up to a symplectic change of variable, that H is of the form

$$H = \sum_{i \geq 0} \alpha_i \tau_i + \sum_{ij} \alpha_{ij} \tau_i \tau_j + o(\|\tau\|^2), \ \alpha_{ij} \in \mathbb{R}.$$

In 1963, Arnold investigated invariant tori for such a Hamiltonian system. He showed that any neighbourhood of the critical point has a positive measure set of invariant tori provided that the matrix (α_{ij}) is non-degenerate [2] (see also [1, 14, 18]).

Following Moser and Pöschel, one may look up for a parametric variant of Arnold's theorem regarding both the τ_i's and the frequency as new variables [22]. In this paper, I will show a result similar to the one proven by Arnold in this context (Theorem 3.3). The approach is however slightly different from that of Arnold, since it depends on finding a KAM type normal form at the singularity similar to the Birkhoff normal form in the formal case. The solution to the Herman invariant tori conjecture is a consequence of this result [9].

The real structure will be treated as an arbitrary anti-holomorphic involution, in particular, we will disregard the signature of the critical point. This means that the real parts of our complex invariant Lagrangian manifolds can be either n-dimensional tori or diffeomorphic to cylinders over some torus of dimension less than n.

The parametric variant of the KAM theorem was first introduced to simplify the proof using the fact that the parametric statement is equivalent to the original one. For degenerations, this is no longer the case and the difference can already be seen in the integrable case. The family of Lagrangian submanifolds

$$L_\varepsilon = \{(q, p) \in \mathbb{C}^{2n} : p_1^2 + q_1^2 = \varepsilon_1, \dots, p_n^2 + q_n^2 = \varepsilon_n\},$$

degenerates into a cone over a polytope with n vertices. Vey's theorem states that all integrable Hamiltonian system having this degeneration are isomorphic [25]. However, this theorem does not guarantee the stability for arbitrary Lagrangian deformations. A first attempt to prove this more general theorem was made by Nguyen D'uc and Pham, but they mistakenly used

Mather's preparation theorem in their proof [13, 16]. The differential relations involved in the definition of a Lagrangian manifolds do not permit to apply directly Mather-Thom theory. Using Sevenheck-van Straten's Lagrangian deformation complex, I was able to prove the stability of the deformation (L_ε) among Lagrangian deformations [7, 23] (see also [24]).

The result of this paper shows that not only the deformation (L_ε) is stable among Lagrangian singularities but that it is also stable as a degeneration of invariant Lagrangian manifolds in a Hamiltonian system.

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1. Lagrangian deformations

Deformations of Poisson algebras is a well established subject going back to the work of Lichnerowicz [15].

Let A be an algebra over $\mathbb{C}$. We say that A is a *Poisson algebra* if it is endowed of a linear antisymmetric biderivation

$$A \times A \longrightarrow A,\ (f, g) \mapsto \{f, g\}$$

which satisfies the Jacobi identity. The tensor product of a Poisson algebra A with an algebra B is a Poisson algebra for the bracket:

$$\{a_1 \otimes b_1, a_2 \otimes b_2\} := \{a_1, a_2\} \otimes (b_1 b_2).$$

It is called the *central extension* of A with respect to B and we say that A is a Poisson algebra *over* B. In the sequel, if A has a Poisson structure, then we implictly consider $A \otimes B$ with this central-extension Poisson structure.

The most standard examples of Poisson algebras over $\mathbb{C}$ are the algebras of polynomials $\mathbb{C}[q, p]$, formal power series $\mathbb{C}[[q, p]]$ and analytic power series $\mathbb{C}\{q, p\}$ in the $2n$ variables

$$q = (q_1, \dots, q_n),\ p = (p_1, \dots, p_n)$$

together with the symplectic Poisson structure

$$\{f, g\} = \sum_{i=1}^{n} \partial_{q_i} f \partial_{p_i} g - \partial_{q_i} g \partial_{p_i} f.$$

In this paper, we will only be concerned with central extensions of symplectic structures.

An ideal I of a Poisson algebra is called *involutive* if:

$$\{I, I\} \subset I.$$

and we consider flat deformations of involutive ideals inside Poisson algebra.

So if $f_1, \dots, f_n$ generates an involutive ideal I then there exists $c_{ij}^k \in A$ such that

$$\{f_i, f_j\} = \sum_{k \geq 0} c_{ij}^k f_k.$$

In practise, our ideals are complete intersection ideals, thus flat involutive deformations are simply deformations of functions generating the ideal which remain in involution.

If we start from a Liouville integrable system $f_1, \dots, f_n \in \mathbb{C}[q, p]$, the parametric point of view consists in replacing f by the involutive ideal I in $A := \mathbb{C}[q, p] \otimes \mathbb{C}[\tau]$, $\tau = (\tau_1, \dots, \tau_n)$ generated by the $f_i - \tau_i$'s.

Let now B be an algebra and consider the Poisson algebra $A \otimes B$. Let $\mathcal{I}$ be an involutive flat deformation of I over B. We say that it is a *trivial Poisson deformation* if there exists a Poisson automorphism $\varphi \in \operatorname{Aut}(A \otimes B)$ such that

$$\varphi(\mathcal{I}) = I \otimes 1.$$

We say that I is *rigid over* B if any of its deformation over B is trivial.

These notions extend naturally from polynomial rings to analytic ones. Let us denote by $\mathcal{O}_{\mathbb{C}^k,0}$ or by $\mathbb{C}\{x\}$ when we want to single out the variables, the algebra of convergent power series in the variable $x = (x_1, \dots, x_k)$.

This algebra has a natural topological structure as direct limit of Banach spaces (see [11]). Because the algebra $\mathcal{O}_{\mathbb{C}^k,0} \otimes \mathcal{O}_{\mathbb{C}^l,0}$ is not isomorphic to $\mathcal{O}_{\mathbb{C}^{k+l},0}$, in the sequel, we consider topological tensor products rather than usual tensor products (see [12]). As both $\mathcal{O}_{\mathbb{C}^k,0}\hat{\otimes}\mathcal{O}_{\mathbb{C}^l,0}$ and $\mathcal{O}_{\mathbb{C}^{k+l},0}$ are completions of the space of polynomials, they are isomorphic.

Similarly, if X and Y are compact spaces, by Stone theorem, multiplication gives an isomorphism of topological vector spaces between $C^0(X,\mathbb{R})\hat{\otimes}C^0(Y,\mathbb{R})$ and $C^0(X \times Y, \mathbb{R})$. These are the only properties of topological tensor products that we shall need and can be taken as a definition, if the reader is not comfortable with this notion.

Although deformation theory of involutive ideals in Poisson algebras can be established in great generality, only a few examples are understood. We now give the simplest non-trivial case.

Theorem 1.1 ([7], [13])**.** *Let us consider the ring of analytic power series in* $2n$*-variables* $\mathbb{C}\{q,p\}$ *together with its symplectic Poisson structure. The involutive ideal of* $\mathbb{C}\{\tau,q,p\}$ *generated by the polynomials* $p_1q_1 - \tau_1, \dots, p_nq_n - \tau_n$ *is rigid over* $\mathbb{C}\{t_1,\dots,t_k\}$ *for any* $k \geq 0$.

In concrete terms, if we take convergent power series

$$F_1, \dots, F_n \in \mathbb{C}\{t,\tau,q,p\}$$

with $F_i(t = 0, \tau, q, p) = p_iq_i - \tau_i$ that generate an involutive ideal then there exists a Poisson automorphism

$$\varphi : \mathbb{C}\{t,\tau,q,p\} \longrightarrow \mathbb{C}\{t,\tau,q,p\}$$

such that

$$\begin{pmatrix} \varphi(F_1) \\ \dots \\ \varphi(F_n) \end{pmatrix} = M \begin{pmatrix} p_1q_1 - \tau_1 \\ \dots \\ p_nq_n - \tau_n \end{pmatrix}$$

where M is an invertible $n \times n$ matrix with coefficients in $\mathbb{C}\{t,\tau,q,p\}$.

As shown by Miranda and Vu Ngoc, the analoguous statement is wrong in the C^∞ category [17].

2. Invariant Lagrangian ideals

KAM theory involves complicated algebras of functions over Cantor sets, so it is useful to have a formal point of view on symplectic and Poisson geometry which allows to deal with formal varieties and nilpotents.

Let A be a Poisson algebra over an algebra B. For any involutive ideal $I \subset A$ and any element $H \in I$, we have a derivation

$$A/I \longrightarrow A/I,\ f \mapsto \{H, f\}$$

and this derivation is unchanged if we add an element of I^2:

$$\{H + \sum_{i=1}^k a_ib_i, f\} = \{H, f\} + \sum_{i=1}^k a_i\{b_i, f\} + \sum_{i=1}^k b_i\{a_i, f\} = \{H, f\} \pmod I$$

for any $a_1, \dots, a_k,\ b_1, \dots, b_k \in I$. This shows that the conormal ideal I/I^2 is mapped to the derivations of A/I via the mapping

$$I/I^2 \longrightarrow \mathrm{Der}\,(A/I),\ H \mapsto \{H, -\}.$$

More generally, if $\{H, I\} \subset I$ then we say that I is *H-invariant*. The derivation $\{H, -\} \in \mathrm{Der}\,(A/I)$ is then defined by the class of H modulo the vector space $I^2 \oplus B$. In case, A is polynomial algebra, any H-invariant involutive ideal defines a family of invariant varieties for H parametrised by B but the notion is more general and holds for any Poisson algebra.

The simplest case of an invariant ideal is the ideal generated by H itself which corresponds to the conservation of energy. On the other extreme, if we start from a Liouville integrable system

$$H = f_1,\ f_2, \dots, f_n \in \mathbb{C}[q, p],\ \{f_i, f_j\} = 0,$$

then the ideal I generated by the $f_i - \tau_i$'s in $A := \mathbb{C}[\tau, q, p]$ is H-invariant and we get a commutative diagram where vertical arrows are isomorphisms

$$\begin{tikzcd} A/I \arrow[r, "\{H,\cdot\}"] \arrow[d] & A/I \arrow[d] \\ \mathbb{C}[q,p] \arrow[r, "\{H,\cdot\}"] & \mathbb{C}[q,p] \end{tikzcd}$$

From the parametric point of view, such an integrable system is a Lagrangian scheme over $\mathrm{Spec}\,(\mathbb{C}[\tau])$. In a joint work with van Straten, we used this formulation to unify Arnold-Liouville-Mineur theorems with Darboux-Weinstein ones [10]

Let us consider the case where A is the ring of formal power series $\mathbb{C}[[t, \tau, q, p]]$ in $4n$-variables with $t = (t_1, \dots, t_n)$. We use the grading where the degrees are:

$$\deg(q_i) = \deg(p_i) = 1,\ \deg(\tau_i) = 2,\ \deg(t_i) = 0$$

together with the associated filtration. We write $f = g + o(k)$ if $f - g$ contains only terms of degree higher than k. If $f = o(k)$ we say that it is of *order* k. We will also use the notation $P \prec k$ if $P \in \mathbb{C}[t, \tau, q, p]$ is a polynomial of weighted homogeneous degree less than k and $P \succcurlyeq k$ if it is of order at least k.

Note that the derivations of a ring graded by integers

$$A = \bigoplus_{i \in \mathbb{N}} A_i$$

is a graded module: *a derivation v is homogeneous of degree k* if it maps A_i to A_{i+k}. Similarly the derivations of a filtred ring

$$A = F^0 \subset F^1 \subset F^2 \subset \dots$$

is a filtred module: *a derivation v is of order k* if it maps F^i to F^{i+k}.

Given a ring of formal power series $A = \mathbb{C}[[x]]$, the exponential gives a mapping from the space of derivations, which vanish at the origin, to that of automorphisms. In case, the series is convergent it can be interpreted as the time one flow of the corresponding vector field.

If A is now a Poisson algebra, then the Hamiltonian derivations, i.e., the derivations which preserve the Poisson bracket, are mapped to Poisson mappings.

Proposition 2.1. *Let $H \in \mathbb{C}[[t, \tau, q, p]]$ be such that*

$$H(t, \tau, q, p) = \sum_{i=1}^{n} (\alpha_i + t_i) p_i q_i + o(2),\ \alpha_i \in \mathbb{C}$$

If the α_i's are linearly independent over $\mathbb{Q}$ then there exists a sequence of polynomial Hamiltonian derivations $u = (u_k)$ of the form

$$u_k = \{-, h_k\} + \sum_{i=1}^{k} g_i^k(t, \tau) \partial_{t_i}$$

with $2^k \preccurlyeq u_k \prec 2^{k+1}$ *such that the sequence*

$$H_{k+1} = e^{-u_k} H_k,\ H_0 = H$$

is of the form

$$H_k = \sum_{i=1}^n (\alpha_i + t_i) p_i q_i + o(2^{k+1} + 1) (\mathrm{mod}\, I^2 \oplus \mathbb{C}[[t, \tau]])$$

where $I \subset \mathbb{C}[[t, \tau, q, p]]$ *is the ideal generated by* $p_1 q_1 - \tau_1, \dots, p_n q_n - \tau_n$.

Proof. Write

$$H = \sum_{i=1}^n (\alpha_i + t_i) p_i q_i + R_k + o(2^{k+1} + 1),\ R_k \in \mathbb{C}[[t, \tau, q, p]].$$

with $2^k + 2 \preccurlyeq R_k \preccurlyeq 2^{k+1} + 1$. We prove the proposition by induction on k.

As the α_i's are linearly independent, we may find a Hamiltonian derivation

$$u_k = \sum_{i=1}^n a_i \partial_{t_i} + \{F, -\},\ a_i \in \mathbb{C}[[t, \tau]],$$

with $2^k \preccurlyeq u_k \prec 2^{k+1}$, and $S \in I^2 \oplus \mathbb{C}[[t, \tau]]$ such that:

$$u_k \left(\sum_{i=1}^n (\alpha_i + t_i) p_i q_i \right) = R_k + S.$$

As u_k is Hamiltonian, the automorphism e^{-u_k} is a Poisson automorphism and

$$e^{-u_k} H = \sum_{i=1}^n (\alpha_i + t_i) p_i q_i + S + o(2^{k+1} + 1).$$

This proves the proposition. □

The proposition implies that any power series $H \in \mathbb{C}[[t, \tau, q, p]]$ of the form

$$\sum_{i=1}^n (\alpha_i + t_i) p_i q_i + o(2)$$

with $[\mathbb{Q}(\alpha_1, \dots, \alpha_n) : \mathbb{Q}] = n$ admits an invariant ideal of dimension n over $\mathbb{C}[[t, \tau]]$ generated by power series of the form $p_i q_i - \tau_i + o(2)$, $i = 1, \dots, n$. Thus the Hamiltonian system admits a family of formal Lagrangian varieties parametrised by $\mathbb{C}[[t, \tau]]$.

Our main result is an analytic variant of this proposition. Due to Poincaré non-integrability theorem [20], it is hopeless to search for an analytic family so, as usual in KAM theory, we search for a family parametrised by some closed subset of positive measure.

Before proceeding to KAM theory, let us observe that the above proposition is a parametric variant of the Birkhoff normal form. Indeed, if H is reduced to its Birkhoff normal form:

$$H = B(p_1 q_1, \dots, p_n q_n),\ B \in \mathbb{C}[[X_1, \dots, X_N]].$$

Consider the natural embedding of algebras

$$\varphi : \mathbb{C}[[q, p]] \longrightarrow \mathbb{C}[[\tau, q, p]].$$

By Taylor's formula we have

$$\varphi(H) = \sum_{i=1}^n \partial_{\tau_i} B(\tau_1, \dots, \tau_n) p_i q_i \mod (I^2 \oplus \mathbb{C}[[\tau]])$$

or equivalently

$$\varphi(H) = \sum_{i=1}^{n} (\alpha_i + t_i) p_i q_i \mod (I^2 \oplus \mathbb{C}[[\tau]])$$

with $t_i = \partial_{\tau_i} B(\tau) - \partial_{\tau_i} B(0)$. Via the identification $\mathbb{C}[[\tau, q, p]]/I \approx \mathbb{C}[[q, p]]$, we get an equality between Hamiltonian derivations

$$\{-, H\} = \{-, \sum_{i=1}^{n} (\alpha_i + t_i) p_i q_i\}.$$

3. Statement of the theorem

We first define arithmetic classes.

Denote by $(\cdot, \cdot)$ the Euclidean scalar product in $\mathbb{C}^n$. For any vector $\alpha \in \mathbb{C}^n$, we define the sequence $\sigma(\alpha)$ by:

$$\sigma(\alpha)_k := \min\{|(\alpha, i)| : i \in \mathbb{Z}^n \setminus \{0\}, \|i\| \leq 2^k\}.$$

Definition 3.1. *The arithmetic class in $\mathbb{C}^n$ associated to a real decreasing sequence $a = (a_k)$ is the set*

$$\mathcal{C}(a) := \{\alpha \in \mathbb{C}^n : \forall k \geq 0, \sigma(\alpha)_k \geq a_k\}.$$

Although the arithmetic class depends on the dimension n, we do not specify it in our notation. Locally an arithmetic class might have zero measure around some point, the following elementary result provides a useful criterion to guarantee the construction of positive measure sets of invariant Lagrangian manifolds:

Proposition 3.2 ([8]). *Consider a real positive decreasing sequence $a = (a_k)$ and let $\rho = (\rho_k)$ be a real positive sequence such that $(2^{kn}\rho_k^{n-1})$ is summable. The density of the set $\mathcal{C}(\rho a)$ is equal to 1 at any point of $\mathcal{C}(a)$.*

For a locally closed subset $X \subset \mathbb{C}^n$ and a given $l \in \mathbb{N} \cup \{\infty\}$, we denote by C^l_X the sheaf of complex valued Whitney C^l functions on X [26]. As any direct limit of locally convex vector spaces, the stalk of C^l_X at a point is a locally convex space [4, 11]. There is a natural restriction mapping

$$r : \mathcal{O}_{\mathbb{C}^n} \longrightarrow C^\infty_X.$$

If E, F are topological vector spaces, we denote by $\mathcal{L}(E, F)$ the space of bounded linear mappings endowed with the strong topology.

We denote by $D^k_r \subset \mathbb{C}^k$ the polydisk of polyradius r centred at the origin, and by $\mathcal{O}_{\mathbb{C}^k}$ the sheaf of holomorphic function on $\mathbb{C}^k$. The main result of this paper is the following KAM version of Proposition 2.1:

Theorem 3.3. *Let $I \subset \mathbb{C}\{\tau, q, p\} \hat{\otimes} \mathcal{O}_{\mathbb{C}^n,\alpha}$ be the involutive ideal generated by the $p_i q_i - \tau_i$'s and consider a holomorphic function of the type*

$$H = \sum_{i=1}^{n} (\alpha_i + t_i) p_i q_i + o(2) \in \mathbb{C}\{\tau, q, p\} \hat{\otimes} \mathcal{O}_{\mathbb{C}^n,\alpha}.$$

There exists a sequence of polynomial Hamiltonian derivations $u = (u_k)$ with $2^{k-1} \preccurlyeq u_k \prec 2^k$ depending only on H with the following property. For any decreasing positive sequence $a = (a_i)$ such that

$$\sum_{i \geq 0} \frac{\log a_i}{2^i} > -\infty$$

and $\alpha \in \mathbb{C}^n$;

1) *the sequence* $\varphi_k = (e^{u_k} \dots e^{u_0})$ *is well-defined and converges in*

$$\mathcal{L}(\mathbb{C}\{\tau, q, p\} \hat{\otimes}\, \mathcal{O}_{\mathbb{C}^n,\alpha},\, \mathbb{C}\{\tau, q, p\} \hat{\otimes}\, C^\infty_{X,\alpha})$$

to a Poisson morphism φ *such that*

$$\varphi(H) = \sum_{i=1}^{n} (\alpha_i + t_i) p_i q_i \ (\mathrm{mod}\, (r(I^2) \oplus \mathbb{C}\{\tau\} \hat{\otimes}\, C^\infty_{X,\alpha});$$

2) *if* f *is holomorphic in* $(\alpha + D_r^n) \times D_r^{3n}$ *then* $\varphi(f) \in \mathcal{O}_{\mathbb{C}^{3n}}(D_s^{3n}) \hat{\otimes} C^\infty_{X,\alpha}(D_s^n \cap \mathcal{C}(a))$ *with*

$$s = 2^{-10n-40} \left(\prod_{i \geq 0} a_i^{2^{-i}} \right)^5 r;$$

3) *if* H *is real for some antiholomorphic involution then* u *can also be chosen real.*

Note that the condition

$$\sum_{i \geq 0} \frac{\log a_i}{2^i} > -\infty$$

ensures that $\prod_{i \geq 0} a_i^{2^{-i}}$ is finite.

The sequence u_k will be constructed explicitly. The theorem implies that the ideal $r(I)$ is invariant for $\varphi(H)$. In particular, the Hamiltonian flow of a representative of H admits a family of invariant Lagrangian manifolds parametrised by a neighbourhood of $(0, \alpha) \in \mathbb{C}^n \times \mathcal{C}(a)$. This set might of course be empty and there are additional conditions, such as Proposition 3.2, under which it will have positive measure.

Regularity results show that it is sufficient to prove the variant of the theorem where we replace $C^\infty_{X,\alpha}$ by $C^0_{X,\alpha}$ (see Appendix).

If we start with a real analytic function H having an elliptic critical point at the origin, the real parts of these invariant Lagrangian manifolds define a family of n-dimensional tori which degenerate. In other words, any representative of

$$H = \sum_{i=1}^{n} (\alpha_i + t_i)(p_i^2 + q_i^2) + o(2) \in \mathbb{C}\{t, \tau, q, p\}$$

admits a positive measure set of invariant tori if the vector $(\alpha_1, \dots, \alpha_n)$ belongs to some arithmetic class which satisfies the condition of the theorem. Note that if we assume H to be real but not necessarily elliptic, the real parts of these complex invariant Lagrangian manifolds are diffeomorphic to cylinders

$$\mathbb{R}^j \times (S^1)^k,\ j + k = n,$$

and $k = n$ in the elliptic case.

In the formulation of the theorem, the fact that H depends, in first approximation, linearly on t is similar to the isochronous non-degeneracy condition of the standard KAM theorem. As we shall now see, the above theorem is a consequence of abstract KAM theory constructed in [6].

4. The category Kolmogorov spaces

Definition 4.1. *An* S*-Kolmogorov space is a directed system of Banach spaces* $E = (E_s, |\cdot|_s)$ *indexed by the interval* $]0, S]$ *such that the maps of the directed system are injective with norm at most one.*

A *graded morphism* of S-Kolmogorov spaces

$$u : E \longrightarrow F$$

is a linear continuous map, which commutes with the morphisms of the directed systems, such that for each t, there exists a unique $\phi(t)$ with

$$u(E_t) \subset F_s$$

and the map ϕ is non-decreasing.

This means that for any $t_1 > t_2 > 0$, we have a commutative diagram of Banach spaces

$$\begin{tikzcd} E_{t_1} \arrow[r] \arrow[d, "u"'] & E_{t_2} \arrow[d, "u"] \\ F_{s_1} \arrow[r] & F_{s_2} \end{tikzcd}$$

A morphism of Kolmogorov spaces is a finite sum of graded morphisms. This defines the *category of Kolmogorov spaces*.

We extend the norm of E_s to E as follows : for

$$x = x_1 + \cdots + x_n \in \bigoplus_{i=1}^{n} E_{t_i},$$

we define

$$|x|_s := \begin{cases} |f_{t_1 s}(x_1) + \cdots + f_{t_n s}(x_n)|_s & \text{if } \mathrm{t_i \geq s},\ \forall i \\ +\infty & \text{otherwise.} \end{cases}$$

Unless specific mention, we denote the norms in a Kolmogorov space by $|\cdot|_s$ without specifying the space E in the notation. For instance, given a map

$$u : E \longrightarrow F$$

we write $|x|_s$ and $|u(x)|_s$ rather than $|x|_{E,s}$ and $|u(x)|_{F,s}$.

A Kolmogorov space E admits many types of decreasing filtrations. The most simple one, that we will call the *canonical filtration*, is given by the vector subspaces

$$E^{(k)} = \{x \in E : \exists C, \tau,\ |x|_s \leq C s^k,\ \forall s \leq \tau\},\ k \in \mathbb{R}_{\geq 0}$$

which, as we shall see, is a generalisation of the filtration of the ring of convergent power series by powers of its maximal ideal.

The following two definitions give an abstract form of differential operators.

Definition 4.2 ([6]). *Let E, F be S-Kolmogorov spaces. A complete morphism is a family of graded morphisms*

$$u_\lambda : E \longrightarrow F,\ \lambda \in]0,1[$$

such that we have commutative diagrams

$$\begin{tikzcd} E_t \arrow[r] \arrow[d, "u_\lambda"'] & E_s \arrow[d, "u_\lambda"] \\ F_{\lambda t} \arrow[r] & F_{\lambda s} \end{tikzcd}$$

and

$$\begin{tikzcd} E_t \arrow[d, "u_\lambda"'] \arrow[dr, "u_\mu"] & \\ F_{\lambda t} \arrow[r] & F_{\mu t} \end{tikzcd}$$

for any $s < t \leq S$ *and* $\mu < \lambda$.

We abusively write complete morphisms as morphisms and not as families: $E \xrightarrow{u} F$.

Definition 4.3 ([6]). *Let* E, F *be Kolmogorov spaces. A complete morphism* u *is called* k*-bounded if there exists a real number* $C > 0$ *such that:*

$$|u(x)|_s \leq \frac{C}{(t-s)^k}|x|_t, \text{ for any } \; s < t \leq S, \; x \in E_t.$$

For simplicity, we will assume that, for $k = 0$, the condition also holds for $s = t$, so that u maps E_t to F_t. This assumption is unessential but it simplifies some of the notations.

We denote by $\mathcal{B}^k(E, F)_\tau$ the space of complete k-bounded morphisms from $E[\tau]$ to $F[\tau]$.

Let $|u|_\tau$ the smallest constant C which satisfy the estimate in Definition 4.3 divided by e^2:

$$|u|_\tau := \sup\{(t-s)^k \frac{e^2|u(x)|_s}{|x|_t} : s < t \leq \tau, x \in E_t\}.$$

The map

$$\mathcal{B}^k(E, F)_\tau \longrightarrow \mathbb{R}_+, \; u \mapsto |u|_\tau$$

defines a Banach space structure on $\mathcal{B}^k(E, F)_\tau$. Moreover there is a natural restriction mapping

$$\mathcal{B}^k(E, F)_\tau \longrightarrow \mathcal{B}^k(E, F)_\sigma, \; \sigma \leq \tau,$$

thus we have defined an S-Kolmogorov space denoted by $\mathcal{B}^k(E, F)$ and called the *space of* k*-bounded morphisms* [6].

5. The Kolmogorov spaces C_n^ω, $L_n^{p,\omega}$.

Definition of C_n^ω. For any open subset $U \subset \mathbb{C}^n$, the space of holomorphic functions in U is endowed with the topology of uniform convergence on compact subsets of U. If the open subset U contains the origin, we get a restriction mapping

$$r : \mathcal{O}_{\mathbb{C}^n}(U) \longrightarrow \mathcal{O}_{\mathbb{C}^n,0}.$$

Such mappings induce a direct limit topology on $\mathcal{O}_{\mathbb{C}^n,0}$ (see [11] for details).

We now construct a smaller directed system of vector spaces having $\mathcal{O}_{\mathbb{C}^n,0}$ as limit. We denote by $(C_n^\omega)_s$, $s \in]0, 1]$, the vector space of continuous functions in the polydisc

$$D_s := \{z \in \mathbb{C}^n : \sup_{i=1,\dots,n} |z_i| \leq s\}$$

which are holomorphic in its interior

$$(C_n^\omega)_s = \mathcal{O}(\mathring{D}_s) \cap C^0(D_s, \mathbb{C}).$$

The C^0-norm

$$|f|_s := \sup_{z \in D_s} |f(z)|$$

endows $(C_n^\omega)_s$ of a Banach space structure. The inclusion $D_s \subset D_t, t > s$ induces a directed system

$$(C_n^\omega)_t \longrightarrow (C_n^\omega)_s$$

which forms a Kolmogorov space. This directed system is of course standard [5, 19]. On can define in a similar way the Kolmogorov spaces $C^{k,\omega}$ by replacing continuous functions by k-differentiable ones and taking the C^k-norm.

Definition of $L_n^{p,\omega}$. We endow $\mathbb{C}^n \approx \mathbb{R}^{2n}$ with the measure $\pi^{-n/2}dV$ where dV is the Euclidean volume and consider the Kolmogorov spaces $L_n^{p,\omega}$:

$$(L_n^{p,\omega})_s := \Gamma(\operatorname{int}(D_s), \mathcal{O}_{\mathbb{C}^n}) \cap L^p(D_s, \mathbb{C})$$

with $s \in]0,1]$.

For $p = 2$, each of these spaces has a Hilbert space structure defined by the hermitian form

$$(f, g) \mapsto \frac{1}{\pi^n} \int_{D_s} f(z)\bar{g}(z)dV,$$

In this Kolmogorov space, the projection on a closed vector subspace is 0-bounded. We now wish to extend this property to the Kolmogorov space C_n^ω.

Identity morphisms. As any continuous function on a compact set is integrable, we get a canonical mapping:

$$I : C_n^\omega \longrightarrow L_n^{2,\omega},\ f \mapsto f.$$

This mapping can be distinguished from the identity only because the source and target spaces are different Kolmogorov spaces. The direct limit functor sends the map I to the identity map of $\mathbb{C}\{z\}$. A morphism or more generally a family of complete morphisms induced by restriction mappings will be called an *Identity morphism.*

As

$$\left(\frac{1}{\pi^n} \int_{D_s} |f(z)|^2 dV\right)^{1/2} \leq \sup_{z \in D_s} |f(z)|\, s^n,$$

this identity morphism I is 0-bounded and

$$|I|_\tau = \tau^n.$$

In particular as $\tau \leq 1$, we have $|I|_\tau \leq 1$.

Conversely, take a function $f \in (L_n^{2,\omega})_t$ and $s < t$. As f is holomorphic inside the disk of radius t, it is in particular continuous thus we have

$$f_{|D_s} \in (C_n^\omega)_s.$$

Consequently the inclusions

$$D_{\lambda s} \subset D_s,\ \lambda \in]0,1[$$

induce restriction mappings

$$(L_n^{2,\omega})_s \longrightarrow (C_n^\omega)_{\lambda s}$$

and a family of complete morphisms

$$I^{-1} : L_n^{2,\omega} \longrightarrow C_n^\omega$$

which is also an identity morphism.

Proposition 5.1. *The identity morphism I^{-1} is 1-bounded of norm at most equal to one.*

Proof. Take $f \in (L_n^{2,\omega})_t$, the Taylor expansion at $w \in D_s$ with $s < t$ gives

$$f(z) = \sum_{j \geq 0} a_j (z - w)^j,\ a_j \in \mathbb{C}.$$

Now let Γ_w be the polydisk centred at w with radius $\sigma = t - s$. We have

$$\frac{1}{\pi^n} \int_{\Gamma_w} |f(z)|^2 dV = \sum_{j \geq 0} |a_j|^2 \sigma^{2j+2}$$

and

$$\frac{1}{\pi^n}\int_{\Gamma_w}|f(z)|^2dV \leq \frac{1}{\pi^n}\int_{D_t}|f(z)|^2dV = |f|_t^2$$

This shows that

$$|f(w)| = |a_0| \leq \frac{1}{\pi^n\sigma}\left(\int_{\Gamma_w}|f(z)|^2dV\right)^{1/2} \leq \sigma^{-1}|f|_{s+\sigma}$$

for any $w \in D_s$ and proves the proposition. □

Corollary 5.2. *Let F be a closed subspace of the Kolmogorov space C_n^ω compatible with the maps of the directed system:*

$$\begin{array}{ccc} F_t & \longrightarrow & F_s \\ \downarrow & & \downarrow \\ (C_n^\omega)_t & \longrightarrow & (C_n^\omega)_s \end{array}$$

for any $t > s$. There exists a one-bounded projection $C_n^\omega \longrightarrow F$.

Proof. Denote by $\widetilde{F}$ the completion of F in $L_n^{\omega,2}$ and consider the orthogonal projection

$$\pi : L_n^{\omega,2} \longrightarrow \widetilde{F}.$$

For any $s < t$, we define a one-bounded projection from C_n^ω to F using the commutative diagram:

$$\begin{array}{ccc} (C_n^\omega)_t & \longrightarrow & F_s \\ {\scriptstyle I}\downarrow & & \uparrow{\scriptstyle I^{-1}_{|\widetilde{F}_t}} \\ (L_n^{\omega,2})_t & \xrightarrow{\pi} & \widetilde{F}_t \end{array}$$

□

Definition of $L_n^{\infty,\omega}$. Let $(L_n^{\infty,\omega})_s$ be the space of convergent power series in $z_1, \dots, z_n$ such that the quantity

$$|f|_s = \sup_i |a_i|s^{|i|}, |i| := i_1 + i_2 + \cdots + i_n$$

is finite. This defines a Kolmogorov space, indexed by $]0, 1]$, that we denote by $L_n^{\infty,\omega}$. Recall that a series is called of *order* N if, in its Taylor expansion, all terms of degree less than N vanish. Our aim is to relate approximations by Taylor series with norms in Kolmogorov spaces.

Lemma 5.3. *If $f \in L_n^{\infty,\omega}$ is of order N then*

$$|f|_s \leq |f|_{s+\sigma}\left(\frac{s}{s+\sigma}\right)^N$$

Proof. Put $f = \sum_{|i|\geq N} a_i z^i$, we have

$$|a_i|s^{|i|} = |a_i|(s+\sigma)^{|i|}\left(\frac{s}{s+\sigma}\right)^{|i|} \leq |f|_{s+\sigma}\left(\frac{s}{s+\sigma}\right)^N$$

□

This lemma relates the filtration by the maximal ideal with the L^∞-norms. We now wish to have a similar result for C_n^ω. This will be obtained by studying the identity morphism between this spaces.

For this, we consider the identity morphism

$$I : C_n^\omega \longrightarrow L_n^{\infty,\omega}.$$

This identity morphism factors through the identity morphism
$$C_n^\omega \longrightarrow L_n^{2,\omega}.$$
It is therefore 0-bounded of norm at most equal to 1.

Proposition 5.4. *The identity morphism*
$$I^{-1} : L_n^{\infty,\omega} \longrightarrow C_n^\omega$$
is n-bounded with norm at most equal to 1*.*

Proof. Take $f \in (L_n^{\infty,\omega})_t$ and choose $z \in D_s$ with $t > s$. We have:
$$|f(z)| \leq \sum_i |a_i| s^{|i|} = \sum_i |a_i| t^{|i|} \left(\frac{s}{t}\right)^{|i|}.$$
Using the estimate
$$|a_i| t^{|i|} \leq |f|_t,$$
we get that
$$|f(z)| \leq \left(\frac{t}{t-s}\right)^n |f|_t \leq \frac{1}{(t-s)^n} |f|_t.$$
□

As the filtration by the maximal ideal in C_n^ω coincides with the canonical one (see Section 4), Lemma 5.3 and Proposition 5.4 show that

Corollary 5.5. *For any $N > 0$ and any $f \in (C_n^\omega)_t^{(2^N)}$ and $s < t \leq 1$ we have:*
$$|f|_s \leq \frac{1}{(t-s)^n} |f|_t \left(\frac{s}{t}\right)^{2^N}.$$

6. Arnold spaces

Definition 6.1. *An S pre-Arnold space $E_\bullet$ is a product of S-Kolmogorov spaces indexed by $\overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$:*
$$E_\bullet := \prod_{i \in \overline{\mathbb{N}}} E_i.$$

As for Kolmogorov spaces, we sometimes omit the index S. We endow pre-Arnold spaces of the product topology.

A map of pre-Arnold spaces $u_\bullet : E_\bullet \longrightarrow F_\bullet$ is a *morphism* if

i) $u_\bullet(E_i) \subset F_i, \forall i \in \mathbb{N}$;
ii) the map $u_\bullet$ induces morphisms of Kolmogorov spaces $u_i : E_i \longrightarrow F_i$.

This defines the *category of pre-Arnold spaces.* In pre-Arnold spaces, τ-morphisms and k-bounded τ-morphism are defined componentwise. We define the *norm* of a k-bounded τ-morphism
$$u_\bullet : E \longrightarrow F$$
as the sequence
$$|u_\bullet|_\tau := |u_i|_\tau.$$
So it is not a norm in the usual sense of it, but rather a sequence of norms.

Filtrations defined for Kolmogorov spaces extend naturally to pre-Arnold spaces, for instance:
$$x_\bullet \in E_\bullet^{(k)} \iff x_i \in (E_i)^{(k)}, \ \forall i \in \mathbb{N}.$$

Definition 6.2. *An S-Arnold space $E_\bullet$ (or simply an Arnold space) is a pre-Arnold space together with* 0*-bounded morphisms*

$$r_{ij} : E_i \longrightarrow E_j,\ i,j \in \mathbb{N} \cup \{\infty\},\ i < j$$

with norm at most one.

So an Arnold space is a special type of directed system of Banach spaces indexed by the product of $\mathbb{N} \cup \{\infty\}$ with an interval.

Note that E_∞ is the limit of the directed system of Kolmogorov spaces (E_i), $k \in \mathbb{N} \cup \{\infty\}$.

The maps r_{ij} are called *restriction morphisms*. For simplicity, they are assumed to be 0-bounded, this condition can be relaxed by k-bounded for arbitrary $k \geq 0$.

The *category of Arnold spaces* is the full subcategory of pre-Arnold spaces having for objects Arnold spaces. In particular, a morphism of Arnold spaces does NOT necessarily commute with restriction mappings.

Here are some conventions to simplify the notations:

i) we use the notation r for the restriction map from E_i to E_∞ instead of $r_{i\infty}$;
ii) for $x \in (E_i)_s$, we denote its norm simply by $|x|_s$;
iii) given morphisms $u : E_i \longrightarrow E_i$, $v : E_{i+j} \longrightarrow E_{i+j}$ we write vu for the composed map $v\, r_{i+j}\, u$.

Note that there is a natural functor from the category of Kolmogorov spaces to that of Arnold spaces by taking the product with itself

$$F : KS \longrightarrow AS,\ E \mapsto E_\bullet := E \times E \times E \times \cdots$$

where restriction mappings are simply all equal to the identity.

The Arnold space $C^\omega_\alpha(a)_\bullet$. We now fix a real positive decreasing sequence $a = (a_j)$ bounded by 1. We define the closed subsets

$$\mathcal{C}(a)_i := \{\alpha \in \mathbb{C}^n : \sigma(\alpha)_j \geq a_j,\ \forall j \leq 2^i\},\ i \in \mathbb{N} \cup \{\infty\}$$

so that $\mathcal{C}(a)$ is equal to $\mathcal{C}(a)_\infty$.

Take $\alpha \in \mathcal{C}(a)$ and let $D_s(\alpha) \subset \mathbb{C}^n$ be the closed polydisk of radius s centred at α. Put

$$K_{i,s} = \mathcal{C}((1-s)a)_i \cap D_s(\alpha).$$

The parametrisation is chosen so for any $x \in K_{i,s}$ and any $t > s$, the polydisk of radius $a_i/2^i(t-s)$ is contained inside $K_{i,t}$ (we will use this property in the Appendix).

We now define the space $C^\omega_\alpha(a)_{i,s}$ by taking continuous on $K_{i,s}$ and holomorphic in its interior:

$$C^\omega_\alpha(a)_{i,s} = \mathcal{O}(\mathring{K}_{i,s}) \cap C^0(K_{i,s}).$$

The supremum norm induces a Banach space structure on $C^\omega_\alpha(a)_{i,s}$.

For each i, the inclusions $K_{i,s} \subset K_{i,t}, t > s$, $K_{i+1,s} \subset K_{i,s}$ induce a doubly directed system

$$\begin{tikzcd} \cdots \arrow[r] & C^\omega_\alpha(a)_{i,t} \arrow[r] \arrow[d] & C^\omega_\alpha(a)_{i+1,t} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & C^\omega_\alpha(a)_{i,s} \arrow[r] & C^\omega_\alpha(a)_{i+1,s} \arrow[r] & \cdots \end{tikzcd}$$

In this way, we defined the *Arnold space* $C^\omega_\alpha(a)_\bullet$ which generalises the construction given by Arnold in his proof of the KAM theorem [1]. If $\alpha \in \mathbb{C}^n$ is the origin and a is the zero sequence then

$$C^\omega_\alpha(a)_\bullet = C^\omega_n.$$

There are, of course, Arnold spaces $C^{l,\omega}_{\alpha}(a)$ obtained by replacing C^0-norms $|\cdot|^0_{i,s}$ by C^l-norms $|\cdot|^l_{i,s}$:

$$|f|^l_{i,s} := \max_{|j|\leq l} \frac{1}{|j|!}|\partial^j f|^0_{i,s}$$

where $j = (j_1, \dots, j_n)$ and $|j| = j_1 + j_2 + \cdots + j_n$.

7. The abstract KAM theorem

We say that an increasing real positive sequence $p := (p_i)$ is *tamed* if

$$\sum_{i\geq 0} \frac{\log p_i'}{2^i} < +\infty,\ \ p_i' := \max(1, p_i).$$

This condition was introduced by Brjuno in the context of Diophantine approximation for linearising vector fields [3].

Definition 7.1. *A k-bounded morphism of Arnold spaces*

$$u_\bullet = (u_i) : E_\bullet \longrightarrow F_\bullet$$

is called k-tamed if the sequence $|u_\bullet|_\tau = (|u_i|_\tau)$ is tamed.

We denote by $\mathcal{M}^k(E_\bullet, F_\bullet)$ the vector space of k-tamed morphisms and for $E_\bullet = F_\bullet$, we write $\mathcal{M}^k(E_\bullet)$ instead of $\mathcal{M}^k(E_\bullet, E_\bullet)$. It is a vector subspace of k-bounded morphisms, therefore it admits a natural filtration:

$$\mathcal{M}^k(E_\bullet, F_\bullet) \subset \mathcal{M}^k(E_\bullet, F_\bullet)^{(1)} \subset \mathcal{M}^k(E_\bullet, F_\bullet)^{(2)} \subset \cdots$$

Observe that we have an inclusion of vector spaces:

$$\mathcal{M}^k(E_\bullet, F_\bullet) \subset \mathcal{B}^k(E_\bullet, F_\bullet)$$

for any $k \geq 0$.

The above definition can be extended to polynomials and more generally to arbitrary mappings:

Definition 7.2. *A map of Arnold spaces (non necessarily linear):*

$$f : E_\bullet \longrightarrow F_\bullet$$

is k-tamed by a tamed sequence (p_i) if:

$$|x|_{i,t} \leq R(t-s)^k \implies |f(x)|_{i,s} \leq Rp_i$$

for any $R \geq 1$.

We now define approximated inverses of linear mappings in Arnold spaces. To do this, we first introduce a new filtration on a Kolmogorov space: the *harmonic filtration*. It generalises both the filtration by Fourier harmonics for functions on a torus and the filtration by the degree for the Taylor expansion of a germ (cf. Corollary 5.5).

This filtration depends on the choice of $d \geq 0$. For a given Kolmogorov space E, the terms of the filtration are defined by

$$\mathcal{H}_d(E_i) = \{x \in E_i : |x|_s \leq \frac{1}{(t-s)^d}\left(\frac{s}{t}\right)^{2^i} |x|_t,\ \forall s < t\}.$$

We may now proceed to the definition of right quasi-inverses (the definition for left quasi-inverses is similar but we do not need it):

Definition 7.3. *Let $u : E_\bullet \longrightarrow F_\bullet$ be a morphism of Arnold spaces. A right d-quasi inverse of a morphism $u : E_\bullet \longrightarrow F_\bullet$ is a morphism $v : F_\bullet \longrightarrow E_\bullet$ such that*

$$y - uv(y) \in \mathcal{H}_d(F_i), \text{ for all } y \in F_i.$$

There is a natural notion of bounded splitting in an Arnold space that we shall use in the formulation of the theorem: if $E_\bullet$ is an Arnold space, we say that $(F_\bullet, G_\bullet)$ define a *k-bounded splitting of $E_\bullet$* if

i) $E_\bullet$ is the direct sum of $F_\bullet$ and $G_\bullet$;
ii) the subspaces $F_\bullet$ and $G_\bullet$ are closed ;
iii) the projections on each factor π_F, π_G are k-bounded with norm at most one.

Theorem 7.4 ([6])**.** *Let $E_\bullet$ be an Arnold space, $M_\bullet$ a closed subspace of $E_\bullet$, $(F_\bullet, G_\bullet)$ a bounded splitting of $M_\bullet$. Consider a vector subspace $\mathfrak{g}_\bullet \subset \mathcal{M}^1(E_\bullet)^{(2)}$ and let $H \in (E_0)_t$ be such that $\mathfrak{g}_\bullet$ maps $H + F_\bullet$ in $G_\bullet$. We consider the linear maps*

$$\rho(\alpha) : \mathfrak{g}_\bullet \longrightarrow G_\bullet, u \mapsto u(H + \alpha)$$

with $\alpha \in F_\bullet$ and assume that there is a right quasi-inverse $j(\alpha)$ to $\rho(\alpha)$. Denote by π_F, π_G m-bounded projections on F, G for some $m \geq 0$. Take $R \in (M_0)_t$ and define inductively $\beta_\bullet \in E_\bullet$, $u_\bullet \in \mathfrak{g}_\bullet$ by putting

$$\beta_0 = R, \; u_0 = j(0)(\beta_0);$$

and

$$\begin{cases} \beta_{i+1} &= & e^{-u_i}(a_i + \beta_i) - a_{i+1}; \\ u_{i+1} &= & j(\sum_{i=0}^{n} \alpha_i)\,(\pi_G(\beta_{i+1})) \end{cases}$$

where

$$\begin{cases} \alpha_i &= & \pi_F(u_i(a_i) - \beta_i); \\ a_{i+1} &= & a_i + \alpha_i. \end{cases}$$

Assume moreover that

A) $\beta_i \in \mathcal{H}_d(E_i)$
B) *for some $k \geq 0$, $j(\alpha) \in \mathcal{M}^k(G_\bullet, \mathfrak{g}_\bullet)$;*
C) *the map $j : F_\bullet \longrightarrow \mathcal{M}^k(G_\bullet, \mathfrak{g}_\bullet)$, $\alpha \mapsto j(\alpha)$ is l-tamed;*

then for any $R \in M_0$ and any $A \in]1, 2[$, there exists a constant $\mu > 0$ depending only on $H, R, d, k, l, m,$ and converging sequences $\alpha_\bullet \in F_\bullet$, $\beta_\bullet \in M_\bullet$ such that the morphism $u_\bullet = j(\alpha_\bullet)(\beta_\bullet)$ satisfies

i) $|u_i|_s < \mu e^{-A^i} s$ *for any* $s < 2^{-10(d+k+l+m+1)} (\prod_{i \geq 0} p_i^{-2^{-n}})^5 t$;
ii) $g(H + R) = r(H) (\operatorname{mod} F_\infty)$ *where g is the limit of the sequence*

$$(re^{u_i} e^{u_{i-1}} \dots e^{u_1} e^{u_0})_{i \in \mathbb{N}} \subset \mathcal{L}(E_0, E_\infty).$$

Our main theorem (Theorem 3.3) is a consequence of the abstract KAM theorem to the following spaces of functions in the $4n$ variables (t, τ, q, p):

$$E_\bullet := C^\omega_\alpha(a)_\bullet \hat{\otimes} C^\omega_{3n}, \; M_\bullet = E^{(3)}_\bullet, \; F_\bullet := E^{(3)}_\bullet \cap \left(I^2 \oplus (C^\omega_n \hat{\otimes} C^\omega_\alpha(a)_\bullet) \right)$$

The subspace $\mathfrak{g}_\bullet$ consists of sequences of Hamiltonian derivations. Corollary 5.5 implies condition A). So, to prove the theorem, it remains to construct a quasi-inverse j to the infinitesimal action ρ.

8. Construction of the quasi-inverse

Let us denote by R the Arnold space $C^\omega_\alpha(a)_\bullet \hat{\otimes} C^\omega_n$ with coordinates functions $t_1, \dots, t_n$, $\tau_1, \dots, \tau_n$.

We define a complement $G_\bullet$ to $F_\bullet$ as a sum of three R-modules

$$G_\bullet = A_\bullet \oplus B_\bullet \oplus C_\bullet$$

with (we use multi-index notations):

(1) $A_\bullet = \overline{\bigoplus_{i=1}^n R(p_i q_i - \tau_i)}$;
(2) $B_\bullet = \overline{\bigoplus_{I-J\neq 0} R p^I q^J}$;
(3) $C_\bullet := \bigoplus_{i=1}^n (p_i q_i - \tau_i) B_\bullet$.

where $\overline{\cdot}$ denotes the closure.

According to Corollary 5.2, $(F_\bullet, G_\bullet)$ define a 1-bounded splitting. Let us denote by $\star$ the Hadamard product for series, that is, the series obtained by taking the products coefficientwise:

$$\left(\sum_{i\geq 0} a_i z^i\right) \star \left(\sum_{i\geq 0} b_i z^i\right) = \sum_{n\geq 0} a_i b_i z^i$$

Lemma 8.1. *The Hadamard products*

$$\star h_\bullet : R \hat{\otimes} L^{2,\omega}_n \longrightarrow R \hat{\otimes} L^{2,\omega}_n$$

with the functions

$$h_k(\alpha, q, p) := \sum_{\|i-j\|=1}^{2^k} \frac{1}{(\alpha, i-j)} q^i p^j, \ a_i \in \mathbb{C}$$

define a 0*-tamed morphism whose norm is bounded from above by the sequence* a^{-1}.

Proof. Write

$$f = \sum_{i\geq 0} f_{ij} q^i p^j, \ f_{ij} \in R.$$

We have

$$f \star h_k(q,p) = \sum_{\|i-j\|=1}^{2^k} \frac{f_{ij}}{(\alpha, i-j)} q^i p^j,$$

thus

$$|f \star h_k|_{k,s} \leq \frac{1}{\sigma(\alpha)_k} \sqrt{\sum_{\|i-j\|=1}^{2^k} |f_{ij}|^2_{k,s} |q^i p^j|_s} \leq \frac{1}{a_k} |f|_{k,s}.$$

This proves the lemma. □

In the decomposition $A_\bullet \oplus B_\bullet \oplus C_\bullet$, the operator $\rho_\bullet(f)$ admits the lower triangular decomposition

$$\begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \star g & 0 \\ 0 & \{-, f\} & \star g \end{pmatrix}$$

with

$$g := \sum_{i \neq j} (\alpha + t, (i-j)) q^i p^j.$$

As the matrix is lower triangular, we can compute the inverse explicitly over formal power series. We truncate these series and define a right quasi-inverse $j_\bullet(f)$ to $\rho(f)$ by putting

$$j_k(f) := \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mu_k & 0 \\ 0 & \mu_k\{-,f\}\mu_k & \mu_k \end{pmatrix}$$

where $\mu_k = - \star h_k$. Corollary 5.5 shows that it is indeed a quasi-inverse to $\rho(f)$.

Lemma 8.1 and Proposition 5.1 imply that the Hadamard product with $h_\bullet$ is 1-tamed with norm bounded from above by the sequence a^{-1}. This shows that our quasi-inverse satisfies condition A) of the abstract KAM theorem.

Let us now check condition B). The map

$$f \mapsto j_k(f)$$

involves first order derivatives of f. Due to Cauchy inequalities, it is therefore 1-tamed. This shows condition B) of the abstract KAM theorem and concludes the proof of the theorem.

The proof of the abstract KAM theorem is constructive and therefore we have explicit bounds for the estimate i) in Theorem 7.4 which implies point 2) of the theorem and point 3) is obvious since the construction of the quasi-inverse preserves real structures when H is real.

Appendix A. Regularity in $C^{l,\omega}_\alpha(a)$.

We give an abstract version the regularity of KAM tori discovered by Pöschel [21]. Gevrey regularity can be treated in a similar way.

The identity morphism

$$I : C^{l,\omega}_\alpha(a) \longrightarrow C^{0,\omega}_\alpha(a)$$

is 0-bounded with norm at most one. The Cauchy inequalities give a partial converse:

Lemma A.1. *The identity morphism*

$$I^{-1} : C^{0,\omega}_\alpha(a) \longrightarrow C^{l,\omega}_\alpha(a)$$

is l-bounded and its norm is bounded by the sequence $(2^{il}a_i^{-l})$

Proof. Take $s < t$ and $f \in (C^{l,\omega}_\alpha(a))_{i,t}$. Put $K_{i,s} = \mathcal{C}((1-s)a)_i \cap D_s(\alpha)$. For any $x \in K_{i,t}$, the ball of radius $a_i/2^i(t-s)$ is contained inside $K_{i,s}$. Thus, the Cauchy inequalities imply that

$$|I(f)|_s \leq \frac{2^{il}}{(t-s)^l a_i^l}|f|_t.$$

This proves the lemma. □

In particular, any k-bounded $u \in \mathcal{B}^k_\tau(C^{0,\omega})$ induces a $(k+l)$-bounded morphism

$$v \in \mathcal{B}^{k+l}_\tau(C^{l,\omega}_\alpha(a))$$

for which there is a commutative diagram

$$\begin{tikzcd} C^{l,\omega}_\alpha(a) \arrow[r, "v"] \arrow[d, "I"] & C^{l,\omega}_\alpha(a) \arrow[d, "I"] \\ C^{0,\omega}_\alpha(a) \arrow[r, "u"] & C^{0,\omega}_\alpha(a) \end{tikzcd}$$

Corollary A.2. *Let $a = (a_i)$ be a positive decreasing sequence and l a positive real number. For any k-bounded τ-morphism $u \in \mathcal{B}^k_\tau(C^{0,\omega}(a))$, the norm of the induced morphism $v \in \mathcal{B}^{k+l}_\tau(C^{l,\omega}_\alpha(a))$ satisfies the estimate*

$$|v_i|_\tau \leq \frac{2^{(k+l)(i+1)}}{a_i^{k+l}}|u_i|_\tau$$

for any $i \geq 0$.

Proof. Take $s < t \leq \tau$ and $f \in (C^{l,\omega}_\alpha(a))_{i,t}$. We cut the interval $t - s$ into two equal pieces and write $v_\bullet = I^{-1}u_\bullet I$. The previous lemma shows that:

$$|I^{-1}u_i I(f)|_s \leq \frac{2^{l(i+1)}}{(t-s)^l a_i^l}|u_i I(f)|_{s+\sigma}$$

with $\sigma = (t - s)/2$. As $u_\bullet$ is k-bounded, we have that

$$|u_i I(f)|_{s+\sigma} \leq \frac{2^{k(i+1)}}{(t-s)^k a_i^k}|u_i|_t\,|I(f)|_t$$

As I is 0-bounded with norm 1, we have

$$|I(f)|_t \leq |f|_t.$$

This proves the corollary. □

We may now consider the following particular case :

Corollary A.3. *Let $a = (a_i)$ be a decreasing positive sequence and u a k-bounded τ morphism of $C^{0,\omega}_\alpha(a)$, for some $\tau > 0$. Assume that $(|u_i|_\tau)$ decreases faster than any power of $(2^{-i}a_i)$:*

$$|u_i|_\tau = o(2^{-ij}a_i^j), \forall j > 0.$$

For any $l > 0$, the norm of the morphism

$$v_\bullet : C^{l,\omega}_\alpha(a) \longrightarrow C^{l,\omega}_\alpha(a)$$

induced by $u_\bullet$ has the same property:

$$|v_i|_\tau = o(2^{-ij}a_i^j),\ \forall j > 0.$$

This shows that the $C^{0,\omega}_\alpha(a)$-bounded morphism $u_\bullet$ of the abstract KAM theorem induces a $C^{\infty,\omega}_\alpha(a)$-bounded morphism $v_\bullet$ whose norm decrease exponentially fast. By [6, Theorem 4.1], the sequence $(e^{v_i} \dots e^{v_0})$ converges to a limit with C^∞ dependence on the t parameters.

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