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SINGULARITIES OF COMPLEX VECTOR FIELDS HAVING MANY CLOSED ORBITS

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Abstract. A well-known result of Mattei and Moussu ([18]) states that a germ of a holomorphic vector field at the origin $0 \in \mathbb{C}^2$ admits a holomorphic first integral if, and only if, the orbits are closed off the origin and only finitely many of these accumulate (only) at the origin. In this paper we investigate possible versions of such a result in terms of the measure of the set of closed orbits. We prove that if the set of closed leaves is a positive, i.e., a non-zero measure subset and the set of leaves accumulating only at the origin is a zero measure subset, then either there is a holomorphic first integral or the germ is formally linearizable as a suitable non-resonant singularity. The result is sharp as we show through some examples.

1. Introduction

The problem of deciding whether a vector field or, more generally, an ordinary differential equation can be integrated by studying its number of non-transcendent solutions goes back to H. Poincaré, Dulac ([12]) and other authors. More recently the classical theorem of G. Darboux ([16] pages 80 and 135) states that a polynomial vector field in the complex plane admits a rational first integral if, and only if, it admits infinitely many algebraic solutions. The class of analytic equations seems to be the one where the above problem makes more sense. Moreover, with the arrival of the theory of foliations the use of geometrical/topological methods has given an important contribution to the comprehension of the problem as well as some important results. The local framework is not less important than the global (algebraic) case. In this sense we have the remarkable theorem of Mattei-Moussu (Theorem B [18]) that states that *a germ of a holomorphic vector field at the origin of $\mathbb{C}^2$ admits a holomorphic first integral if, and only if, it has only finitely many leaves accumulating at the singularity and all other leaves are closed.* Recall that by a *holomorphic first integral* for a germ of a vector field, we shall mean a germ of a holomorphic function, which is not locally constant, but which is locally constant along the orbits of the vector field.

Following the convention in [10], in this paper *we say that a subset $\Omega \subset \mathbb{C}^n$ has positive measure if it is not a zero measure subset of $\mathbb{C}^n$, in the usual Lebesgue measure sense.*

In [1] the authors prove the existence of a holomorphic first integral, under the hypothesis of existence of a uniform bound for the volume of the orbits of the vector field, and some additional condition that restricts the so called “dicritical case”.

A holomorphic vector field X defined in a neighborhood $U \subset \mathbb{C}^2$ of the origin $0 \in \mathbb{C}^2$, with an isolated singularity at the origin, defines a unique germ of holomorphic foliation $\mathcal{F}(X)$ with a singularity at the origin in a natural way: the non-singular orbits of X are the (representatives of) the leaves of $\mathcal{F}(X)$. Conversely, any germ $\mathcal{F}$ of holomorphic foliation with a singularity at the origin is defined in a small enough open neighborhood of the origin by a holomorphic vector field, i.e., $\mathcal{F} = \mathcal{F}(X)$ for some vector field X as above. This is a consequence of Hartogs' extension

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theorem ([14]). Thus we shall refer to *leaf* as well as to *orbit* in our considerations. Given such a pair (X, U), we denote by $\Omega(X, U) \subset U$ the union of orbits of the restriction $X\big|_U$ which are closed in U. Also we denote by $\mathrm{Sep(X, U)} \subset \mathrm{U}$ the union of orbits of $X\big|_U$ that accumulate only at the singularity. By Remmert-Stein's classical theorem, each leaf in $\mathrm{Sep(X, U)}$ is contained in an analytic invariant curve, called a *separatrix* of $\mathcal{F}$. Recall that a germ of an isolated singularity at $0 \in \mathbb{C}^2$ is called *dicritical* if it exhibits infinitely many separatrices.

It is not difficult, by using the reduction of singularities (see Section 5) to conclude that a germ is dicritical if, and only if, for any arbitrarily small neighborhood of the origin, its set of separatrices in U has non-empty interior and therefore, positive measure (cf. Lemma 5.1).

Mattei-Moussu's above mentioned theorem then states that the germ of a vector field X admits a holomorphic first integral if, and only if, for some small enough neighborhood U of the origin we have: (i) $\mathrm{Sep(X, U)}$ is a finite union of analytic curves and (ii) $\Omega(X, U) = U \setminus \mathrm{Sep(X, U)}$. This implies that (a) $\mathrm{Sep(X, U)}$ *is a zero measure subset* and (b) $\Omega(X, U)$ *has positive measure*. We shall say that a germ of a holomorphic foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ is the *germ of a (PCO) singularity* if for some neighborhood U of the origin, $\mathcal{F}$ is given by a vector field X such that $\Omega(X, U)$ has positive measure.

At this point one may ask whether conditions (a) and (b) above are enough to assure the existence of a holomorphic first integral. In other words:

"Does a non-dicritical (PCO) germ of a holomorphic foliation admit a holomorphic first integral?"

As we shall see, the answer to the above question is not always positive, for there may be regions of non-closed orbits, having positive measure. This is because of the very particular local dynamics of certain germs of complex diffeomorphisms, exhibiting some non-resonance properties. Let us start with this class of maps. A germ of a complex diffeomorphism at the origin $0 \in \mathbb{C}$ is given by a convergent power series $f(z) = az + \sum\limits_{j=2}^{\infty} a_j z^j$, where $a = f'(0) \neq 0$. Given a small enough neighborhood $0 \in U \subset \mathbb{C}$ we can choose a representative $f \colon U \to f(U)$ for the germ. The set of closed orbits of f in U is the set $\Omega(f, U) \subset U$ of points $x \in U$ such that $O_f(x) \cap U$ is closed. Here, by $O_f(x)$ we denote the pseudo-orbit of x under the action of f. The map f is *non-resonant* if $a = e^{2\pi i \lambda}$ where $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Such a map is not necessarily analytically linearizable (cf. [3],[22]). Following the classical terminology, a non-linearizable non-resonant map will be called a *Cremer* map. Details of such dynamics will be given later (see Section 2). As for now we observe that a Cremer map germ is always formally linearizable ([2, 3]). Such a map will be called *(PCO)* (from positive-closed-orbits) if for arbitrarily small neighborhoods of the origin, the set of closed orbits in such a neighborhood has positive measure. Thanks to the notion of holonomy of a separatrix, there is a strict connection between these Cremer maps and a suitable class of singularities of holomorphic vector fields. This is detailed in Section 2 (cf. Remark 3.7). As for now, an isolated singularity of a holomorphic vector field in dimension two will be called a *Cremer type singularity* if it is a non-resonant singularity, i.e., of the form $\mathcal{F} : xdy - \lambda ydx + \text{h.o.t.} = 0$, $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, which is also not analytically linearizable. (This already implies $\lambda \in \mathbb{R}_-$ since for $\lambda \in \mathbb{R}_+ \setminus \mathbb{Q}$, by Poincaré linearization theorem, the singularity is always analytically linearizable). For such a non-resonant singularity it is well known that we have exactly two transverse separatrices. Actually, we can change coordinates in order to write it as $\mathcal{F} : x(1 + a(x, y))dy - \lambda ydx = 0$, where $a(x, y)$ is holomorphic and vanishes at $(0, 0)$. The holonomy map h of the separatrix $(y = 0)$ is well defined up to conjugacy in $\mathrm{Diff}(\mathbb{C}, 0)$. Thanks to the relation between the analytic classification of the singularity and that of the holonomy map of a separatrix ([19, 18]) we conclude that a non-resonant singularity is of Cremer type if, and only if, the corresponding holonomy map h is a Cremer map germ. A Cremer type singularity is then called a *(PCO) Cremer type singularity* if the mentioned holonomy map is also a (PCO) Cremer map. In terms of the foliation itself, this is is equivalent to saying that the singularity is non-resonant and not analytically linearizable, but with a positive measure set of closed orbits/leaves on each arbitrarily small neighborhood of the singularity. Thus a (PCO) Cremer type singularity is just a Cremer type singularity which is the germ of a (PCO) singularity in the sense defined above.

There are other examples of singularities with positive measure sets of closed orbits as we pause to describe. A germ of a foliation $\mathcal{F}$ is a *holomorphic pull-back* of a Cremer type singularity, if there is a germ of a Cremer type singularity $\mathcal{G} : \Omega = xdy - \lambda ydx + \text{h.o.t.} = 0$ and a germ of a holomorphic map $\Phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $\mathcal{F} = \Phi^*(\mathcal{G})$, i.e., $\mathcal{F}$ is defined by the germ of a holomorphic one-form $\Phi^*(\Omega)$. In particular, if the Cremer type singularity is a (PCO) singularity then the same holds for $\mathcal{F}$. (A closed leaf will be analytic and the same holds for its pre-image under the map Φ).

Since Cremer type singularities are always formally linearizable, there are formal functions $\hat{f}_j \in \mathcal{O}_2$ in two complex variables and complex numbers $\lambda_j, j = 1, 2$; such that $\Omega \wedge \hat{\omega} = 0$ where $\widehat{\omega} = \sum\limits_{j=1}^{2} \lambda_j d\hat{f}_j/\hat{f}_j$. Briefly, the foliation is *defined* by a formal closed one-form with simple poles. Thus, (PCO) singularities can be seen as a particular case of a bigger class as follows. A germ of a holomorphic foliation $\mathcal{F}$ is *formally equivalent* to a *Darboux foliation* (or also to a *logarithmic foliation*) if it can be defined by a formal closed meromorphic one-form with simple poles $\mathcal{F} : \hat{\omega} = \sum\limits_{j=1}^{r} \lambda_j d\hat{f}_j/\hat{f}_j$ for some $\lambda_j \in \mathbb{C} \setminus \{0\}$ and formal functions $\hat{f}_j$.

As an extension of the above mentioned result, the following theorem is proved in this paper.

Theorem 1.1. *Let X be a holomorphic vector field defined in an open neighborhood of the origin with a non-dicritical singularity at $0 \in \mathbb{C}^2$. Then the following conditions are equivalent:*

(1) *For any sufficiently small neighborhood U of the origin the union of closed orbits $\Omega(X, U)$ is a positive measure subset.*
(2) *The corresponding germ of a holomorphic foliation $\mathcal{F}(X)$ admits a holomorphic first integral or it is formally equivalent to a Darboux type singularity having some (PCO) Cremer type singularity in its reduction of singularities.*

Remark 1.2. We refer to Section 5 for the description of the reduction of singularities mentioned above. The proof of Theorem 1.1 actually shows that *the existence of holomorphic first integral is equivalent to (1) plus the fact that the reduction of singularities of $\mathcal{F}(X)$ exhibits no non-resonant singularity, i.e.*, no singularity of the form $xdy - \lambda ydx + \text{h.o.t.} = 0$ with $\lambda \in \mathbb{R} \setminus \mathbb{Q}$.

The existence of (PCO) Cremer type singularities in shown in Section 2. From Theorem 1.1 and from the considerations in Section 2 and Remark 3.7, we can state, still for foliation germs at the origin $0 \in \mathbb{C}^2$:

Corollary 1.3. *A foliation germ $\mathcal{F} = \mathcal{F}(X)$ exhibits a holomorphic first integral if and only if for any sufficiently small neighborhood U of the origin:*

(1) $\Omega(X, U)$ *has positive measure.*
(2) $\mathrm{Sep(X, U)}$ *has zero measure.*
(3) *There is no recurrent orbit or no orbit properly accumulating at* $\mathrm{Sep(X, U)}$.

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2. Cremer maps: according to Pérez-Marco

Expand a germ of a complex diffeomorphism f at the origin $0 \in \mathbb{C}$ as

$$f(z) = e^{2\pi i\lambda}z + a_{k+1}z^{k+1} + \dots .$$

The multiplier $f'(0) = e^{2\pi i\lambda}$ does not depend on the coordinate system. We shall say that the germ $f \in \mathrm{Diff}(\mathbb{C}, 0)$ is *non-resonant* if $\lambda \in \mathbb{C} \setminus \mathbb{Q}$. If $\lambda \notin \mathbb{R}$ then $|f'(0)| \neq 1$ and the germ is *hyperbolic*. In particular, it is either an attractor or a repeller. In the hyperbolic case the diffeomorphism is *analytically linearizable*, i.e., conjugated to its linear part by a germ of a map ([12]). If $|f'(0)| = 1$, then we have $f'(0) = e^{2\pi i\lambda}$ for some $\lambda \in \mathbb{R}$. If $f'(0)$ is a root of unity (i.e., if $\lambda \in \mathbb{Q}$) then the dynamics of f is well-known from a theorem due to C. Camacho ([4], or also [3] page 38 and for dynamics of holomorphic maps in dimension one). In particular, close to the origin, none of the orbits off the origin is periodic. If $f'(0)$ is not a root of unity then we have $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. In this case we shall say that the diffeomorphism is *non-resonant*. Cremer gave the first proof of the existence of non-linearizable non-resonant map germs in 1927 [11]. Then his results were followed by those of Siegel, Brujno, Yoccoz and other authors. Most of these results are associated to diophantine conditions on the multiplier of the map. Such conditions are stated in terms of the convergence or divergence of certain series. A very nice description of the dynamics of such maps is given by Pérez-Marco in [24, 25]. Recall that if f is a diffeomorphism map germ, given a representative $f\colon U \to f(U)$ defined in an open connected subset $0 \in U \subset \mathbb{C}$ then the *stable set* of f in U is defined by the intersection

$$K(U, f) = \bigcap_{j=0}^{\infty} f^{-j}(U).$$

According to Pérez-Marco ([23], [22]) if f is a Cremer map, and given a representative defined in an open connected subset $0 \in U \subset \mathbb{C}$, then:

(i) The stable set $K(U, f)$ is compact, contains a connected totally invariant compact K_0, full (i.e., $U \setminus K(U, f)$ is connected), it is not reduced to $\{0\}$, and it is not locally connected at any point distinct from the origin.

(ii) Any point of $K(U, f) \setminus \{0\}$ is recurrent (that is, a limit point of its orbit).

(iii) There is an orbit in $K(U, f)$ which accumulates at the origin, but no non-trivial orbit converges to the origin.

Recall that a (PCO) *Cremer map* germ is a Cremer map germ, such that its representatives exhibit positive measure sets of closed orbits, in arbitrarily small neighborhoods of the origin.

Proposition 2.1 (Existence of (PCO) Cremer maps and (PCO) Cremer type -singularities)**.** *There exist (PCO) Cremer map germs as well as (PCO) Cremer type singularities.*

Proof. We first show the existence of (PCO) Cremer map germs. According to Pérez-Marco (cf. his talk about the "Siegel problem" at the "Bourbaki seminar" [22], pages 281, 282), by "reversing" the geometric proof of Siegel-Brujno's linearization theorem (see [22] pages Chapter 4), using the notion of renormalization, it is possible to construct Cremer maps exhibiting a sequence of periodic points converging to the origin. Actually, as already pointed by Peréz-Marco, the construction is very flexible and allows us to fix with liberty the dynamics of return around these periodic orbits, but not around the origin (see the first paragraph of page 282 of [22]). To be more precise, let $F\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be such a Cremer map, exhibiting a sequence of periodic points $\{p_i\}_{i\in\mathbb{N}}$, converging to $0 \in \mathbb{C}$. Denoting by n_i the period of p_i, it is clear that the sequence $\{n_i\} \subset \mathbb{N}$ goes to infinity. From the renormalization techniques detailed in Chapter 4 in [22] (or also in [31]) we know that the *renormalized* dynamics f^{n_i} about p_i can arbitrarily be fixed on a sufficiently small neighborhood of p_i. In particular, they can be chosen to locally coincide with the identity. Thus the resulting local diffeomorphism is also a Cremer map (see the first paragraph in [22] page 287) and possesses a set of periodic points having positive measure.

Now we show the existence of (PCO) Cremer type singularities. First we notice that, in case we have a (PCO) Cremer map $f\colon (\mathbb{C},0) \to (\mathbb{C},0)$, then, by a result due to Pérez-Marco and Yoccoz [26] the map germ f is conjugate to the local holonomy of a separatrix associated to a germ of a holomorphic foliation $\mathcal{F}(f)$ possessing a Siegel-type singular point at the origin. Then, by construction, the germ of a foliation $\mathcal{F}(f)$ is non-dicritical, non-resonant and possesses a set of closed leaves having positive measure. (These leaves are in natural correspondence with periodic points of f). It is however clear that the foliation in question only admits constant holomorphic first integrals. (Indeed, not all leaves are closed off the origin or else, using [18], it has a non-linearizable holonomy for one separatrix). This shows the existence of (PCO) Cremer type singularities.

□

3. Groups with positive measure set of closed orbits

Let $\mathrm{Diff}(\mathbb{C},0)$ denote the group of germs of holomorphic diffeomorphisms at the origin $0 \in \mathbb{C}$. The following is a well-known result:

Lemma 3.1. *Let $G \subset \mathrm{Diff}(\mathbb{C},0)$ be a finite group of germs of complex diffeomorphisms. Then G is analytically conjugate to a cyclic group generated by a rational rotation, i.e., up to a holomorphic change of coordinates we have $G = \{z \mapsto e^{2k\pi i/\nu} z,\ k = 0,1,...,\nu-1\}$ for some $\nu \in \mathbb{N}$.*

Proof. The proof is well-known, the linearization is given by the map $\Phi\colon (\mathbb{C},0) \to (\mathbb{C},0)$ defined by $\Phi(z) = \frac{1}{|G|} \sum\limits_{g\in G} g(z)/g'(0)$, where g runs through the (finite) list of elements of G and $|G|$ denotes the order of G. □

Definition 3.2 (resonant group)**.** A germ of a complex diffeomorphism $g \in \mathrm{Diff}(\mathbb{C},0)$ is called *resonant* if its multiplier is a root of unity, i.e., $g'(0) = e^{2\pi ik/\ell}$ for some $k, \ell \in \mathbb{N}$. A group $G \subset \mathrm{Diff}(\mathbb{C},0)$ of germs of holomorphic diffeomorphisms will be called *resonant* if each map $g \in G$ is a resonant germ. This is equivalent to the fact that G has a set of generators consisting only of resonant maps.

The next result is, for the case of resonant groups, a generalization of a result in [18] and of a result found in [29].

We shall say that a subgroup $G \subset \mathrm{Diff}(\mathbb{C},0)$ *has the (PCO) property* if any sufficiently small neighborhood U of the origin $0 \in \mathbb{C}$, the set $\Omega(G,U)$ of points having closed pseudo-orbit has positive measure in U.

Lemma 3.3. *If $G \subset \mathrm{Diff}(\mathbb{C},0)$ has the (PCO) property then $G \cap \{\mathrm{Id} + \text{h.o.t.}\} = \{\mathrm{Id}\}$. In particular G is abelian.*

Proof. Indeed, pick a non-trivial element $g \in G$ of the form $g(z) = z + a_{k+1} z^{k+1} + \text{h.o.t.}$, $a_{k+1} \neq 0$. According to [4] the pseudo-orbits of this element are neither closed nor finite. Thus necessarily $G \cap \{\mathrm{Id} + \text{h.o.t.}\} = \{\mathrm{Id}\}$. Given two element $f, g \in G$ the commutator belongs to $G \cap \{\mathrm{Id} + \text{h.o.t.}\}$ so f and g commute. Hence G is abelian. □

Lemma 3.4. *Let $G \subset \mathrm{Diff}(\mathbb{C},0)$ be a finitely generated <u>resonant</u> subgroup with the (PCO) property. Then G is finite cyclic and analytically conjugate to a group of rational rotations.*

Proof. By Lemma 3.3 above the group is abelian and all of its elements tangent to the identity are trivial. Since by hypothesis any element has a periodic linear part we conclude that:

Claim 3.5. *Any element $g \in G$ has finite order.*

Since G is abelian and finitely generated, the claim implies that G itself is finite. According to Lemma 3.1, G must be a group of rational rotations up to analytic conjugation. $\square$

As a consequence of the above considerations we obtain:

Lemma 3.6. *Let $G \subset \text{Diff}(\mathbb{C}, 0)$ be a finitely generated subgroup with the (PCO) property. Then either G is a cyclic finite (resonant) group or it is an abelian formally linearizable group, containing some (PCO) Cremer diffeomorphism.*

Proof. By Lemma 3.4 we can assume that G contains some non-resonant map. Since hyperbolic and linearizable maps with non-periodic linear part exhibit no closed orbit off the origin, we can then assume that G contains some Cremer map say $f_0 \in G$. By Lemma 3.3 G is indeed abelian with no element tangent to the identity and all resonant elements are periodic. Since $f_0 \in G$ is formally linearizable and non-resonant, any map in G is also formally linearizable in the same formal coordinate that linearizes f_0. Thus G is formally linearizable. $\square$

Remark 3.7 (Closed orbits versus periodic orbits for Cremer maps)**.** Notice that, fixed a neighborhood $0 \in U \subset \mathbb{C}$, where we have defined a representative $f\colon U \to f(U) \subset \mathbb{C}$, of a Cremer map, the stable set $K(f, U)$ is the set of points in U which have all iterates $f^j(p), j \in \mathbb{N}$, defined and contained in U. The fact that a point $p \in U$ has all its iterates in U and has a closed orbit, implies (since $K(f, U)$ is compact), that the orbit of p is periodic. Such points do not exist in U, because all the points in $K(f, U) \setminus \{0\}$ are recurrent (cf. Section 2). Nevertheless, there can be points with closed orbits, which are not periodic in U. These points have only finitely many of their iterates defined and contained in U. Actually, this situation has already been addressed by Mattei-Moussu in their original article ([18]), in the proof of their Theorem 2.1 page 478. Indeed, they consider a fixed disc D where the map is defined, and write (under the hypothesis that all orbits of the considered map are finite, but not necessarily contained in the fixed disc), the disc as $D = P \cup F \cup I$. In their terminology we have: P is the set of points periodic points in D, F is the set of points with periodic orbits, but not fully contained in D, and I is the set of points with infinitely many iterates in D. So they prove $D = P \cup F \cup I$ and conclude that if the map is not periodic (finite order map) then I is non-countable and contains the origin in its adherence, which is enough for their purposes. In his revisiting paper ([21]) Moussu proves an analogue of the same theorem (see the Proposition at page 477) but using Pérez-Marco's work. (Namely, the key point is the fact that a Cremer map exhibits a stable set, having the origin as an accumulation point, which does not reduce to the origin, and where the orbits are not finite.)

4. Holonomy and virtual holonomy groups

Let now $\mathcal{F}$ be a holomorphic foliation with singularities on a complex surface M. Denote by $\text{Sing}(\mathcal{F})$ the singular set of $\mathcal{F}$. It is known that $\text{Sing}(\mathcal{F})$ can be assumed to be a discrete set of points in M. Given a leaf L_0 of $\mathcal{F}$ we choose any base point $p \in L_0 \subset M \setminus \text{Sing}(\mathcal{F})$ and a transverse disc $\Sigma_p \Subset M$ to $\mathcal{F}$ centered at p. The holonomy group of the leaf L_0 with respect to the disc Σ_p and to the base point p is the image of the representation $\text{Hol}\colon \pi_1(\mathrm{L_0, p}) \to \text{Diff}(\Sigma_\mathrm{p}, \mathrm{p})$ obtained by lifting closed paths in L_0 with base point p, to paths in the leaves of $\mathcal{F}$, starting at points $z \in \Sigma_p$, by means of a transverse fibration to $\mathcal{F}$ containing the disc Σ_p ([6]). Given a point $z \in \Sigma_p$ we denote the leaf through z by L_z. Given a closed path $\gamma \in \pi_1(L_0, p)$ we denote by $\tilde{\gamma}_z$ its lift to the leaf L_z and starting from the point z. Then the image of the corresponding holonomy map is $h_{[\gamma]}(z) = \tilde{\gamma}_z(1)$, i.e., the final point of the lifted path $\tilde{\gamma}_z$. This defines a diffeomorphism germ map $h_{[\gamma]}\colon (\Sigma_p, p) \to (\Sigma_p, p)$ and also a group homomorphism $\text{Hol}\colon \pi_1(\mathrm{L_0, p}) \to \text{Diff}(\Sigma_\mathrm{p}, \mathrm{p})$. The image $\text{Hol}(\mathcal{F}, \mathrm{L_0}, \Sigma_\mathrm{p}, \mathrm{p}) \subset \text{Diff}(\Sigma_\mathrm{p}, \mathrm{p})$ of such homomorphism is called the *holonomy group* of the leaf L_0 with respect to Σ_p and p. By considering any parametrization $z\colon (\Sigma_p, p) \to (\mathbb{D}, 0)$ we may identify (in a non-canonical way) the holonomy group with a subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$. It is clear from the construction that the maps in the holonomy group preserve the traces of the leaves of the foliation in the given transverse section. Nevertheless, this property can be shared by a larger group that may therefore contain more information about the foliation in a neighborhood of the leaf. The *virtual holonomy group* of the leaf with respect to the transverse section Σ_p and base point p is defined as ([7] Definition 2, page 432 or also in [9])

$$\mathrm{Hol^{virt}}(\mathcal{F}, \Sigma_\mathrm{p}, \mathrm{p}) = \{\mathrm{f} \in \mathrm{Diff}(\Sigma_\mathrm{p}, \mathrm{p}) \big| \widetilde{\mathrm{L}}_\mathrm{z} = \widetilde{\mathrm{L}}_{\mathrm{f(z)}}. \forall \mathrm{z} \in (\Sigma_\mathrm{p}, \mathrm{p})\}$$

The virtual holonomy group contains the holonomy group and consists of the map germs that preserve the traces of the leaves of the foliation in the given transverse section.

5. Reduction of singularities in dimension two ([30])

Given a foliation $\mathcal{F}$ of dimension one on a complex surface $\overline{M}$ with singular set $\mathrm{Sing}\mathcal{F}$, the reduction theorem of Seidenberg ([30]) asserts the existence of a proper holomorphic map $\pi\colon \widetilde{M} \to \overline{M}$ which is a finite composition of blow-ups at the singular points of $\mathcal{F}$ in $\overline{M}$ such that the pull-back foliation $\widetilde{\mathcal{F}} := \pi^*\mathcal{F}$ of $\mathcal{F}$ by π satisfies:

(a) $\mathrm{Sing}\widetilde{\mathcal{F}} \subset \pi^{-1}(\mathrm{Sing}\mathcal{F})$, and

(b) any singularity $\tilde{p} \in \mathrm{Sing}\widetilde{\mathcal{F}}$ belongs to one of the following categories (called *irreducible singularities*):

(i) $xdy - \lambda ydx + \text{h.o.t.} = 0$ and λ is not a positive rational number, i.e. $\lambda \notin \mathbb{Q}_+$ (*simple* or *non-degenerate singularity*),

(ii) $y^{k+1}dx - [x(1 + \lambda y^k) + p(x, y)]dy = 0$, where $k \geq 1$ and $p(x, y)$ is holomorphic of order $\geq k$ at the origin. This case is called a *saddle-node*. The separatrix $\{y = 0\}$ is called the *strong manifold* or *strong separatrix* of the saddle-node. Its local holonomy map is strongly linked to the analytical classification of the saddle-node (cf. [20]).

We call the lifted foliation $\widetilde{\mathcal{F}}$ the *desingularization* or *reduction of singularities* of $\mathcal{F}$.

The *exceptional divisor* $D = \pi^{-1}(\mathrm{Sing}\mathcal{F}) \subset \widetilde{M}$ of the resolution π can be written as

$$D = \bigcup_{j=1}^{m} D_j,$$

where each D_j is diffeomorphic to an embedded projective line $\mathbb{CP}^1$ introduced as a divisor of the successive blow-ups ([13]). The D_j are called *components* of the divisor D. From the final picture of the reduction of singularities we conclude that a singularity $q \in \mathrm{Sing}\mathcal{F}$ is nondicritical if, and only if, $\pi^{-1}(q)$ is invariant by $\widetilde{\mathcal{F}}$. Such a germ is called a *generalized curve* if no saddle-nodes appear in the reduction of singularities. Any two components D_i and D_j, $i \neq j$, of the exceptional divisor intersect (transversely) at at most one point, which is called a *corner*. There are no triple intersection points. An irreducible singularity $xdy - \lambda ydx + \text{h.o.t.} = 0$ is in the *Poincaré domain* if $\lambda \notin \mathbb{R}_-$ and it is in the *Siegel domain* otherwise. For singularities in the Poincaré domain, the non-resonance condition ($\lambda \notin \mathbb{Q}$) actually implies hyperbolicity ($\lambda \in \mathbb{C} \setminus \mathbb{R}$) and by Poincaré linearization theorem the singularity is analytically linearizable (cf.[12],[5]). For singularities in the Siegel domain, the non-resonance condition ($\lambda \notin \mathbb{Q}_-$) implies formal linearization for the singularity (cf.[5]). Nevertheless, such a non-resonant singularity is analytically linearizable if and only if it is topologically linearizable, i.e., conjugated to its linear part by a homeomorphism between neighborhoods of the origin ([5], [2]).

Fix now a germ of holomorphic foliation with a singularity at the origin $0 \in \mathbb{C}^2$. Choose a representative $\mathcal{F}_U$ for the germ $\mathcal{F}$, defined in an open neighborhood U of the origin. A leaf of

$\mathcal{F}_U$ accumulating only at p is closed off p, thus by Remmert-Stein extension theorem ([15]) it is contained in an irreducible analytic curve through p. Such a curve is called a local *separatrix* of $\mathcal{F}$ through p. A singularity is dicritical if and only if it exhibits infinitely many separatrices. Actually, we have:

Lemma 5.1. *A germ of a holomorphic foliation singularity $\mathcal{F}(X)$ at $0 \in \mathbb{C}^2$ is non-dicritical if, and only if, for some neighborhood $U \ni 0$, the set of separatrices* $\mathrm{Sep(X,U)}$ *is a zero measure set.*

Proof. Indeed, notice that a neighborhood of some point on some projective line in a finite sequence of blow-ups starting at the origin corresponds to what we call *sector* with vertex at the origin. Thus, from the above mentioned Theorem of reduction of singularities ([30]), a dicritical singularity always exhibits a "sector" of separatrices with vertex at the singular point. Such a "sector" has non-empty interior and therefore positive measure. □

By Newton-Puiseaux parametrization theorem, every separatrix is biholomorphic to a disc. Further, the separatrix minus the singularity is biholomorphic to a punctured disc. In particular, given a separatrix S_p through a singularity $p \in \mathrm{Sing}(\mathcal{F})$, we may choose a loop $\gamma \in S_p \setminus \{p\}$ generating the (local) fundamental group $\pi_1(S_p \setminus \{p\})$. The corresponding holonomy map h_γ is defined in terms of a germ of complex diffeomorphism at the origin of a local disc Σ transverse to $\mathcal{F}$ and centered at a non-singular point $q \in S_p \setminus \{p\}$. This map is well-defined up to conjugation by germs of holomorphic diffeomorphisms, and is generically referred to as *local holonomy* of the separatrix S_p with respect to the singularity p.

Definition 5.2 (fully resonant)**.** A germ of a generalized curve $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$ will be called *fully resonant* if every singularity arising in the final step of the reduction of singularities is a resonant singularity, i.e., a singularity of the form $xdy - \lambda ydx + \text{h.o.t.} = 0$ with $\lambda \in \mathbb{Q}_-$.

The following remark is important in the approach we use:

Remark 5.3. By definition, if $\mathcal{F}$ is a germ of a foliation at a fully resonant singularity then every local holonomy arising in the reduction of singularities of $\mathcal{F}$ is resonant. Therefore, since the components of the exceptional divisor are projective lines (and so homeomorphic to the 2-sphere S^2), we conclude that *every component of the exceptional divisor has a resonant holonomy group.* Nevertheless, *a priori*, it is not clear that the same holds for the virtual holonomy groups. Indeed, such groups may in principle be much larger than the holonomy groups and generated by some non-resonant maps as well. The notion of *invariance group* is introduced in [18] page 521, which is used to "glue" in a compatible way, different local first integrals in the exceptional divisor. As it follows from [18], the finiteness of these groups implies the existence of a holomorphic first integral. Nevertheless, again, it is then necessary to prove that these invariance groups are resonant. This is done in the proof of Theorem 1.1 (cf. Claim 6.2).

Lemma 5.4. *Let $\mathcal{F}$ be a holomorphic foliation defined in a neighborhood U of the origin $0 \in \mathbb{C}^2$, with an isolated singularity at the origin. A closed leaf of $\mathcal{F}$ in U is analytic. A leaf that accumulates only at the singular point is contained in an invariant analytic curve. Let $p \in U \setminus \{0\}$ and Σ_p a small disc transverse to the foliation and centered at p. Then any closed leaf of $\mathcal{F}$ intersecting Σ_p induces a closed orbit for the holonomy group* $\mathrm{Hol}(\mathcal{F}, \mathrm{L_p}, \Sigma_\mathrm{p}, \mathrm{p})$ *of the leaf L_p and the same holds for the virtual holonomy group* $\mathrm{Hol}^{\mathrm{virt}}(\mathcal{F}, \mathrm{L_p}, \Sigma_\mathrm{p}, \mathrm{p})$.

Proof. Indeed, by Remmert-Stein extension theorem ([15]), a leaf which is closed in U is analytic. Also by Remmert-Stein extension theorem, a leaf L such that $\overline{L} \setminus L = \{0\}$ has analytic closure, because $\dim L = 1 > 0 = \dim(\overline{L} \setminus L)$. The last part follows from the fact that the intersection of two transverse analytic sets of dimension one in $\mathbb{C}^2$ is a discrete set of points. □

Lemma 5.5. *Let $\mathcal{F}$ be a germ of a holomorphic foliation in a neighborhood of the origin $0 \in \mathbb{C}^2$. Assume that, for some representative $\mathcal{F}_U$ of $\mathcal{F}$ defined in a neighborhood U of the origin, the set $\Omega(\mathcal{F}, U)$ of closed leaves of $\mathcal{F}_U$ in U has positive measure and the set of leaves accumulating only at the origin has zero measure. Then $\mathcal{F}$ is a generalized curve. Moreover it is either fully resonant or it exhibits some (PCO) Cremer type singularity in its reduction of singularities.*

Proof. By hypothesis the set $\Omega(\mathcal{F}, U)$ of closed leaves of $\mathcal{F}_U$ in U has positive measure and the set of leaves accumulating only at the origin has zero measure. This last hypothesis already implies that the foliation is non-dicritical (cf. Lemma 5.1). In order to prove that the singularity is a fully resonant generalized curve, we proceed by induction on the number $r \in \{0, 1, 2, ...\}$ of blow-ups in the reduction of singularities for the germ $\mathcal{F}$.

Case 1. $r = 0$. In this case the singularity is already irreducible and we have two possibilities:
(i) The singularity is non-degenerate of the form $xdy - \lambda ydx + \text{h.o.t.} = 0$ for some $\lambda \in \mathbb{C} \setminus \mathbb{Q}_+$. By hypothesis, there is a neighborhood U of the origin where the set of closed leaves has positive measure and the set of leaves accumulating at the origin has zero measure.

• If the singularity is in the Poincaré domain, i.e., $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$, then it is not a resonance: indeed, if $\lambda \in \mathbb{N}$ or $1/\lambda \in \mathbb{N}$ then $\lambda \in \mathbb{Q}_+$, which does not correspond to the final picture of the reduction of singularities. Therefore, by Poincaré linearization theorem, the singularity is analytically linearizable. We may therefore choose local coordinates $(x, y) \in (\mathbb{C}^2, 0)$ such that the germ can be written as $xdy - \lambda ydx = 0$. The holonomy of the coordinate axis $(y = 0)$ with respect to a small disc $\Sigma : \{x = a\}$ is given by $h(y) = e^{2\pi i\lambda}y$. Suppose that $\lambda \notin \mathbb{R}$. In this case the map h is hyperbolic and satisfies $\lim_{n\to\infty} h^n(y) \to 0$ or $\lim_{n\to\infty} h^{-n}(y) \to 0$. Thus all leaves (not contained at the separatrices) accumulate at the separatrices. Therefore these leaves are not closed. This case is excluded by our hypotheses. Assume now that $\lambda \in \mathbb{R}$. Then $\lambda \in \mathbb{R}_+ \setminus \mathbb{Q}_+$. In this case the holonomy map $h(y) = e^{2\pi i\lambda}y$ is an irrational rotation and therefore none of its orbits off the origin is closed. Again this case is excluded.

• If the singularity is in the Siegel domain, i.e., $\lambda \in \mathbb{R}_-$, then we can put it in the form $x(1 + yA(x, y))dy - \lambda y(1 + xB(x, y))dx = 0$ for some holomorphic functions $A(x, y), B(x, y)$ defined in a neighborhood U of the origin. In this case the holonomy of one of the coordinate axes is a map whose analytic classification is strictly related to the analytic classification of the foliation ([18]). Let us fix a transverse disc $\Sigma = \{x = a\}$ for some $a \in \mathbb{C}$ close to zero, but different from zero. The holonomy map of the separatrix $\{y = 0\}$ can be defined for $|y| < \epsilon$ for some $\epsilon > 0$, and gives as a map $h\colon (\Sigma, 0) \to (\Sigma, 0)$. Because of the hypothesis, the holonomy map has closed orbits on a positive measure set of points of the disc Σ. This map has a multiplier of the $h'(0) = e^{2\pi i\lambda}$. Suppose that $\lambda \notin \mathbb{Q}$. In this case, the holonomy of a separatrix is a non-resonant map germ. According to Lemma 3.6 this is a (PCO) Cremer type singularity.
(ii) The singularity is a saddle-node. In this case according to [20] there are at most two leaves which are closed off the singularity and the other leaves are not closed, they do accumulate at the strong separatrix. Thus this case is excluded.

Case 2. Assume that the result is proved for foliation germs that admit a reduction of singularities with a number of blow-ups less than or equal to r. Suppose that the fixed germ $\mathcal{F}$ admits a reduction of singularities consisting of $r+1$ blow-ups. Then we perform a first blow-up $\pi_{(1)}\colon \widetilde{U}^{(1)} \to U$ at the origin and obtain a lifted foliation $\widetilde{\mathcal{F}}^{(1)} = (\pi_{(1)})^*(\mathcal{F})$ with an exceptional divisor $E^{(1)} = \pi_{(1)}^{-1}(0)$ consisting of a single embedded projective line. It is enough to prove that every singularity of $\widetilde{\mathcal{F}}^{(1)}$ in $E^{(1)}$ is a generalized curve which is fully-resonant or has Cremer (PCO) singularities in its reduction of singularities. Given a singularity $\widetilde{p} \in \text{Sing}(\widetilde{\mathcal{F}}^{(1)}) \subset \text{E}^{(1)}$ of $\widetilde{\mathcal{F}}^{(1)}$, this singularity admits a reduction of singularities consisting of less than $r+1$ blow-ups. Thus, by the induction hypothesis, in order to conclude that $\widetilde{p}$ is a fully-resonant singularity it is enough to prove that the germ of $\widetilde{\mathcal{F}}^{(1)}$ at $\widetilde{p}$ has a zero measure set of separatrices and a positive measure set of closed leaves.

Given a leaf L of $\mathcal{F}$ in U we denote by $\widetilde{L}(1)$ the lifting $\widetilde{L}(1) = \pi_{(1)}^{-1}(L)$ of L to $\widetilde{U}^{(1)}$. By hypothesis, the set of leaves of $\mathcal{F}$ accumulating at the origin has zero measure. This implies that the exceptional divisor is not generically transverse to $\widetilde{\mathcal{F}}^{(1)}$. Therefore, the exceptional divisor is invariant by $\widetilde{\mathcal{F}}^{(1)}$. Given a singularity $\widetilde{p} \in \operatorname{Sing}(\widetilde{\mathcal{F}}^{(1)}) \subset \mathrm{E}^{(1)}$ of $\widetilde{\mathcal{F}}^{(1)}$, we can conclude that for any small enough neighborhood $\widetilde{W}_{\widetilde{p}}$ of $\widetilde{p}$ in $\widetilde{U}^{(1)}$, a leaf $\widetilde{L}_0$ of the restriction $\widetilde{\mathcal{F}}^{(1)}\big|_{\widetilde{W}_{\widetilde{p}}}$ that accumulates only at the singularity $\widetilde{p}$, necessarily projects into a piece of leaf $\pi(\widetilde{L}_0)(1)$ which is contained in a leaf L of $\mathcal{F}$ that accumulates only at the origin. Therefore, by the hypothesis on $\mathcal{F}$ in U, the set of leaves of $\widetilde{\mathcal{F}}^{(1)}\big|_{\widetilde{W}_{\widetilde{p}}}$ that accumulate only at the singularity $\widetilde{p}$ has zero measure in $\widetilde{W}$. Now, if a leaf L of $\mathcal{F}$ in U is closed, then its lift $\widetilde{L}(1)$ is closed in $\widetilde{U}^{(1)}$. Such a leaf cannot contain a separatrix of a singularity $\widetilde{p}$. Moreover, because of the invariance of $E^{(1)}$, if the leaf $\widetilde{L}(1)$ intersects $\widetilde{W}_{\widetilde{p}}$ then this intersection $\widetilde{L}(1) \cap \widetilde{W}_{\widetilde{p}}$ corresponds to a finite number of closed leaves of $\widetilde{\mathcal{F}}^{(1)}\big|_{\widetilde{W}_{\widetilde{p}}}$ in $\widetilde{W}_{\widetilde{p}}$. On the other hand, given any leaf $\widetilde{L}_0$ of $\widetilde{\mathcal{F}}^{(1)}\big|_{\widetilde{W}_{\widetilde{p}}}$, this leaf projects into a piece of leaf L of $\mathcal{F}$ in U and if the leaf $\widetilde{L}_0(1)$ is not closed in U then, because of the invariance of $E^{(1)}$, the leaf $\widetilde{L}_0(1)$ is not closed either. Therefore, we conclude that the set of leaves of $\widetilde{\mathcal{F}}^{(1)}\big|_{\widetilde{W}_{\widetilde{p}}}$ which are closed in $\widetilde{W}_{\widetilde{p}}$ has positive measure.

Thus, by the induction hypothesis, the germ of $\widetilde{\mathcal{F}}^{(1)}$ at each singular point in $E^{(1)}$ is either a fully-resonant generalized curve or it is a generalized curve exhibiting some (PCO) Cremer type singularity in its reduction of singularities. By the induction hypothesis the germ $\mathcal{F}$ is a generalized curve, either fully-resonant or exhibits some (PCO) Cremer type singularity in its reduction of singularities. This proves the lemma. ☐

6. Proof of Theorem 1.1

Let us now prove our main result Theorem 1.1.

Proof of Theorem 1.1. Let us first prove that (1) implies (2). By hypothesis, there is a neighborhood U of the origin where the set of closed leaves has positive measure and the set of leaves accumulating only at the origin has zero measure. By Lemma 5.1 the singularity is non-dicritical. Moreover, by Lemma 5.5, either the foliation is a fully-resonant generalized curve or it admits a (PCO) Cremer type singularity in its reduction of singularities. Again we proceed by induction on the number $r \in \{0, 1, 2, ...\}$ of blow-ups in the reduction of singularities for the germ $\mathcal{F}$.

Case 1 ($r = 0$). In this case the singularity is already irreducible and from the above, it is not a saddle-node. Thus the singularity is non-degenerate of the form $xdy - \lambda ydx + \text{h.o.t.} = 0$ for some $\lambda \in \mathbb{C} \setminus \mathbb{Q}_+$ and we have two possibilities. Either singularity is resonant or it is a non-resonant Cremer type singularity. In the resonant case, by Lemma 3.6 the local holonomy of a separatrix is finite periodic and therefore the singularity admits a holomorphic first integral (cf. [18]). In the Cremer type singularity case the foliation is formally linearizable.

Case 2 (induction step). Assume that the result is proved for foliation germs that admit a reduction of singularities with a number of blow-ups less than or equal to r. Suppose that the fixed germ $\mathcal{F}$ admits a reduction of singularities consisting of $r+1$ blow-ups. Then we perform a first blow-up $\pi_{(1)} \colon \widetilde{U}^{(1)} \to U$ at the origin and obtain $\widetilde{\mathcal{F}}(1)$ and $E^{(1)}$ as in the proof of Lemma 5.5. Nevertheless, unlike in that proof, it is not enough to prove that the desired property holds for each singularity $\widetilde{p} \in \operatorname{Sing}(\widetilde{\mathcal{F}}^{(1)}) \subset \mathrm{E}^{(1)}$. Indeed, for instance, the existence of a holomorphic first integral is not just a semi-local matter. (According to Lins Neto [17] we can construct germs of generalized curves which can be reduced with a single blow-up, having a pre-determined set of generators for the holonomy group of the exceptional divisor. Thus we can consider a holonomy group with periodic generators, but which is not abelian and therefore not finite). This shows the necessity of considering some globalization arguments. Arguing as in the proof of Lemma 5.5 we conclude that $E^{(1)}$ is $\widetilde{\mathcal{F}}^{(1)}$-invariant. Furthermore, given a singularity $\widetilde{p} \in \mathrm{Sing}(\widetilde{\mathcal{F}}^{(1)}) \subset \mathrm{E}^{(1)}$ of $\widetilde{\mathcal{F}}^{(1)}$, for any small enough neighborhood $\widetilde{W}_{\widetilde{p}}$ of $\widetilde{p}$ in $\widetilde{U}^{(1)}$ (i) the set of leaves of $\widetilde{\mathcal{F}}^{(1)}\big|_{\widetilde{W}_{\widetilde{p}}}$ that accumulate at the singularity $\widetilde{p}$ has zero measure in $\widetilde{W}$ and (ii) the set of leaves of $\widetilde{\mathcal{F}}^{(1)}\big|_{\widetilde{W}_{\widetilde{p}}}$ which are closed in $\widetilde{W}_{\widetilde{p}}$ has positive measure. By the induction hypothesis, this implies that the singularity $\widetilde{p}$ fits into one of the two following cases:

(a) (the germ of $\widetilde{\mathcal{F}}$ at) $\widetilde{p}$ is fully resonant and admits a holomorphic first integral say $\widetilde{f}_{\widetilde{p}}$ defined in $\widetilde{W}_{\widetilde{p}}$ if this last is small enough.

(b) (the germ of $\widetilde{\mathcal{F}}$ at) $\widetilde{p}$ exhibits some (PCO) Cremer type singularity in its reduction of singularities.

Case (i). Assume that the germ of $\mathcal{F}$ at the origin is fully-resonant. Now we analyze the holonomy of the leaf $E_0(1) := E(1) \setminus \mathrm{Sing}(\widetilde{\mathcal{F}}^{(1)})$. Choose a regular point $\widetilde{q} \in E_0(1)$ and a small transverse disc Σ to $E_0(1)$ centered at $\widetilde{q}$. The corresponding holonomy group representation will be denoted by $H := \mathrm{Hol}(\widetilde{\mathcal{F}}^{(1)}, \Sigma, \widetilde{\mathrm{q}}) \subset \mathrm{Diff}(\Sigma, \widetilde{\mathrm{q}})$. We know that this group is finitely generated and by the invariance of $E^{(1)}$ and the above argumentation we know that actually, *the holonomy group H of the exceptional divisor has a positive measure set of closed orbits*. Since the virtual holonomy group preserves the leaves of the foliation, the arguments above already show that *the virtual holonomy group H^{virt} of the exceptional divisor has a positive measure set of closed orbits*. The problem is we still do not know that the virtual holonomy group is resonant so that we cannot conclude that this virtual holonomy group is finite. Nevertheless, from Lemma 3.4 we obtain:

Claim 6.1. *Any resonant finitely generated subgroup of the virtual holonomy group H^{virt} is a finite group.*

Let us then proceed as follows: given the singularities $\{\widetilde{p}_1, ..., \widetilde{p}_m\} = \mathrm{Sing}(\widetilde{\mathcal{F}}^{(1)}) \subset \mathrm{E}^{(1)}$, by induction hypothesis each singularity admits a local holomorphic first integral. Thus, there are small discs $D_j \subset E^{(1)}$, centered at the $\widetilde{p}_j$ and such that in a neighborhood V_j of $\widetilde{p}_j$ in the blow-up space $\widetilde{\mathbb{C}_0^2}$, of product type $V_j = D_j \times \mathbb{D}_\epsilon$, we have a holomorphic first integral $g_j \colon V_j \to \mathbb{C}$, with $g_j(\widetilde{p}_j) = 0$. Fix now a point $\widetilde{p}_0 \in E(1) \setminus \mathrm{Sing}(\widetilde{\mathcal{F}}^{(1)})$. Since $E^{(1)}$ is homeomorphic to S^2, there is a simply-connected domain $A_j \subset E^{(1)}$ such that $A_j \cap \{\widetilde{p}_0, \widetilde{p}_1, ..., \widetilde{p}_m\} = \{\widetilde{p}_0, \widetilde{p}_j\}$, for every $j = 1, ..., m$. Since A_j is simply-connected, we may extend the local holomorphic first integral g_j to a holomorphic first integral $\widetilde{g}_j$ for $\widetilde{\mathcal{F}}^{(1)}$ in a neighborhood U_j of $D_j \cup A_j$, and we may assume that U_j contains V_j. Now, given a local transverse section Σ_0 centered at $\widetilde{p}_0$ and contained in U_j, we may introduce the *invariance group* of the restriction $g_j^0 := \widetilde{g}_j\big|_{\Sigma_0}$ as the group $\mathrm{Inv(g_j^0)} := \{\mathrm{f} \in \mathrm{Diff}(\Sigma_0, \mathrm{p}_0),\ \mathrm{g_j^0} \circ \mathrm{f} = \mathrm{g_j^0}\}$. In other words, the invariance group of g_j^0 is the group $\mathrm{Inv(g_j^0)} \subset \mathrm{Diff}(\Sigma_0, \mathrm{p}_0)$ of map germs that preserve the fibers of g_j^0. Clearly $\mathrm{Inv(g_j^0)}$ is a finite (resonant) group. Let us now denote by $\mathrm{Inv}(\widetilde{\mathcal{F}}^{(1)}, \Sigma_0) \subset \mathrm{Diff}(\Sigma_0, \mathrm{p}_0)$ the subgroup generated by the invariance groups $\mathrm{Inv(g_j^0), j = 1, ..., m}$. We call $\mathrm{Inv}(\widetilde{\mathcal{F}}^{(1)}, \Sigma_0)$ the *global invariance group* of $\widetilde{\mathcal{F}}^{(1)}$ with respect to (Σ_0, p_0). Then, from the above we immediately obtain:

Claim 6.2. *The global invariance group* $\mathrm{Inv}(\widetilde{\mathcal{F}}^{(1)}, \Sigma_0)$ *is a resonant group.*

Since $\mathrm{Inv}(\widetilde{\mathcal{F}}^{(1)}, \Sigma_0)$ preserves the leaves of $\widetilde{\mathcal{F}}^{(1)}$ we have by Claim 6.1 that $\mathrm{Inv}(\widetilde{\mathcal{F}}^{(1)}, \Sigma_0)$ is a finite group. Notice that this global invariance group contains in a natural way the local

invariance groups of the local first integrals g_j. Therefore, as observed in [18], once we have proved that the global invariance group $\mathrm{Inv}(\widetilde{\mathcal{F}}^{(1)}, \Sigma_0)$ is finite, together with the fact that the singularities in $E^{(1)}$ exhibit local holomorphic first integrals, we conclude as in [29] (or as in the original proof in [18]) that the foliation $\widetilde{\mathcal{F}}^{(1)}$ and therefore the foliation $\mathcal{F}$ has a holomorphic first integral.
Case (ii). Now we assume that the germ of $\mathcal{F}$ at the origin is not fully-resonant. This means that some singularity $\widetilde{p} \in \mathrm{Sing}(\widetilde{\mathcal{F}}(1))$ in the first blow-up is not fully-resonant. Such a singularity, by the induction hypothesis, exhibits some real non-resonant singularity in its own reduction of singularities. Given any local separatrix $\Gamma_{\widetilde{p}}$ through $\widetilde{p}$, and a transverse disc Σ meeting $\Gamma_{\widetilde{p}}$ at a point $\widetilde{p} \neq \widetilde{q} = \Sigma \cap \Gamma_{\widetilde{p}}$, the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\widetilde{\mathcal{F}}(1)_{\widetilde{\mathrm{p}}}, \Sigma, \widetilde{\mathrm{q}})$ is a group containing a (PCO) Cremer map germ.

Claim 6.3. *Given any separatrix Γ through the origin, and a transverse disc Σ meeting Γ at a point $0 \neq q = \Sigma \cap \Gamma$, the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\mathcal{F}, \Sigma, \mathrm{q})$ is an abelian (formally linearizable) group containing some (PCO) Cremer map germ.*

Proof of Claim 6.3. Consider any separatrix Γ of $\mathcal{F}$ through the origin. Since the projective line $E^{(1)}$ in the first blow-up is invariant, the lift $\widetilde{\Gamma}$ is the separatrix of some singularity $\widetilde{p}_1$ of $\widetilde{\mathcal{F}}(1)$. If $\widetilde{p}_1$ is not fully-resonant, then by the paragraph preceding Claim 6.3, we conclude that the virtual holonomy group associated to this separatrix Γ contains a Cremer map and has infinitely many periodic orbits. Assume now that $\widetilde{p}_1$ is fully-resonant. In this case we have to show the existence of a Cremer map in the virtual holonomy of the separatrix $\widetilde{\Gamma}$ by "importing" this map from some virtual holonomy of other singularity. Indeed, by hypothesis, some singularity $\widetilde{p}$ in the first blow-up is not fully-resonant. Therefore, its virtual holonomy relatively to the separatrix contained in the projective line, contains a non-resonant map germ. Now, since the projective line $E^{(1)}$ is invariant, given two points $\widetilde{q}$ and $\widetilde{q}_1$, close to $\widetilde{p}$ and $\widetilde{p}_1$ respectively, and transverse discs Σ and Σ_1 meeting $E^{(1)}$ at these points respectively, we can choose a simple path $\alpha\colon [0,1] \to E^{(1)} \setminus \mathrm{Sing}(\widetilde{\mathcal{F}}(1))$ from $\widetilde{q}$ to $\widetilde{q}_1$. The holonomy map $h_\alpha\colon (\Sigma, \widetilde{q}) \to (\Sigma_1, \widetilde{q}_1)$ associated to the path α (recall that $E^{(1)} \setminus \mathrm{Sing}(\widetilde{\mathcal{F}}(1))$ is a leaf of $\widetilde{\mathcal{F}}(1)$), induces a natural morphism for the virtual holonomy groups $\alpha^*\colon \mathrm{Hol}^{\mathrm{virt}}(\widetilde{\mathcal{F}}(1), \Sigma_1, \widetilde{\mathrm{q}}_1) \to \mathrm{Hol}^{\mathrm{virt}}(\widetilde{\mathcal{F}}(1), \Sigma, \mathrm{q})$, by $\alpha^* : h \mapsto h_\alpha^{-1} \circ h \circ h_\alpha$. Since $h_{\alpha^{-1}} = (h_\alpha)^{-1}$ in terms of holonomy maps, we conclude that the above morphism is actually an isomorphism between the virtual holonomy groups. Thus, also the virtual holonomy group associated to the separatrix $\widetilde{\Gamma}$ of $\widetilde{\mathcal{F}}(1)$ through $\widetilde{p}_1$ is a real group of rotations and contains some irrational rotation map. Recall that the blow-up is a diffeomorphism off the origin and off the exceptional divisor, so that the maps in the virtual holonomy of $\widetilde{\Gamma}$ induce maps in the disc Σ transverse to Γ in $\mathbb{C}^2$, but which are defined only in the punctured disc, i.e., off the origin. Nevertheless, since these projected maps are one-to-one, the classical Riemann extension theorem for bounded holomorphic maps shows that indeed such maps induce germs of diffeomorphisms defined in the disc Σ. These diffeomorphisms are virtual holonomy maps of the separatrix Γ of $\mathcal{F}$ evaluated at the transverse section Σ. Hence, by projecting the maps in $\mathrm{Hol}^{\mathrm{virt}}(\widetilde{\mathcal{F}}(1), \Sigma, \widetilde{\mathrm{q}})$ we obtain non-resonant, actually (PCO) Cremer maps in this virtual holonomy group as stated. □

From Claim 6.3, each virtual holonomy group associated to $\mathcal{F}$ is abelian, formally linearizable, containing a non-resonant map. It follows then (as in [7], Proposition 1, page 433 or Lemma 4, page 435, for the convergent case, or also as in [28], see Proposition 3 (a) page 10 and Lemma 3 page 11, for the formal case) that $\mathcal{F}$ is given by a formal closed meromorphic 1-form with simple poles. More precisely, given a holomorphic one-form Ω defining a representative of $\mathcal{F}$ in a neighborhood U of the origin $0 \in \mathbb{C}^2$, there are formal functions $\hat{f}_j \in \mathcal{O}_2$ in two complex variables and complex numbers $\lambda_j, j = 1, ..., r$; such that (in terms of formal expressions) $\Omega \wedge \hat{\omega} = 0$ where $\widehat{\omega} = \sum\limits_{j=1}^{2} \lambda_j d\hat{f}_j/\hat{f}_j$.

Now it remains to prove that (2) implies (1). Indeed, if $\mathcal{F}$ admits a holomorphic first integral then all leaves are closed, except for a finite number of leaves, those containing the separatrices. If $\mathcal{F}$ is formally equivalent to a (non-dicritical) Darboux foliation, admitting a (PCO) Cremer type singularity then clearly also $\mathcal{F}$ is a (non-dicritical) foliation with a positive measure of closed leaves. This ends the proof of Theorem 1.1. □

References

[1] J.C. Alexander, A. Verjovsky: First integrals for singular holomorphic foliations with leaves of bounded volume, Holomorphic dynamics (Mexico, 1986), 1–10, Lecture Notes in Math. 1345, Springer-Verlag, Berlin - New York, 1988.

[2] F. Bracci: Local dynamics of holomorphic diffeomorphisms. Boll. UMI 7-B (2004), 609-636.

[3] F. Bracci: Local holomorphic dynamics of diffeomorphisms in dimension one. Contemporary Mathematics, vol. 525, 2010, pages 1–42. DOI: 10.1090/conm/525/10363

[4] C. Camacho: On the local structure of conformal maps and holomorphic vector fields in $\mathbb{C}^2$, Journées Singuliéres de Dijon (Univ. Dijon, Dijon, 1978, 3, 83–94), Astérisque 59-60, Soc. Math. France, Paris 1978.

[5] C. Camacho, N. Kuiper, J. Palis: The topology of holomorphic flows with singularity. Inst. Hautes Études Sci. Publ. Math., no. 48 (1978), 5–38.

[6] C. Camacho, A. Lins Neto: Geometric theory of foliations, Birkhauser Inc., Boston, Mass., 1985. DOI: 10.1007/978-1-4612-5292-4

[7] C. Camacho, A. Lins Neto, P. Sad: Foliations with algebraic limit sets, Ann. of Math. (2) 136 (1992), 429–446. DOI: 10.2307/2946610

[8] C. Camacho and P. Sad: Invariant varieties through singularities of holomorphic vector fields, Ann. of Math. 115 (1982), 579–595. DOI: 10.2307/2007013

[9] C. Camacho and B. Scárdua: Foliations on complex projective spaces with algebraic limit sets. Géométrie complexe et systèmes dynamiques (Orsay, 1995). Astérisque, 261, 57–88, Soc. Math. France, Paris 2000.

[10] C. Camacho and B. Scárdua: On the existence of stable compact leaves for transversely holomorphic foliations, Topology and its Applications Volume 160, Issue 13, 15 August 2013, Pages 1802-1808. DOI: 10.1016/j.topol.2013.07.012

[11] H. Cremer: Zum Zentrumproblem. Math. An.. 98 (1927), 151-163.

[12] H. Dulac; Solutions d'un système de équations différentielles dans le voisinage des valeus singulières, Bull. Soc. Math. France 40 (1912) 324–383.

[13] P. Griffiths, ; J. Harris: *Principles of algebraic geometry.* Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. xiv + 813 pp.

[14] R.C. Gunning: Introduction to holomorphic functions of several variables, vol. I, Function Theory. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990.

[15] R.C. Gunning: Introduction to holomorphic functions of several variables, vol. II, Local Theory. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1990

[16] J.P. Jouanolou: Équations de Pfaff algèbriques, Lecture Notes in Math. 708, Springer-Verlag, Berlin, 1979.

[17] A. Lins Neto: *Construction of singular holomorphic vector fields and foliations in dimension two.* J. Differential Geom. 26 (1987), no. 1, 1 - 31.

[18] J.F. Mattei and R. Moussu: Holonomie et intégrales premières, Ann. Sci. École Norm. Sup. (4) **13** (1980), 469–523.

[19] J. Martinet and J-P. Ramis: Classification analytique des équations différentielles non linéaires resonnants du premier ordre, Ann. Sci. École Norm. Sup. (4) 16 (1983), 571–621.

[20] J. Martinet and J-P. Ramis: Problème de modules pour des équations différentielles non linéaires du premier ordre, Publ. Math. Inst. Hautes Études Scientifiques, 55 (1982), 63–124. DOI: 10.1007/BF02698695

[21] R. Moussu: *Sur l'existence d'intégrales premières holomorphes.* Annali della Scuola Normale Superiore di Pisa - Classe di Scienze 26.4 (1998): 709-717.

[22] R. Pérez-Marco: Solution complète au problème de Siegel de linéarisation d'une application holomorphe au voisinage d'un point fixe (d'après J.-C. Yoccoz), Astérisque, , #206: pp. No. 753, 4, 273-310, séminaire Bourbaki, Vol. 1991/92, 1992

[23] R. Pérez-Marco: Sur les dynamiques holomorphes non linéarisables et une conjecture de V.I. Arnold. Ann. Sci. École Norm. Sup. 26 (1993), 565-644.

[24] R. Pérez-Marco: Sur une question de Dulac et Fatou. C.R. Acad. Sci. Paris 321 (1995), 1045-1048.

[25] R. Pérez-Marco: Fixed points and circle maps. Acta Math. 179 (1997), 243-294.

[26] R. Pérez-Marco, J.-C. Yoccoz: Germes de feuilletages holomorphes à holonomie prescrite. Complex analytic methods in dynamical systems (Rio de Janeiro, 1992), Astérisque 222 (1994), 7, 345-371

[27] J. Raissy: Geometrical methods in the normalization of germs of biholomorphisms, Ph.D. Thesis, Università di Pisa 2010.

[28] B. Scárdua: Integration of complex differential equations. Journal of Dynamical and Control Systems, issue 1, vol. **5**, pp.1-50, 1999.

[29] B. Scárdua: Complex vector fields having orbits with bounded geometry, Tohoku Math. J. (2) 54 (2002), no. 3, 367–392.

[30] A. Seidenberg: Reduction of singularities of the differential equation $Ady = Bdx$, Amer. J. Math. 90 (1968), 248–269.

[31] J.-C. Yoccoz: Théorème de Siegel, nombres de Bruno et polynomes quadratiques. Astérisque 231 (1995), 3-88.

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