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VECTOR FIELDS TANGENT TO FOLIATIONS AND BLOW-UPS

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Dedicated to Xavier Gómez-Mont on the occasion of his 60th birthday.

1. Introduction

In this note we consider germs of holomorphic vector fields at the origin of $(\mathbb{C}^3, 0)$

$$\xi = a(x,y,z)\frac{\partial}{\partial x} + b(x,y,z)\frac{\partial}{\partial y} + c(x,y,z)\frac{\partial}{\partial z}$$

having a formal invariant curve $\widehat{\Gamma}$ that is totally transcendental, that is $\widehat{\Gamma}$ is not contained in any germ of analytic hypersurface of $(\mathbb{C}^3, 0)$.

It is known (see [4, 6, 7]) that among such vector fields we find the only ones that cannot be desingularized by birational blow-ups in the sense that it is not possible to obtain *elementary singularities* (non nilpotent linear part).

On the other hand, not all germs of vector fields are tangent to a codimension one holomorphic foliation of $(\mathbb{C}^3, 0)$.

We present here a result relating the above two properties

Theorem 1. *Let ξ be a germ of vector field on $(\mathbb{C}^3, 0)$ having a totally transcendental formal invariant curve $\widehat{\Gamma}$ and let D be a normal crossings divisor of $(\mathbb{C}^3, 0)$. Denote by $\mathcal{L}$ the foliation by lines induced by ξ. Assume that there is a germ of codimension one holomorphic foliation $\mathcal{F}$ of $(\mathbb{C}^3, 0)$ such that ξ is tangent to $\mathcal{F}$. Then there is a finite sequence of local blow-ups*

$$(\mathbb{C}^3, 0) = (M_0, p_0) \overset{\pi_1}{\leftarrow} (M_1, p_1) \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_n}{\leftarrow} (M_n, p_n) \tag{1}$$

with the following properties:

(1) *The center Y_{i-1} of π_i is a point or a germ of non-singular analytic curve invariant for the transformed foliation by lines $\mathcal{L}_{i-1}$ of $\mathcal{L}$. Moreover Y_{i-1} has normal crossings with the total transform D_{i-1} of D.*
(2) *The points p_i belong to the strict transform $\widehat{\Gamma}_i$ of $\widehat{\Gamma}$.*
(3) *The final transform $\mathcal{L}_n$ is generated by an elementary germ of vector field.*

As it has been noted by F. Sanz and F. Sancho, (in [3] one find a first reference to this example) there are examples of germs of vector fields ξ such that it is not possible to find a sequence as in Equation 1 with the above properties (1),(2) and (3). This is the starting point of the non-birational strategy of Panazzolo in [6]. The specific example is the following one

$$\xi_{\alpha,\beta,\lambda;x,y,z} = x\left(x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y} - \beta z\frac{\partial}{\partial z}\right) + xz\frac{\partial}{\partial y} + (y - \lambda x)\frac{\partial}{\partial z},$$

where $\alpha, \beta \in \mathbb{R}_{\geq 0}$ and $\lambda \in \mathbb{R}_{>0}$. It is an obvious corollary of Theorem 1 that this vector field is not tangent to any codimension one foliation. Anyway, we start this note by giving a direct proof of this fact, based on geometrical arguments and on the behaviour of $\xi_{\alpha,\beta,\lambda;x,y,z}$ under blow-up. The proof of Theorem 1 comes just by remarking that the "bad" behaviour of the Sanz-Sancho vector fields does not occur when ξ is tangent to a codimension one foliation.

As a direct consequence of Theorem 1 we obtain that any germ of vector field tangent to a codimension one foliation can be desingularized.

2. The properties of Sanz-Sancho's example

We recall here the properties of the examples of Sanz-Sancho that allow to assure the non-existence of a desingularization sequence as in Theorem 1.

First of all, the singular locus of $\xi_{\alpha,\beta,\lambda;x,y,z}$ is exactly $x = y = 0$ and the divisor $x = 0$ is invariant.

Proposition 1. *Let $\pi : M \to (\mathbb{C}^3, 0)$ be the blow-up with center the origin of $\mathbb{C}^3$ and let ξ' be the transform of $\xi_{\alpha,\beta,\lambda;x,y,z}$ by π. Denote by $E = \pi^{-1}(0)$ the exceptional divisor and by H' the strict transform of $x = 0$ by π. Then*

(1) *The exceptional divisor E is invariant by ξ'.*
(2) *There is exactly one point $p' \in Sing(\xi') \cap E \setminus H'$ where ξ' has linear part of rank one.*
(3) *The point p' is in the strict transform of the line $y - \lambda x = z - \lambda(\alpha+1)x = 0$.*
(4) *If we take local coordinates x', y', z' at p' given by $x' = x$, $y' = y/x - \lambda$ and $z' = z/x - \lambda(\alpha+1)$, then the germ of ξ' at p' coincides with $\xi_{\alpha',\beta',\lambda';x',y',z'}$ where*
$$\alpha' = \alpha + 1,\ \beta' = \beta + 1,\ \lambda' = \lambda(\alpha+1)(\beta+1).$$
(5) *The singular locus $Sing(\xi') \setminus H'$ outside H' corresponds to the projective straight line $L \subset E$ passing through p' with local coordinates $x' = y' = 0$.*

Proof. Consider coordinates x', y^*, z^* in the *first* chart of the blow-up, given by $x' = x$, $y^* = y/x$ and $z^* = z/x$. The transformed vector field ξ' is given by
$$\xi' = x'\left\{x'\partial/\partial x' - (\alpha+1)y^*\partial/\partial y^* - (\beta+1)z^*\partial/\partial z^*\right\} + x'z^*\partial/\partial y^* + (y^* - \lambda)\partial/\partial z^*.$$
We already see that $\mathrm{Sing}(\xi') \setminus H'$ is given by $x' = 0, y^* - \lambda = 0$. Put $y' = y^* - \lambda$ and $z' = z^* - \mu$, then
$$\begin{aligned}\xi' \quad = \quad & x'\left\{x'\partial/\partial x' - (\alpha+1)(y'+\lambda)\partial/\partial y' - (\beta+1)(z'+\mu)\partial/\partial z'\right\} + \\ & x'(z'+\mu)\partial/\partial y' + y'\partial/\partial z' = \\ & x'\left\{x'\partial/\partial x' - (\alpha+1)y'\partial/\partial y' - (\beta+1)z'\partial/\partial z'\right\} + \\ & x'(z'+\mu-\lambda(\alpha+1))\partial/\partial y' + (y' - \mu(\beta+1)x')\partial/\partial z'.\end{aligned}$$
The value $\mu = \lambda(\alpha+1)$ gives the only point p' with linear part of rank one.

All the statements are now directly induced from the precedent computations. □

Now, let us recall a general fact on line foliations

Proposition 2. *Let $\widehat{\Gamma}$ be a formal curve for $(\mathbb{C}^3, 0)$. Let $\mathcal{L}$ be a foliation by lines of $(\mathbb{C}^3, 0)$ generated by a germ of vector field ξ. Let us consider the sequence of blow-ups corresponding to the infinitely near points of $\widehat{\Gamma}$*

$$\mathcal{S}_{\widehat{\Gamma}}:\ (\mathbb{C}^3, 0) = (M_0, q_0) \overset{\sigma_1}{\leftarrow} (M_1, q_1) \overset{\sigma_2}{\leftarrow} (M_2, q_2) \cdots \tag{2}$$

where the center of σ_i is q_{i-1} and q_i is in the strict transform $\widehat{\Gamma}_i$ of $\widehat{\Gamma}$. Then the following properties are equivalent

(1) *$\widehat{\Gamma}$ is invariant by $\mathcal{L}$.*
(2) *There is an index k_0 such that for all $k \geq k_0$ the point q_k is singular for the transform $\mathcal{L}_k$ of $\mathcal{L}$.*

Proof. See [1, 2] □

Let us start with $\xi_0 = \xi_{\alpha,\beta,\lambda;x,y,z}$. We blow-up to obtain the point p_1 and coordinates x_1, y_1, z_1 as in Proposition 1 where the transform ξ_1 of ξ is given by $\xi_1 = \xi_{\alpha_1,\beta_1,\lambda_1;x_1,y_1,z_1}$. We repeat the procedure indefinitely to obtain $p_0, p_1, p_2, \ldots$. These ones are the infinitely near points of a non singular formal curve $\widehat{\Gamma}$ transversal to $x = 0$. Moreover, by Proposition 2 the curve $\widehat{\Gamma}$ is invariant by ξ_0. In view of Proposition 1 we have that $\widehat{\Gamma}$ is parameterized by

$$y = \lambda x + \sum_{k=2}^{\infty} \lambda_{k-1} x^k;\ z = (\alpha+1)\lambda x + \sum_{k=2}^{\infty} (\alpha_{k-1}+1)\lambda_{k-1} x^k.$$

Remark 1. If we start with $\alpha = \beta = 0$, $\lambda = 1$, we get

$$y = x + \sum_{k=2}^{\infty} (k-1)!(k-1)! x^k;\ z = x + \sum_{k=2}^{\infty} k!(k-1)! x^k$$

that are obviously non convergent formal power series.

Let us give a general proof that $\widehat{\Gamma}$ is not contained in a germ of analytic surface $S \subset (\mathbb{C}^3, 0)$. We are going to do it by using elementary technics of blow-ups and transcendency. Let us work by contradiction by assuming that there is S containing $\widehat{\Gamma}$. First of all let us remark that $\widehat{\Gamma}$ is not a convergent germ of curve, otherwise its plane projection

$$y = \lambda x + \sum_{k=2}^{\infty} \lambda_{k-1} x^k$$

should be convergent. But this is not the case, since

$$\lambda_k = \lambda(\alpha+1)(\beta+1)(\alpha+2)(\beta+2)\cdots(\alpha+k)(\beta+k).$$

Next Lemma is a version of the transcendence argument known as "truc de Moussu" (see for instance [5]).

Lemma 1. *Let $\widehat{\Gamma}$ be a formal non convergent invariant curve of a germ of analytic vector field ξ of $(\mathbb{C}^3, 0)$ such that Sing(ξ) has codimension at least two. Assume that $\widehat{\Gamma}$ is contained in a germ of irreducible surface $(S, 0) \subset (\mathbb{C}^3, 0)$. Then $(S, 0)$ is invariant by ξ.*

Proof. The analytic set of the tangency locus between ξ and S contains $\widehat{\Gamma}$ but it cannot be equal to $\widehat{\Gamma}$. Thus it coincides with S. □

As a consequence of Lemma 1, we deduce that S is invariant by ξ_0. In particular the intersection $S \cap (x = 0)$ must be invariant by $\xi_0|_{(x=0)}$. Now, noting that

$$\xi_0|_{(x=0)} = y\frac{\partial}{\partial z}$$

we deduce that $S \cap (x = 0) = (x = y = 0)$.

By next Lemma 2 we reduce our problem to the case that S is non-singular and with normal crossings with $x = 0$.

Lemma 2. *Let $\widehat{\Gamma}$ be a non convergent formal curve for $(\mathbb{C}^3, 0)$ contained in a surface $S \subset (\mathbb{C}^3, 0)$. Consider the sequence of blow-ups corresponding to the infinitely near points of $\widehat{\Gamma}$*

$$\mathcal{S}_{\widehat{\Gamma}}:\ (\mathbb{C}^3, 0) = (M_0, q_0) \stackrel{\sigma_1}{\leftarrow} (M_1, q_1) \stackrel{\sigma_2}{\leftarrow} (M_2, q_2) \cdots$$

as in Equation 2. There is an index k_0 such that for all $k \geq k_0$ the strict transform S_k of the surface S is non-singular at q_k and has normal crossings with the exceptional divisor.

Proof. The proof is similar to the proof of Proposition 2. We do it for the sake of completeness. Up to a finite number of blow-ups, we can assume that $\widehat{\Gamma}$ is non singular and transversal to $x = 0$. We can take formal coordinates $x, \hat{y}, \hat{z}$ such that $\widehat{\Gamma} = (\hat{y} = \hat{z} = 0)$. Let us express the blow-ups in that coordinates. The first one is given by

$$x = x';\ \hat{y} = x\hat{y}';\ \hat{z} = x\hat{z}'.$$

Now, let $f(x, \hat{y}, \hat{z}) = 0$ be a formal equation of S. We know that $f = \hat{y}f' + \hat{z}f''$, moreover, $\widehat{\Gamma}$ is not in the singular locus of S since it is not convergent. Then, we have that

$$f'(x, 0, 0) = x^s\hat{u}, f''(x, 0, 0) = x^t\hat{v}$$

where either $\hat{u}(0, 0, 0) \neq 0$ or $\hat{v}(0, 0, 0) \neq 0$. To fix ideas, assume that $\hat{u}(0, 0, 0) \neq 0$ and the origin is singular or has no normal crossings with $x = 0$. After one blow-up we get $s' < s$ and this cannot be repeated indefinitely. □

Now, up to blow-up, we can assume that S in non singular at p, has normal crossings with $x = 0$ and moreover $S \cap (x = 0) = (x = y = 0)$. This suggests to blow-up the line $x = y = 0$. We explain the effect of performing this blow-up in next statement.

Proposition 3. *Let $\pi : M \to (\mathbb{C}^3, 0)$ be the blow-up with center $x = y = 0$ and let ξ' be the transform of $\xi_{\alpha,\beta,\lambda;x,y,z}$ by π. Denote by $E = \pi^{-1}(x = y = 0)$ the exceptional divisor and by H' the strict transform of $x = 0$ by π. Then*

1. *The exceptional divisor E is invariant by ξ'.*
2. *There is exactly one point $p' \in Sing(\xi') \cap \pi^{-1}(0) \setminus H'$ where ξ' has linear part of rank one. The point p' is in the strict transform of the plane $y - \lambda x = 0$.*
3. *The singular locus $Sing(\xi') \setminus H'$ outside H' coincides with $\pi^{-1}(0)$.*
4. *If we take local coordinates x', y', z' at p' given by $x' = x$, $y' = z$ and $z' = y/x - \lambda$, then the germ of ξ' at p' coincides with $\xi_{\alpha',\beta',\lambda';x',y',z'}$ where*

$$\alpha' = \beta,\ \beta' = \alpha + 1,\ \lambda' = \lambda(\alpha + 1).$$

Proof. Consider coordinates x', y^*, z^* in the first chart of the blow-up, given by $x' = x$, $y^* = y/x$ and $z^* = z$. The transformed vector field ξ' is given in these coordinates by

$$\xi' = x'\left\{x'\partial/\partial x' - (\alpha + 1)y^*\partial/\partial y^* - \beta z^*\partial/\partial z^*\right\} + z^*\partial/\partial y^* + x'(y^* - \lambda)\partial/\partial z^*.$$

We already see that $\mathrm{Sing}(\xi') \setminus H'$ is given by $x' = z^* = 0$. Put $z' = y^* - \lambda$ and $y' = z^*$, then

$$\xi' = x'\left\{x'\partial/\partial x' - \beta y'\partial/\partial y' - (\alpha + 1)z'\partial/\partial z'\right\} + x'z'\partial/\partial y' + (y' - \lambda(\alpha + 1)x')\partial/\partial z'$$

All the statements are now directly induced from the precedent computations. □

Now Proposition 3 gives a contradiction with the existence of S. In fact, since S has normal crossings with $x = 0$ and $x = y = 0$ is contained in S, the strict transform S' of S by the blow-up π with center $x = y = 0$ does not contain $\pi^{-1}(0)$. We can do the same argument as for S at the point p' to see that $S' \cap E' = \mathrm{Sing}(\xi')$, but $\mathrm{Sing}(\xi') = \pi^{-1}(0)$ (locally at p'). This is the desired contradiction.

Thus, we have proved that $\widehat{\Gamma}$ is totally transcendental.

Proposition 1 and Proposition 3 are the initial remarks of F. Sanz and F. Sancho to show that the vector fields $\xi_{\alpha,\beta,\lambda;x,y,z}$ cannot be desingularized by blow-ups with centers in the singular locus, since the only possibilities are the origin and the line $x = y = 0$, and in both cases we repeat the situation. Anyway, in order to be complete, we need to show that there is no other analytic invariant curve that could be used as a center.

Corollary 1. *The singular locus $x = y = 0$ is the only nonsingular germ of analytic curve invariant by $\xi_{\alpha,\beta,\lambda;x,y,z}$ and having normal crossings with the divisor $x = 0$.*

Proof. Assume that γ is a nonsingular invariant curve having normal crossings with $x=0$ and different from $x=y=0$. The only invariant curve contained in $x=0$ is precisely $x=y=0$, hence γ must be transversal to $x=0$. By blowing-up the origin as in Proposition 1, we see that the strict transform of γ is transversal to the exceptional divisor in a point q' of the singular locus of ξ'. If $q'=p'$, we repeat the procedure. At one moment $q'\neq p'$, since otherwise γ and $\widehat{\Gamma}$ would have the same infinitely near points and thus $\gamma=\widehat{\Gamma}$ and this is not possible since $\widehat{\Gamma}$ is completely transcendental and γ is a germ of analytic curve. Now, assume that $q'\neq p'$. Actually it is enough to show that there is no invariant curve for $\xi_{\alpha,\beta,\lambda;x,y,z}$ in a point of coordinates $x=0, y=0, z=z_0\neq 0$ that is non singular and transversal to $x=0$. This is a consequence of Proposition 3 since blowing-up $x=y=0$, we see that there is no singular points over $(0,0,z_0)$ outside the strict transform of $x=0$. □

3. An example of vector field not tangent to a foliation

In this section we show that $\xi_{\alpha,\beta,\lambda;x,y,z}$ is not tangent to any codimension one foliation of $(\mathbb{C}^3,0)$.

Lemma 3. *Let η be a germ of vector field not collinear with $\xi_{\alpha,\beta,\lambda;x,y,z}$ and let $\mathcal{L}$ be the foliation by lines induced by η. Then*

1. *$\widehat{\Gamma}$ is not an invariant curve of η.*
2. *If we consider the sequence $\mathcal{S}_{\widehat{\Gamma}}$ of the infinitely near points of $\widehat{\Gamma}$ described in Equation 2, there is an index k_0 such that for all $k\geq k_0$ the transform $\mathcal{L}_k$ is generated by a non-singular vector field and the exceptional divisor is invariant.*

Proof. If $\widehat{\Gamma}$ is invariant for η, then it is contained in the set of collinearity of η and $\xi_{\alpha,\beta,\lambda;x,y,z}$, this is an analytic set that should be the whole space, because of the fact that $\widehat{\Gamma}$ is totally transcendental. The second part is a direct consequence of Proposition 2. □

Let us assume now that $\xi_{\alpha,\beta,\lambda;x,y,z}$ is tangent to a codimension one foliation $\mathcal{F}$. Then there is another germ of vector field η tangent to $\mathcal{F}$ and not collinear with $\xi_{\alpha,\beta,\lambda;x,y,z}$. Up to blowing-up points, and in order to find a contradiction, we can assume without loss of generality that η is non singular and tangent to $x=0$. Thus, the foliation $\mathcal{F}$ has dimensional type two, in the sense that it is trivialized by the flow of η, moreover it is singular, otherwise $\widehat{\Gamma}$ should be contained in a germ of hyper-surface. The singular locus $\mathrm{Sing}(\mathcal{F})$ is a curve invariant by η and $\xi_{\alpha,\beta,\lambda;x,y,z}$. The only possibility is then that

$$\mathrm{Sing}(\mathcal{F})=(x=y=0). \tag{3}$$

Now, we perform the blow-up with center $x=y=0$ to obtain transforms $\mathcal{F}'$, $\xi_{\alpha',\beta',\lambda';x',y',z'}$ and η' that we consider locally at the point p' described in Proposition 3. We take notations as in Proposition 3. By the same argument as before, and since η' is still a non singular vector field tangent to $\mathcal{F}'$, we have that

$$\mathrm{Sing}(\mathcal{F}')=(x'=y'=0).$$

But on the other hand, $\mathcal{F}$ has dimensional type two and thus the singular locus of $\mathcal{F}'$ must be etale over $\mathrm{Sing}(\mathcal{F})$ under the blow-up σ. This is not the case, since around p' we have that

$$\sigma(\mathrm{Sing}(\mathcal{F}'))=\{p\}.$$

This is the desired contradiction.

4. Vector fields tangent to a foliation

In this section we give a proof of Theorem 1. Take notations and hypothesis as in Theorem 1. We shall reason by contradiction by showing that if the vector field ξ cannot be desingularized, then it has the properties of Sanz-Sancho's examples that are contradictory with the fact of being tangent to a foliation.

We assume thus that ξ cannot be desingularized and that it is tangent to a foliation $\mathcal{F}$. We also consider the sequence $\mathcal{S}_{\widehat{\Gamma}}$ of infinitely near points of $\widehat{\Gamma}$ as in Equation 2

$$\mathcal{S}_{\widehat{\Gamma}}:\ (\mathbb{C}^3,0)=(M_0,q_0)\overset{\sigma_1}{\leftarrow}(M_1,q_1)\overset{\sigma_2}{\leftarrow}(M_2,q_2)\cdots$$

We know that $\widehat{\Gamma}$ is desingularized by this sequence and thus there is k_0 such that for any $k\geq k_0$ the strict transform $\widehat{\Gamma}_k$ of $\widehat{\Gamma}$ is nonsingular and transversal to the exceptional divisor (this one is also non singular at p_k). We can assume without loss of generality that $k_0=0$ and that the exceptional divisor is given by $x=0$. Now, we can parameterize $\widehat{\Gamma}$ by

$$y=\hat{\phi}(x);\quad z=\hat{\psi}(x).$$

Let us see how is transformed ξ under the sequence $\mathcal{S}_{\widehat{\Gamma}}$. For our purposes we can use the formal coordinates $x,\hat{y}=y-\hat{\phi}(x),\hat{z}=z-\hat{\psi}(x)$. Then all the blow-ups are given by a equation having the same shape, that is we have formal coordinates at q_k given inductively by

$$x_k=x,\ \hat{y}_k=\hat{y}_{k-1}/x,\ \hat{z}_k=\hat{z}_{k-1}/x,$$

starting by $\hat{y}_0=\hat{y},\ \hat{z}_0=\hat{z}$. Let us write the vector field ξ (up to multiplying it by x if it is necessary to keep a logarithmic expression) as

$$\xi=\hat{a}(x,\hat{y},\hat{z})x\frac{\partial}{\partial x}+\hat{b}(x,\hat{y},\hat{z})\frac{\partial}{\partial\hat{y}}+\hat{c}(x,\hat{y},\hat{z})\frac{\partial}{\partial\hat{z}}.$$

Consider the invariant

$$r_0=\min\{\nu_0(\hat{a}),\nu_0(\hat{b})-1,\nu_0(\hat{c})-1\},$$

where $\nu_0(f)$ is the order of f at the origin. Then the transformed line foliation $\mathcal{L}_k$ is given at q_k by

$$\xi_k=\hat{a}_k\left\{x\frac{\partial}{\partial x}-k\hat{y}_k\frac{\partial}{\partial\hat{y}_k}-k\hat{z}\frac{\partial}{\partial\hat{z}_k}\right\}+\hat{b}_k\frac{\partial}{\partial\hat{y}_k}+\hat{c}_k\frac{\partial}{\partial\hat{z}_k}$$

where

$$\hat{a}_{k+1}=\hat{a}_k/x^{r_k};\ \hat{b}_{k+1}=\hat{b}_k/x^{r_k+1};\ \hat{c}_{k+1}=\hat{c}_k/x^{r_k+1},$$

and $r_k=\min\{\nu_{q_k}(\hat{a}_k),\nu_{q_k}(\hat{b}_k)-1,\nu_{q_k}(\hat{c}_k)-1\}$. The starting terms of this induction are evident. Let us note that $r_k\geq 0$ for all k since we are in a singular point of $\mathcal{L}_k$.

Now, we know that $\widehat{\Gamma}=(\hat{y}=\hat{z}=0)$ is invariant and it is not in the singular locus of ξ (otherwise ξ should be identically zero, since $\widehat{\Gamma}$ is completely transcendental). In algebraic terms this is explained by saying that

$$\hat{a}(x,0,0)\neq 0;\quad \hat{b}=\hat{y}\hat{b}'+\hat{z}\hat{b}'',\ \hat{c}=\hat{y}\hat{c}'+\hat{z}\hat{c}''.$$

Write $\hat{a}=x^s\hat{u}+\hat{y}\hat{a}'+\hat{z}\hat{a}''$, with $\hat{u}(0,0,0)\neq 0$. Up to a finite number of steps, we obtain that x^s divides $\hat{a}$ and we can write

$$\hat{a}=x^s\hat{U};\ \hat{U}(0,0,0)\neq 0.$$

Dividing by $\hat{U}$ we may assume that $\hat{a}=x^s$. Now, we conclude that $r_k=0$ for $k>>0$, otherwise s strictly decreases each time and once we obtain $s=0$ we get an elementary singularity, contradiction with our hypothesis. So we assume without loss of generality that $s>0$ and $r_k=0$ for all $k\geq 0$. This implies that

$$\min\{\nu_0(\hat{b}),\nu_0(\hat{c})\}=1.$$

Thus, up to one blow-up, we can write

$$\hat{b}=\alpha\hat{y}+\beta\hat{z}+\hat{y}x\tilde{b}'+\hat{z}x\tilde{b}'',\ \hat{c}=\gamma\hat{y}+\delta\hat{z}+\hat{y}x\tilde{c}'+\hat{z}x\tilde{c}'',$$

where $\alpha,\beta,\gamma,\delta$ are not all zero. Since the linear part must be nilpotent, up to a linear coordinate change in $\hat{y},\hat{z}$ we may assume that

$$\hat{b}=\hat{y}x\tilde{b}'+\hat{z}x\tilde{b}'',\ \hat{c}=\hat{y}+\hat{y}x\tilde{c}'+\hat{z}x\tilde{c}'',$$

and hence ξ has the expression (we take $n\in\mathbb{Z}_{\geq}0$)

$$\xi=x^s\left\{x\frac{\partial}{\partial x}-n\hat{y}\frac{\partial}{\partial\hat{y}}-n\hat{z}\frac{\partial}{\partial\hat{z}}\right\}+\hat{y}\frac{\partial}{\partial\hat{z}}+x\left\{(\hat{y}\tilde{b}'+\hat{z}\tilde{b}'')\frac{\partial}{\partial\hat{y}}+(\hat{y}\tilde{c}'+\hat{z}\tilde{c}'')\frac{\partial}{\partial\hat{z}}\right\}.$$

The singular locus $\mathrm{Sing}(\xi)$ is then $x=\hat{y}=0$.

Recall that we assume ξ to be tangent to the codimension one foliation $\mathcal{F}$. By the same arguments as in the precedent Section 3, up to blow-up some infinitely near points of $\widehat{\Gamma}$, we may assume that $\mathcal{F}$ is if dimensional type two and $x=0$ is invariant by $\mathcal{F}$. In particular it is also true that the singular locus $\mathrm{Sing}\mathcal{F}$ is $x=\hat{y}=0$. Now, let us blow-up this singular locus and let us focus on the transform ξ' of ξ at the origin of the first chart (that corresponds to the strict transform of $\widehat{\Gamma}$). The local coordinates are given by $x=x,\hat{y}=xy',\hat{z}=z'$ and ξ' is given by

$$\begin{aligned}\xi'=\quad & x^s\left\{x\tfrac{\partial}{\partial x}-(n+1)y'\tfrac{\partial}{\partial y'}-nz'\tfrac{\partial}{\partial z'}\right\}+xy'\tfrac{\partial}{\partial z'}+\\ & +(xy'\tilde{b}'+z'\tilde{b}'')\tfrac{\partial}{\partial y'}+x(y'x\tilde{c}'+z'\tilde{c}'')\tfrac{\partial}{\partial z'}.\end{aligned}$$

The new singular locus $\mathrm{Sing}(\xi')$ is $x=0=z'\tilde{b}''$. It contains $x=z'=0$. But this is not possible since $x=\hat{z}=0$ is not contained in the singular locus of the transform $\mathcal{F}'$ of $\mathcal{F}$, because $\mathcal{F}$ has dimensional type two, we have done a blow-up centered at the singular locus $x=\hat{y}=0$ of $\mathcal{F}$ and $x=\hat{z}=0$ projects under this blow-up to the origin and not to the whole singular locus of $\mathcal{F}$. This is the desired contradiction.

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