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ON SMOOTH DEFORMATIONS OF FOLIATIONS WITH SINGULARITIES

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Abstract. We study smooth deformations of codimension one foliations with Morse and Bott-Morse singularities of center-type. We show that in dimensions ≥ 3, every small smooth deformation by foliations of a Morse function with only center type singularities is a deformation by Morse functions. We also show that this statement is false in dimension 2. In the same vein we show that if $\mathcal{F}$ is a foliation with Bott-Morse singularities on a manifold M, all of center type, and if we assume there is a component $N \subset \operatorname{sing}(\mathcal{F})$ of codimension $m \geq 3$ such that $H^1(N, \mathbb{R}) = 0$, then every small smooth deformation $\{\mathcal{F}_t\}$ of $\mathcal{F}$ is compact, stable and given by a Bott-Morse function $f_t \colon M \to [0, 1]$ with only two critical values at 0 and 1. Furthermore, each such foliation $\{\mathcal{F}_t\}$ is topologically equivalent to $\mathcal{F}$. Hence, Bott-Morse foliations with only center-type singularities and having a component $N \subset \operatorname{sing}(\mathcal{F})$ of codimension $m \geq 3$ such that $H^1(N, \mathbb{R}) = 0$, are structurally stable under smooth deformations. These statements are false in general if we drop the codimension $m \geq 3$ condition.

1. Introduction and results

An important problem in geometry and dynamics is studying the stability of singular foliations under deformations. This is classical for 1-dimensional foliations defined by (real or complex) vector fields. For higher dimensional foliations, we need to impose some additional structure on the foliations and/or on the type of singularities, in order to be able to say something about them.

For instance, in the interesting article [4], the authors give extensions of Reeb's Stability Theorem to singular holomorphic foliations of codimension 1 having a meromorphic first integral and defined on projective manifolds $\mathcal{M}$ with $H^1(\mathcal{M}, \mathcal{C}) = 0$. In doing so, the authors study foliations defined by Lefschetz pencils, defined by a general meromorphic function, and prove a stability theorem for these. A key ingredient in the proof of that theorem is looking at the behavior of the foliation near a Kupka component of its singular set.

Let us recall that given any integrable polynomial homogeneous 1-form ω on $\mathbb{C}^{n+1}$ with singular set of codimension ≥ 2, we define the *Kupka singular set of* ω as

$$K(\omega) = \{p \in \mathbb{C}^{n+1} \backslash 0 \mid \omega(p) = 0, d\omega(p) \neq 0\}.$$

The *Kupka singular set of* the corresponding foliation $\mathcal{F} = \mathcal{F}(\omega)$ in $\mathbb{C}P(n)$ is $K(\mathcal{F}) = \pi(K(\omega))$ where π is the projectivization map.

We know from [3, 4, 6, 9] that if $n \geq 3$, then the Kupka set is a locally closed codimension 2 smooth submanifold of $\mathbb{C}P(n)$ which has a local product structure: Given a connected component $K \subset K(\mathcal{F})$ there exist a holomorphic 1-form η, called the transversal type of K, defined on a neighborhood of $0 \in \mathbb{C}^2$ and vanishing only at 0, a covering $\{U_\alpha\}$ of a neighborhood of K in $\mathbb{C}P(n)$ and a family of holomorphic submersions $\varphi_\alpha \colon U_\alpha \to \mathbb{C}^2$ satisfying that $\varphi_\alpha^{-1}(0) = K \cap U_\alpha$ and $\varphi_\alpha^* \eta$ defines $\mathcal{F}$ in U_α. Furthermore, $K(\mathcal{F})$ is persistent under small perturbations of $\mathcal{F}$, namely, fixed any $p \in K(\mathcal{F})$ with defining 1-form $\varphi^*\eta$ as above, and for any foliation $\mathcal{F}'$ sufficiently close to $\mathcal{F}$, there is a holomorphic 1-form η' close to η and a submersion φ' close to φ, such that $\mathcal{F}'$ is defined by $(\varphi')^*\eta'$ near the point p.

In this work we study a different but somehow similar setting. Here we look at the class of codimension one real foliations in smooth manifolds, such that at each point the foliation is locally defined by a Bott-Morse function of center type. The singular set consists of a disjoint union of submanifolds and one has for these, all the properties mentioned above for the Kupka set. We also prove that just as in the case of the Kupka set, all these properties are preserved under appropriate deformations of the foliation.

Before describing with more care what we do, let us recall that probably the most important foundational result in the theory of foliations is the celebrated Local Stability Theorem of Reeb (see for instance [3, 10]): *A compact leaf of a foliation having finite holonomy group is stable, i.e., it admits a fundamental system of invariant neighborhoods where each leaf is compact with finite holonomy group.* This is followed in importance by Reeb's Global Stability Theorem: *If $\mathcal{F}$ is codimension one foliation, of class $C^r, r \geq 1$, on a closed connected manifold M and $\mathcal{F}$ has a compact leaf with finite fundamental group, then all leaves of $\mathcal{F}$ are compact with finite fundamental group. Moreover, if $\mathcal{F}$ is transversely orientable then the leaves of $\mathcal{F}$ have trivial holonomy group and they are the fibers of a locally trivial fibration $M \to S^1$.* In fact, according to Thurston ([14]), the same conclusion holds if $\mathcal{F}$ is transversely orientable and exhibits a compact leaf L with zero first Betti number $H^1(L, \mathbb{R}) = 0$.

Some interesting questions arise when we consider small perturbations of a given foliation. For instance the classical Tischler's fibration theorem ([15]) states that a codimension one foliation induced by a nonsingular closed one-form on a compact manifold, can be approximated by compact foliations induced by closed one-forms, and hence the manifold fibers over the circle. The basic idea is to perturb the closed one-form into a closed one-form with rational periods. On the other hand, it is not true that every compact foliation can be approximated by noncompact foliations, even if the compact foliation is defined by a closed one-form. This was already considered by Reeb, who proved the following classical result concerning stability for perturbations, which strengthens his Local Stability Theorem:

Theorem (Reeb), [3]: *Let $\mathrm{Fol}_k^r(M)$ be the space of codimension $k \geq 1$ foliations of class C^r on M, $2 \leq r \leq \omega$, endowed with the C^0-topology. Let $\mathcal{F}$ be an element in $\mathrm{Fol}_k^r(M)$ with a compact leaf L having finite fundamental group. Then for each neighborhood W of L in M and for each point $q \in L$, there exist an open neighborhood V of q in W and a neighborhood $\mathcal{V}$ of $\mathcal{F}$ in $\mathrm{Fol}_k^r(M)$, such that for each foliation $\mathcal{G} \in \mathcal{V}$, the saturated of V by $\mathcal{G}$ is contained in W and it is a union of compact leaves of $\mathcal{G}$, each leaf being a finite covering of L.*

Using arguments as in Thurston's version of Reeb's Global Stability Theorem, Langevin and Rosenberg gave in [7] a generalization of the preceding result: *Equip the space $\mathrm{Fol}_k^r(M)$ with the C^1 topology and assume $\mathcal{F} \in \mathrm{Fol}_k^r(M)$ has a compact leaf L such that $H^1(L, \mathbb{R}) = 0$ and $\mathrm{Hom}(\pi_1(L), \mathrm{GL}(k, \mathbb{R})) = \mathrm{Id}$. Then we have the same conclusions as in Reeb's theorem of stability for perturbations. Moreover, the compact leaves of $\mathcal{G} \in \mathcal{V}$ close enough to L have trivial holonomy and they are diffeomorphic to L.* In fact when $k = 1$ it is enough to assume $H^1(L, \mathbb{R}) = 0$.

On the other hand, foliations with singularities play a significant role in several areas of mathematics. It is thus natural to search for stability theorems for singular foliations in the spirit of the preceding results, and that was the motivation for this article.

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Our starting point is Reeb's Sphere Recognition theorem (see [11]): If a compact manifold M of dimension $m \geq 3$ admits a foliation with Morse singularities all of center type, then all leaves are compact and diffeomorphic to spheres S^{m-1}, M is homeomorphic to the sphere S^m and the foliation is given by the level surfaces of a Morse function having only two centers as singular set. Such a foliation will be called a *Morse-Reeb fibration*.

Our first result is:

Theorem 1. *In dimension $m \geq 3$ Morse-Reeb fibrations on spheres are stable under small smooth deformations. Moreover, every small enough smooth deformation by foliations of a Morse function with only center type singularities in dimension ≥ 3, is a deformation by Morse functions.*

It is well-known that Morse singularities are stable under deformations as functions. The point here is proving the persistence of center type Morse singularities under smooth deformations as foliations. We give an example showing that this condition fails in dimension 2.

The next step we envisage in this article is considering a compact connected manifold M of dimension $m \geq 2$ and a codimension one smooth (*i.e.*, of class C^∞) foliation $\mathcal{F}$ on M with Bott-Morse singularities in the sense of [12, 13]. This means that its singular set, $\operatorname{sing}(\mathcal{F})$, is a union of a finite number of disjoint compact connected submanifolds, $\operatorname{sing}(\mathcal{F}) = \bigcup_{j=1}^{t} N_j$, each of codimension ≥ 2, and for each $p \in N_j \subset \operatorname{sing}(\mathcal{F})$ there exists a neighborhood V of p in M where the foliation is defined by a Bott-Morse function. That is, there is a diffeomorphism $\varphi\colon V \to D \times B$, where $D \subset \mathbb{R}^{n_j}$, $n_j = \dim N_j$, and $B \subset \mathbb{R}^{m-n_j}$ are open balls centered at the origin, such that φ takes $\mathcal{F}|_V$ into the product foliation $D \times \mathcal{G}$, where $\mathcal{G} = \mathcal{G}(N_j)$ is the foliation on B given by a Morse function with a singularity at the origin.

Given a point $p \in N_j$, we write $\varphi(p) = (x(p), y(p))$, so that the discs $\Sigma_p = \varphi^{-1}(x(p) \times B)$ are transverse to $\mathcal{F}$ outside $\operatorname{sing}(\mathcal{F})$ and the restriction $\mathcal{F}|_{\Sigma_p}$ is an ordinary Morse singularity, whose Morse index $r = r(N_j)$ does not depend on the point p in the component N_j. The restriction $\mathcal{G}(N_j) = \mathcal{F}|_{\Sigma_p}$ is the *transverse type* of $\mathcal{F}$ along N_j; it is a codimension one foliation in the disc Σ_p with an ordinary *Morse singularity* at $\{p\} = N_j \cap \Sigma_p$. A component $N \subset \operatorname{sing}(\mathcal{F})$ is of *center type* (or just *a center*) if the transverse type $\mathcal{G}(N) = \mathcal{F}|_{\Sigma_q}$ of $\mathcal{F}$ along N is a center, *i.e.*, its Morse index is either 0 or $r = \dim \Sigma_q$.

Such a foliation $\mathcal{F}$ is *transversally orientable* if there exists a vector field X on M, possibly with singularities at $\operatorname{sing}(\mathcal{F})$, such that X is transverse to $\mathcal{F}$ outside $\operatorname{sing}(\mathcal{F})$. Throughout this paper, all foliations are assumed to be transversely oriented.

Recall that in the classical framework of nonsingular foliations, a compact leaf is *stable* if it admits a fundamental system of invariant neighborhoods such that on each neighborhood the leaves are compact. In codimension one, Reeb's local stability theorem implies that this is equivalent to finiteness of the holonomy group of the leaf.

One has the similar notion of stability for a center type component $N \subset \operatorname{sing}(\mathcal{F})$ of the singular set of a foliation with Bott-Morse singularities: N is *stable* if it admits a fundamental system of invariant neighborhoods such that on each neighborhood the leaves are compact. The foliation is *stable* if all its leaves are compact and stable and all components of the singular set are of center type and stable.

In [12] the authors prove a natural version of Reeb's global stability theorem in this setting: *Let $\mathcal{F}$ be a foliation with Bott-Morse singularities on a closed oriented manifold M of dimension $m \geq 3$ having only center type components in $\operatorname{sing}(\mathcal{F})$. Assume that $\mathcal{F}$ has some compact leaf L_o with finite fundamental group, or there is a codimension ≥ 3 component N of $\operatorname{sing}(\mathcal{F})$ with finite fundamental group. Then all leaves of $\mathcal{F}$ are compact, stable, with finite fundamental group. If,* *moreover, $\mathcal{F}$ is transversely orientable, then* $\operatorname{sing}(\mathcal{F})$ *has exactly two components and there is a differentiable Bott-Morse function* $f\colon M \to [0,1]$ *whose critical values are* $\{0,1\}$ *and such that* $f\big|_{M\setminus \operatorname{sing}(\mathcal{F})}\colon M \setminus \operatorname{sing}(\mathcal{F}) \to (0,1)$ *is a fiber bundle with fibers the leaves of* $\mathcal{F}$. According to [8] the same conclusion holds if we assume that we have a compact leaf or a codimension ≥ 3 center type component $N \subset \operatorname{sing}(\mathcal{F})$ with first Betti number zero.

In this article we prove the following stability theorem:

Theorem 2. *Let M be a compact oriented connected manifold and $\mathcal{F}$ a foliation with Bott-Morse singularities on M all of center type. Assume there is a component $N \subset \operatorname{sing}(\mathcal{F})$ of codimension $\ell \geq 3$ and such that $H^1(N,\mathbb{R}) = 0$. Given a smooth deformation $\{\mathcal{F}_t\}$, $t \in [0,\epsilon)$ of $\mathcal{F}$ there is $0 < \epsilon_1 < \epsilon$ such that if $0 \leq t \leq \epsilon_1$ then $\mathcal{F}_t$ is compact, stable and given by a Bott-Morse function $f_t\colon M \to [0,1]$ with critical values at 0 and 1.*

Just as for Theorem 1, Example **??** below shows that Theorem 2 is sharp in the sense that one cannot drop the codimension ≥ 3 condition. These two theorems are similar, with the additional condition of the existence of a smooth deformation, to the fact that the class of Morse functions is an open subset in the C^1-topology.

As a corollary of the proof of Theorem 2 we have:

Corollary 1. *Let $\mathcal{F}$ be a foliation with Bott-Morse singularities on a manifold M. Assume there is a center type component $N \subset \operatorname{sing}(\mathcal{F})$ of codimension $\ell \geq 3$ such that $H^1(N,\mathbb{R}) = 0$. Given a smooth deformation $\{\mathcal{F}_t\}$, $t \in [0,\epsilon]$, of $\mathcal{F}$ there is $0 < \epsilon_1 < \epsilon$ such that if $0 \leq t \leq \epsilon_1$ then $\mathcal{F}_t$ also exhibits an stable center type component $N_t \subset \operatorname{sing}(\mathcal{F}_t)$ which is close and isotopic to N, and therefore it is stable.*

Also from the proof of Theorem 2 and from Theorems A, B and C in [12] we have the following weak structural stability:

Corollary 2. *In the situation of Theorem 2 the foliations $\mathcal{F}_t$ are topologically conjugate to $\mathcal{F}$ for t small enough.*

2. An example

Given a foliation $\mathcal{F}$ on M with singular set $\operatorname{sing}(\mathcal{F}) \subset M$, by a C^∞ *deformation* of $\mathcal{F}$ we mean a family $\{\mathcal{F}_t\}_{t\in[0,\epsilon)}$ of foliations $\mathcal{F}_t$ on M, with $\mathcal{F}_0 = \mathcal{F}$ and which is smooth in the sense that for each point $p \in M$, there are an open set $p \in U \subset M$ and a smooth family of differential one-forms $\Omega_t(x) := \Omega(x,t)$ in $U \times [0,\epsilon)$ such that for each t the one-form Ω_t is integrable and defines $\mathcal{F}_t$ in U.

Before proving Theorems 1 and 2, let us show that these results are sharp in the sense that the codimension ≥ 3 condition cannot be dropped. Notice that in [8] examples are given showing that the conditions on the component $N \subset \operatorname{sing}(\mathcal{F})$ cannot be dropped without destroying the stability of the foliation $\mathcal{F}_0$.

Let $\Omega = d(x^2 + y^2)$ and $\Omega_\lambda = xdy - \lambda ydx$ in affine coordinates $(x,y) \in \mathbb{R}^2$, where $\lambda \in \mathbb{R}$ is not zero. Put

$$\Omega_t := \Omega + t\Omega_\lambda = (2x - t\lambda y)dx + (2y + tx)dy.$$

Then $\operatorname{sing}(\Omega_t) = \{(0,0)\}$. For

$$X_t := (2y + tx)\frac{\partial}{\partial x} + (t\lambda y - 2x)\frac{\partial}{\partial y}$$

we have $\Omega_t \cdot X_t = 0$. Thence

$$DX_t(0,0) = \begin{pmatrix} t & 2 \\ -2 & t\lambda \end{pmatrix}.$$

The eigenvalues of X_t at $(0,0)$ are the α given by

$$0 = \mathrm{Det}(DX_t(0,0) - \alpha I) = (t-\alpha)(t\lambda - \alpha) + 4\,.$$

Thus we have

$$\alpha = \frac{(1+\lambda)t \pm \sqrt{t^2(1+\lambda)^2 - 4(4+t^2\lambda)}}{2}\,.$$

For $t = 0$ we have

$$\alpha = \pm 2\sqrt{-1}\,.$$

For $t \approx 0$ but $t \neq 0$ we have $\alpha = a + b\sqrt{-1} \in \mathbb{C}$ where $b \approx 2$ and $0 \neq a \approx 0$ provided that $\lambda \neq -1$. In this case the quotient of eigenvalues of X_t at the origin is of the form

$$\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}} = \frac{a^2 - b^2 + 2\sqrt{-1}ab}{a^2+b^2} \notin \mathbb{R}$$

and therefore X_t has a hyperbolic singularity at the origin. In particular, thanks to the dynamics of such a singularity, *the leaves of* Ω_t *are not closed and the foliation* $\Omega_t = 0$ *exhibits no continuous first integral in a neighborhood of the origin* $(0,0) \in \mathbb{R}^2$.

Now, by gluing two copies of the 2-disk D^2 we obtain the 2-sphere S^2. Endowing each copy of D^2 with a foliation given by $\Omega_t = 0$ we obtain a deformation $\mathcal{F}_t$ of the foliation $\mathcal{F}_0$ by parallels, $\mathcal{F}_0$ is of Morse type with singularities only at the North and South poles, both of center type. The foliation $\mathcal{F}_t$ (obtained indeed as an extension of the foliation in $\mathbb{R}^2$ given by $\Omega_t = 0$) exhibits singularities at the North and South poles either, but these are not of Morse type as seen above. By taking products with a closed manifold N we obtain a foliation $\tilde{\mathcal{F}}_0$ with singularities of Bott-Morse type, all of center type, which is deformed into foliations which are *not* of Bott-Morse type. We can of course take N such that $H^1(N,\mathbb{R}) = 0$, thus showing that the codimension ≥ 3 condition on the singular component $N \subset \mathrm{sing}(\mathcal{F})$ in Theorem 2 cannot be dropped.

3. Deformations of Morse singularities by foliations

Let us consider a differential one-form $\Omega = \sum\limits_{j=1}^{m} f_j\, dx_j$ in coordinates $(x_1, ..., x_n) \in U \subset \mathbb{R}^m$ in an open subset.

Definition 1. The *gradient* vector field of Ω is defined as $\mathrm{grad}(\Omega) := \sum\limits_{j=1}^{m} f_j \frac{\partial}{\partial x_j}$.

This is a differentiable vector field which, away from the (singular) zero-set, is orthogonal to the distribution $\mathrm{Ker}(\Omega)$. Also $\mathrm{sing}(\mathrm{grad}(\Omega)) = \mathrm{sing}(\Omega)$.

Theorem 3. *Let* $\mathcal{F}_t$ *be a smooth deformation of* $\mathcal{F}$ *in an open neighborhood* U *of the origin* $0 \in \mathbb{R}^m$. *Assume* $\mathcal{F}$ *has a Morse singularity of center type at the origin and either* $m \geq 3$, *or else* $m = 2$ *and the leaves of* Ω_t *are compact for* t *small enough. Then there exist* $\epsilon > 0$, *a neighborhood* $V \subset \mathbb{R}^m$ *of* 0 *and a smooth function* $\xi\colon [0,\epsilon) \to V$ *such that:*

(i) $\xi(0) = 0$;
(ii) *For* $t < \epsilon$ *we have* $\mathrm{sing}(\mathcal{F}_t) \cap V = \{\xi(t)\}$ *and the leaves of* $\mathcal{F}_t$ *close enough to* $\xi(t)$ *are compact and diffeomorphic to the sphere* S^{m-1};
(iii) *Moreover, for each such* t, $\xi(t)$ *is a center type Morse singularity of* $\mathcal{F}_t$: *there is a smooth map* $\rho_t\colon V \to \mathbb{R}$ *with* $\rho_t(\xi(t)) = 0$, *which is a first integral for* $\mathcal{F}_t$ *in* V *and has a nondegenerate critical point at* $\xi(t)$ *of center-type.*

First part of the proof of Theorem 3. Let $\{\Omega_t\}_{t\in[0,\epsilon)}$ be a smooth family of integrable one-forms in the neighborhood U of the origin such that $\mathcal{F}_t$ is defined by the one-form Ω_t and $\mathcal{F}_0 = \mathcal{F}$. Since $\mathcal{F}$ has a center-type singularity at 0, there is a neighborhood $W \subset U$ of the origin where we can choose local coordinates $(x_1, ..., x_m) \in W$ such that Ω is of the form:

$$\Omega = gd(\sum_{j=1}^{m} x_j^2) = \sum_{j=1}^{m} 2gx_j dx_j \,.$$

We set $\Omega_t(x_1, ..., x_m) = \sum\limits_{j=1}^{m} a_j(t,x)dx_j$, then each a_j is smooth. Define a smooth map

$$F\colon [0,\epsilon) \times W \to \mathbb{R}^m$$

by $F(t,x) = (a_1(t,x), ..., a_m(t,x))$. Then we have

$$\frac{\partial}{\partial(x_1, ..., x_m)}\Big|_{t=0} F(t,x) = D(a_1(0,x), ..., a_m(0,x)).$$

This last is a diagonal matrix and its determinant is $(2g(0))^m \neq 0$. Since $F(0,0) = 0$, by the Implicit Function theorem, if $\epsilon > 0$ is small enough, there is a smooth map $\xi\colon [0,\epsilon) \to W$ such that $\xi(0) = 0$, $F(t,\xi(t)) = 0$ and $\text{sing}(\Omega_t) \cap W = \{\xi(t)\}$. Moreover, the partial derivative $\frac{\partial F}{\partial(x_1,...,x_m)}(t,\xi(t))$ is non-singular, so that Ω_t has a nondegenerate singularity at $\xi(t)$. In order to prove that $\xi(t)$ is an stable singularity (i.e, a singularity surrounded by compact leaves with finite holonomy) of Ω_t, we proceed as follows. The leaves of $\mathcal{F}$ in W are spheres of dimension $m-1 \geq 2$. Choose a small neighborhood $V \subset W$ of the origin, invariant by $\mathcal{F}$. Fix a leaf $L_0 \in \mathcal{F}$ such that L_0 bounds a region (a ball) contained in V. By Reeb's stability for perturbations theorem if $\epsilon > 0$ is small enough then $\mathcal{F}_t$ exhibits a compact leaf L_t close to L_0, contained in W. Denote by $R(L_t) \subset W$ the region (diffeomorphic to a closed ball), containing the origin and therefore the singularity $\xi(t) \in \text{sing}(\mathcal{F}_t)$, bounded by the leaf L_t. By Reeb's complete stability theorem all leaves of $\mathcal{F}_t$ in $R(L_t)$ are compact diffeomorphic to L_t. This proves (i) and (ii) in Theorem 3. □

Now we consider the family of vector fields $X_t := -\,\text{grad}(\Omega_t)$ in W (cf. Definition 1). Then X_t is a smooth deformation of the vector field $X_0 = -\,\text{grad}(\Omega_0) = -2g\vec{R}$ where $\vec{R}$ is the radial vector field. Using what we have seen above we have:

Lemma 1. *Assume that the dimension $m \geq 3$ is odd. Then for t small enough the vector field X_t exhibits a smooth separatrix through its unique singularity $\xi(t)$ close to the origin.*

Proof. Concerning the existence of separatrices, we may indeed assume that $g = \frac{1}{2}$ and $X_0 = -\vec{R}$. Denote by $\xi(t)$ the singular point of X_t close to the origin $0 = \xi(0)$ in $\mathbb{R}^m$. Then the derivative $DX_t(\xi(t))$ is a perturbation of the derivative $DX_0(0) = \text{Id} \in \text{GL}(m,\mathbb{R})$. This implies that its characteristic equation $P_t(\lambda) = \text{Det}(DX_t - \lambda\,\text{Id}) = 0$ is a perturbation of the characteristic equation $P_0(\lambda) = \text{Det}(DX_0 - \lambda I) = (1-\lambda)^m = 0$. By continuity, for t small enough, the eigenvalues of DX_t at $\xi(t)$ have positive real part, in particular X_t has a hyperbolic singularity at $\xi(t)$ (see Hartman [5]). Since by hypothesis m is odd, there is at least one real eigenvalue and therefore (by the classical Hartman-Grobman theorem [5]) we have at least one smooth unstable separatrix through the singular point $\xi(t)$. □

Using now the fact that X_t is transverse to the leaves of $\mathcal{F}_t$ which are compact manifolds filling up a neighborhood of the singularity $\xi(t)$, we obtain the following fact:

Lemma 2. *The vector field X_t exhibits a smooth separatrix Γ_t through the singularity $\xi(t)$.*

Proof. If m is odd then we apply Lemma 1. Assume now that $m \geq 4$ is even. The one-form Ω_t defines a compact foliation $\mathcal{F}_t$ with a non-degenerate singularity at $\xi(t)$, with $\Omega_0 = g_0\, d(\sum\limits_{j=1}^{m} x_j^2)$ and $\xi(0) = 0$. We may assume that for each leaf L_t of $\mathcal{F}_t$, the vector field X_t points inwards the region $R(L_t)$, bounded by L_t, that contains the singularity $\xi(t)$. Since the regions $R(L_t)$ form a fundamental system of neighborhoods of $\xi(t)$ we conclude that the singularity $\xi(t)$ is asymptotically stable with respect to X_t.

Claim 1. *The spectrum* $\mathrm{Spec}(DX_t(\xi(t))) \subset \mathbb{C}$ *of* X_t *at* $\xi(t)$ *exhibits some real eigenvalue.*

Proof. Write $X_t = (a_1^t, b_1^t, ..., a_n^t, b_n^t)$ where $n = m/2$. Put $Y_t := X_t^{\perp} = (-b_1^t, a_1^t, ..., -b_n^t, a_n^t)$. Then Y_t is orthogonal to X_t and therefore its orbits are tangent to the leaves of Ω_t. In particular, the orbits of Y_t are contained in compact manifolds. The nonsingular orbits of Y_t do not accumulate at the singularity $\xi(t)$. Suppose by contradiction that the characteristic polynomial P_t of $DX_t(\xi(t))$ is of the form $P_t(\lambda) = \prod\limits_{j=1}^{m/2} (\lambda^2 + a_j\lambda + b_j)$ in irreducible polynomials over $\mathbb{R}[\lambda]$. We have several cases to consider.

If there are no multiple eigenvalues then we can write $DX_t(\xi(t))$ as a diagonal matrix of $m/2$ blocks B_j of the form

$$B_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$$

where $\beta_j \neq 0$. Assume for simplicity that $DX_t(\xi(t) = B_j$ is one block. We write $X_t = (a_1^t, b_1^t)$ and $Y_t = X_t^{\perp} = (-b_1^t, a_1^t)$. Then the same linear coordinates that give $DX_t(\xi(t)) = B_j$ give $DY_y(\xi(t)) = B_j^{\perp}$ which is defined as

$$B_j^{\perp} = \begin{pmatrix} \beta_j & -\alpha_j \\ \alpha_j & \beta_j \end{pmatrix}$$

The linear system

$$\dot{x} = DY_t(\xi(t)) \cdot x = B_j^{\perp} \cdot x$$

has its solutions given explicitly in terms of $\exp(\beta_j t)\cos(\alpha_j t)x_j$ and $\exp(\beta_j t)\sin(\alpha_j t)x_j$ so that, thanks to the terms in $\exp(\beta_j t)$ we conclude that the solutions of $\dot{x} = DY_t(\xi(t))x$ cannot be contained in a compact manifold surrounding the singularity $\xi(t)$, they must instead accumulate at the singular point $\xi(t)$. By Hartman-Grobman theorem this same statement holds for the solutions of Y_t. This excludes this "diagonalizable" case.

Now we consider the case where $DX_t(\xi(t))$ is a matrix with blocks of the form

$$\begin{pmatrix} B_j & O \\ I_2 & B_j \end{pmatrix}$$

where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore we have

$$DY_t(\xi(t)) = \begin{pmatrix} B_j^{\perp} & O \\ J_2 & B_j^{\perp} \end{pmatrix},$$

where

$$J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Again we conclude that the linear system

$$\dot{x} = Y_t(x) = B_j^{\perp} \cdot x$$

has solutions that cannot be contained in a compact manifold surrounding the singularity $\xi(t)$. They must instead accumulate at the singular point $\xi(t)$. By the classical Hartman-Grobman linearization theorem this gives a final contradiction and proves the claim in this case. The cases of even dimension ≥ 6 are similar and can be proved in the same way.

□

Let us now finish the proof of Lemma 2. Because the spectrum of $DX_t(\xi(t))$ contains some real eigenvalue, it contains some negative eigenvalue and then X_t exhibits some smooth stable separatrix Γ_t through the singularity $\xi(t)$ thanks to the Stable manifold theorem. This proves the lemma in the even dimensional case. □

End of the proof of Theorem 3. The trace of Γ_t is (diffeomorphic to) an interval $[0,1]$ with the origin corresponding to the singularity $\xi(t)$, and transverse to each leaf of $\mathcal{F}_t$ in $R(L_t)$. We may take any smooth function $\rho_t\colon [0,1] \to \mathbb{R}$ such that $\rho_t(0) = 0$ and extend ρ_t to $R(L_t)$ as constant through the leaves of $\mathcal{F}_t$ in $R(L_t)$. Now, if we choose $\rho_t\big|_{\Gamma_t}$ such that it has an order two zero at the origin then we claim that the extension $\rho_t\colon R(L_t) \to \mathbb{R}$ has a nondegenerate singularity at $\xi(t)$. Indeed, since ρ_t is a first integral for Ω_t we have $\Omega_t \wedge d\rho_t = 0$, and since Ω_t has a nondegenerate singularity at $\xi(t)$ we can write $d\rho_t = h_t.\Omega_t$ for some smooth function h_t. In coordinates we have $d\rho_t = h_t \cdot \sum\limits_{j=1}^{m} a_j(t,x)dx_j$ so that $\frac{\partial \rho_t}{\partial x_j} = h_t \cdot a_j(x,t)$, $\forall j = 1,\ldots,m$. Since $a_j(t,\xi(t)) = 0$ we conclude that $\frac{\partial^2 \rho_t}{\partial x_i \partial x_j} = h_t(\xi(t)) \cdot \frac{\partial a_j(t,\xi(t))}{\partial x_i}$. If $h_t(\xi(t)) = 0$ then $D^2\rho_t(\xi(t)) = 0$ what is a contradiction to our original choice of ρ_t as having an order two zero at the origin. Therefore, $h_t(\xi(t)) \neq 0$ and the Hessian of ρ_t at $\xi(t)$ is nonsingular. This implies that ρ_t has a nondegenerate Morse type singularity at $\xi(t)$ and, since the leaves of $\mathcal{F}_t$ in $R(L_t)$ are compact, this singularity is a center. □

4. Integrable deformations of non-isolated singularities

As for the non-isolated case we have the following version of the first part of Theorem 3.

Lemma 3. *Let $\mathcal{F}$ be a foliation on M having a Bott-Morse component $N \subset \operatorname{sing}(\mathcal{F})$ of center type and* $\operatorname{codim} N = \ell \geq 3$. *Let now $\mathcal{F}_t$ be a C^∞ deformation of $\mathcal{F} = \mathcal{F}_0$, where $t \in [0,\epsilon)$. There are a neighborhood W of N in M and $0 < \epsilon_1 < \epsilon$ such that if $t \leq \epsilon_1$ then:*

(1) $\operatorname{sing}(\mathcal{F}_t) \cap W = N_t$ *is a compact nondegenerate component, diffeomorphic to N_0.*
(2) N_t is isotopic to N_0.
(3) $N_t \subset \operatorname{sing}(\mathcal{F}_t)$ is a Bott-Morse component of center type.

Proof. The same ideas as in the proof of Theorem 3 apply here. Indeed, let N be a codimension ℓ component of the singular set of $\mathcal{F}$. Given a point $p \in N$ there is a neighborhood U of p in M diffeomorphic to the product $D^\ell \times D^{m-\ell}$ of discs $D^\ell \subset \mathbb{R}^\ell$ and $D^{m-\ell} \subset \mathbb{R}^{m-\ell}$, such that the restriction $\mathcal{F}\big|_U$ is equivalent to the product foliation $D^{m-\ell} \times \mathcal{F}_1$, where $\mathcal{F}_1$ is a foliation on the disc D^ℓ with an isolated Morse type singularity of center type at the origin $0 \in D^\ell$. At each disc $\mathcal{D}_q := \{q\} \times D^\ell$, for any point $q \in D^{m-\ell}$, $\mathcal{F}$ induces an ordinary Morse singularity of center type, isomorphic to $\mathcal{F}_1$. Given any smooth deformation $\mathcal{F}_t$ of $\mathcal{F} = \mathcal{F}_0$, for small t the foliation $\mathcal{F}_t$ is transverse to the discs $\mathcal{D}_q$ and induces by restriction a smooth deformation $\mathcal{F}_t\big|_{\mathcal{D}_q}$ of $\mathcal{F}\big|_{\mathcal{D}_q}$. Since $\dim \mathcal{D}_q = \ell \geq 3$, by Theorem 3 above there is a smooth function $\xi_q(t)$ of the parameter t such that $\xi_q(t)$ is the only singularity of $\mathcal{F}\big|_{\mathcal{D}_q}$. Moreover, this singularity is of Morse center type.

Finally, the map ξ_q depends also smoothly on the point q as it follows from the Implicit function theorem, where we consider $q \in D^{m-\ell}$ as a parameter on which the coefficients of the map F (which is just the map having as coordinate functions the coefficients of the form $\Omega(t,x) = \Omega_t(x)$, in the proof of Theorem 3) depend smoothly. This shows that there is a neighborhood W of N in U such that for t small enough $\text{sing}(\mathcal{F}_t) \cap W = N_t$ is center type Bott-Morse component, mapped as the graph of a smooth map $\xi(q,t)$ taking values on the transverse disc $\mathcal{D}_q$. By uniqueness these maps glue and this shows that for a suitable neighborhood V of N in M and for small t, the singular set $\text{sing}(\mathcal{F}_t) \cap V$ is a nondegenerate component N_t, diffeomorphic (indeed isotopic) to $N_0 = N$. This shows (1) and (2) in the lemma. Assume now that $\ell = \text{codim}\, N$ is odd. Then the above arguments show, as in the proof of Theorem 3, that for t small enough we may choose local defining functions ρ_t for $\mathcal{F}_t$ around the points of N_t such that each ρ_t has a center type Bott-Morse singularity at the points in N_t. This proves (3). □

Definition 2. Given an integrable one-form Ω in a manifold M we say that Ω *has nondegenerate singularities* if its singular set $\text{sing}(\Omega)$ is a disjoint union of closed submanifolds $N \subset M$ such that for each point $p \in N \subset \text{sing}(\Omega)$ there are local coordinates $(x_1, ..., x_\ell, x_{\ell+1}, ..., x_m)$ for M, centered at p, such that $N : (x_{\ell+1} = ... = x_m = 0)$ and writing $\Omega = \sum_{j=1}^{m} a_j(x) dx_j$ we have $\text{Det}\left(\frac{\partial a_i}{\partial x_j}\right)_{i,j=\ell+1}^{m}(0) \neq 0$.

5. Proof of the results

Let us now prove our main results.

Proof of Theorem 1. Let $\mathcal{F}_t$ be a smooth deformation of a Morse-Reeb fibration on a compact manifold M of dimension $m \geq 3$. We claim that for t small enough the foliation $\mathcal{F}_t$ is a Morse-Reeb fibration. Indeed, by Theorem 3 , for t small enough, the foliation $\mathcal{F}_t$ is a foliation with nondegenerate singularities of center type, and the leaves close to the singularities are spheres. By Reeb's theorem in [11] $\mathcal{F}_t$ is a Morse-Reeb fibration. □

The proof of Theorem 2 relies on the following local stability result, similar to Thurston's version of Reeb local stability (Corollary 1 in [14]).

Proposition 1 (Proposition 1 in [8]). *Let $\mathcal{F}$ be a transversely orientable codimension one foliation with Bott-Morse singularities on a manifold M. Assume that $N \subset \text{sing}(\mathcal{F})$ is a center type component with $H^1(N;\mathbb{R}) = 0$. Then N is stable. Indeed, there is a fundamental system of saturated neighborhoods W of N in M such that each leaf $L \subset W$ is compact with $H^1(L;\mathbb{R}) = 0$. Moreover, the holonomy of the component N is trivial and there is a Bott-Morse function $f\colon W \to \mathbb{R}$, defined in an invariant neighborhood W of N, which defines $\mathcal{F}$ in W.*

Proof of Theorem 2. Consider a deformation $\mathcal{F}_t$, $t \in [0, \epsilon)$, of the Bott-Morse foliation $\mathcal{F}$ having a component $N \subset \text{sing}(\mathcal{F})$ with $H^1(N,\mathbb{R}) = 0$. By Proposition 1 we may apply Lemma 3 and conclude that if $\epsilon > 0$ is small enough then the singular set of $\mathcal{F}_t$ exhibits a center type component N_t isotopic to $N = N_0$. In particular, for $t < \epsilon$ small enough the foliation $\mathcal{F}_t$ is a Bott-Morse foliation having all singularities of center type and some component $N_t \subset \text{sing}(\mathcal{F}_t)$ such that $H^1(N_t,\mathbb{R}) = 0$. Then, according to Theorem 1 in [8] there is a Bott-Morse function $f_t\colon M \to \mathbb{R}$ that defines $\mathcal{F}_t$. □

Proof of Corollary 2. For t small enough in Theorem 2 the foliation $\mathcal{F}_t$ is a Bott-Morse foliation with only center type singularities. Moreover, there is a Bott-Morse function $f_t\colon M \to \mathbb{R}$ that defines $\mathcal{F}_t$. From Theorem A in [12] the singular set $\text{sing}(\mathcal{F}_t)$ has only two components say

N_1^t, N_2^t and $f_t\big|_{M\setminus(N_1^t\cup N_2^t)}$ is a fibre bundle over $(0,1)$ with fibers the leaves of $\mathcal{F}_t$. Moreover, by Lemma 3, each component N_j^t is isotopic (and therefore homeomorphic) to $N_j^0 = N_j \subset \operatorname{sing}(\mathcal{F})$. Finally, from Reeb stability theorem, all leaves of $\mathcal{F}_t$ are diffeomorphic to a (typical) leaf $L^t \in \mathcal{F}_t$ and each leaf L_t is homeomorphic to the (typical) leaf $L^0 \in \mathcal{F}$. Thus the bundles $\mathcal{F}_t\big|_{M\setminus(N_1^t\cup N_2^t)}$ and $\mathcal{F}\big|_{M\setminus(N_1\cup N_2)}$ are topologically equivalent. This and the product type of $\mathcal{F}_t$ around the singularities N_j^t give the topological equivalence between $\mathcal{F}_t$ and $\mathcal{F}$. □

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