Journal *of* Singularities
Volume 9 (2014), 219-225

Proc. of Algebraic Methods in Geometry, Guanajuato, 2011
DOI: 10.5427/jsing.2014.9q

HYPERSURFACES IN $\mathbb{P}^5$ CONTAINING UNEXPECTED SUBVARIETIES

I. VAINSENCHER

ABSTRACT. Smooth cubic 4-folds in $\mathbb{P}^5$ containing a general pair of 2-planes are known to be rational. They form a family of codimension 2 in $\mathbb{P}^{55}$. We find a polynomial which encodes, for all $d \geq 3$, the degrees of the loci of hypersurfaces in $\mathbb{P}^5$ of degree d containing some plane-pair.

Dedicated to Xavier Gómez-Mont Ávalos
on the occasion of his 60th birthday.

1. INTRODUCTION

The Noether-Lefschetz theorem tells us that a surface of degree at least four is not supposed to contain curves besides its intersection with another surface. Asking surfaces of a given degree to contain say, a (few) line(s), or a conic, a twisted cubic, etc., defines subvarieties, the so called Noether-Lefschetz loci, in the appropriate projective space. There are *polynomial* formulas for their degrees, [4], [14].

Our motivation here stems from a tale told by Joe Harris we were fortunate to attend (cf. [9]). The theme was the lack of knowledge about the rationality of cubic 4-folds in $\mathbb{P}^5$. As an elementary dimension count shows, a general cubic 4-fold $F_3 \subset \mathbb{P}^5$ contains no plane $\mathbb{P}^2 \subset \mathbb{P}^5$. Those F_3 that do contain some $\mathbb{P}^2$ are easily seen to form a hypersurface in $\mathbb{P}^{55}$. Now the smooth cubic 4-folds containing *two* disjoint planes are known to form a family (of dimension 53) of *rational* hypersurfaces. Indeed, through a general point of such hypersurface F, there is exactly one line which meets both planes; conversely, given a choice of a point on each plane, the line joining them meets F in a third point, thus establishing a birational map $F \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$. See [11, 1.33, p. 24].

Our aim is to find the degree of the family of cubic 4-folds containing some pair of disjoint planes in $\mathbb{P}^5$. In fact, the answer is given by a polynomial in d which encodes, for each $d \geq 3$, the degree of the locus of hypersurfaces in $\mathbb{P}^5$ of degree d containing some plane-pair, cf. (2).

A naïve, direct application of the formula of double points gives a *wrong* answer, **5 752 908**, instead of **3 371 760**, presently dedicated to Xavier. In fact, it turns out that the cubic 4-folds containing some pair of incident planes contribute a full component to the double point locus. Ditto for those containing a pair of planes meeting along a line.

We employ the tools pioneered by Geir Ellingsrud and Stein A. Strømme, cf. [6] and masterfully used by Maxim Kontsevich in [12].

Let us summarize the main construction. Write $\mathbb{G}$ for the Grassmann variety of planes in $\mathbb{P}^5$. The family of plane-pairs can be parameterized by a double blowup $\widehat{\mathbb{X}} \to \widetilde{\mathbb{X}} \to \mathbb{G} \times \mathbb{G}$, first along the diagonal, then along the strict transform of the locus of plane-pairs containing a line. The resulting variety $\widehat{\mathbb{X}}$ comes equipped with vector bundles $\mathcal{V}_d$, $d \geq 2$. The fiber of $\mathcal{V}_d$ over each

1991 *Mathematics Subject Classification.* 14N05, 14N15 (Primary).

Key words and phrases. intersection theory, Noether-Lefschetz locus, enumerative geometry, rational cubic fourfold.

Partially supported by CNPQ-Brasil.

$q \in \widehat{\mathbb{X}}$ is equal to the vector space of homogeneous polynomials of degree d lying in the ideal of the plane-pair q (or a flat specialization thereof). There are natural $\mathbb{C}^\star$-actions with finitely many explicit fixed points in $\widehat{\mathbb{X}}$. Bott's formula applies to compute the answers for as many values of d as needed to interpolate and find an explicit polynomial. The *a priori* polynomial nature of the answers is deduced from Grothendieck-Riemann-Roch.

2. NOTATION AND PRELIMS

2.1. Let us start with Fermat's $F_3 := x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0$ just to get a feeling. It contains a few pairs of disjoint planes in $\mathbb{P}^5$, *e.g.,* $q = \langle p_1, p_u \rangle$ with

$$\begin{cases} p_1 : x_0 + x_1 = x_2 + x_3 = x_4 + x_5 = 0, \\ p_u : x_0 - ux_1 = x_2 - ux_3 = x_4 - ux_5 = 0, \ (u^3 = -1, u \neq -1). \end{cases}$$

A little calculation shows that in the affine neighborhood of the grassmannian of planes in $\mathbb{P}^5$ defined by

$$p_a : \begin{cases} a_{1,1}x_3 + a_{1,2}x_4 + a_{1,3}x_5 + x_0, \\ a_{2,1}x_3 + a_{2,2}x_4 + a_{2,3}x_5 + x_1, \\ a_{3,1}x_3 + a_{3,2}x_4 + a_{3,3}x_5 + x_2, \end{cases} \tag{1}$$

there are 162 planes contained in F_3, *e.g.,* $x_0 + x_3, x_1 + x_5, x_2 + x_4$, or $x_0 + x_3, x_1 + x_4, ux_5 + x_2$, with $u^2 + u + 1 = 0$.

2.2. Cubic 4-folds in $\mathbb{P}^5$ depend on $\binom{3+5}{3} - 1 = 55$ parameters. Those containing say the plane $p_0 : x_0 = x_1 = x_2 = 0$ can be written uniquely as $a_0x_0 + a_1x_1 + a_2x_2$ where the a_i are homogeneous polynomials of degree 2. We may assume

$$\begin{cases} a_0 = & \text{any quadric,} & \binom{7}{2} = 21 \text{ free coefficients,} \\ a_1 = & \text{has no term with } x_0, & \binom{6}{2} = 15, \\ a_2 = & \text{has no term with } x_0, x_1, & \binom{5}{2} = 10. \end{cases}$$

Hence we get a $\mathbb{P}^{45}$-bundle over the grassmannian $\mathbb{G}$=$\mathbb{G}$r[2,5],

$$X = \{(p, F_3) \in \mathbb{G} \times \mathbb{P}^{55} \mid F_3 \supset p\}.$$

The dimension of the total space is 54. We expect and get a hypersurface in $\mathbb{P}^{55}$ consisting of cubic 4-folds which contain some $\mathbb{P}^2 \subset \mathbb{P}^5$. This comes from an argument of Castelnuovo-Mumford regularity (cf. [5]).

2.2.1. **Proposition.** *Hypersurfaces of degree d containing some $\mathbb{P}^2$ in $\mathbb{P}^5$ form a subvariety of codimension $\binom{d+2}{2} - 9$ and degree*

$7\binom{d+5}{8}\big(50d^{19} + 750d^{18} + 3300d^{17} + 1800d^{16} - 7800d^{15} + 5400d^{14} - 141400d^{13} - 367800d^{12} - 215115d^{11} - 2480805d^{10} + 2686380d^9 - 538110d^8 + 12830747d^7 + 86752281d^6 - 18022266d^5 + 703254420d^4 - 305343432d^3 + 2350054944d^2 - 787739904d + 2821754880\big) / \big(2^9 \cdot 3^{10} \cdot 5^2\big).$

In particular, cubic 4−folds containing some $\mathbb{P}^2$ in $\mathbb{P}^5$ form a hypersurface in $\mathbb{P}^{55}$ of degree **3402**.

Proof. Consider the tautological exact sequence of vector bundles over the grassmannian $\mathbb{G}$,

$$\mathcal{S} \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{Q}$$

where $\mathcal{F}$ stands for the trivial bundle with fiber $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$ whereas $\mathcal{S}$ is the subbundle of rank 3 with fiber over each $p \in \mathbb{G}$ equal to the space of linear forms cutting the plane $p := \mathbb{P}^2$ in $\mathbb{P}^5$. The fiber of the quotient bundle is $\mathcal{Q}_p = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Taking symmetric power, we get the exact sequence

$$\mathcal{S}^{(d)} := \ker \rho \rightarrowtail \operatorname{Sym}_d \mathcal{F} \xrightarrow{\rho} \operatorname{Sym}_d \mathcal{Q} .$$

Here ρ stands for the map which sends homogeneous forms of degree d in $\mathbb{P}^5$ to their restrictions to a varying $\mathbb{P}^2 \subset \mathbb{P}^5$. The fiber of $\mathcal{S}^{(d)}$ over $p \in \mathbb{G}$ is equal to the space of forms of degree d lying in the homogeneous ideal of the plane p in $\mathbb{P}^5$. The projectivization $\mathbb{P}(\mathcal{S}^{(d)}) \subset \mathbb{G} \times \mathbb{P}^N$ consists of pairs (p, F_d) such that the plane p lies in the hypersurface F_d. The map $\mathbb{P}(\mathcal{S}^{(d)}) \to \mathbb{P}^N$ induced by the projection $p_2 : \mathbb{G} \times \mathbb{P}^N \to \mathbb{P}^N$ is generically injective for all $d \geq 3$ due to an argument of regularity discussed in §2.3.1. Put

$$m := \dim \mathbb{P}(\mathcal{S}^{(d)}) = 9 + \binom{d+5}{5} - \binom{d+2}{2} - 1.$$

The image $\mathbb{Y}_d$ of $\mathbb{P}(\mathcal{S}^{(d)})$ in $\mathbb{P}^N$ has the same dimension m. It consists of all $F_d \subset \mathbb{P}^5$ containing some $\mathbb{P}^2$. The degree of $\mathbb{Y}_d$ is given by $\int h^m \cap [\mathbb{Y}_d]$, where $h := c_1 \mathcal{O}_{\mathbb{P}^5}(1)$ is the hyperplane class. By the projection formula, we have $\deg \mathbb{Y}_d = \int p_2^\star h^m \cap [\mathbb{P}(\mathcal{S}^{(d)})]$. Pushing forward via the structure map $\mathbb{P}(\mathcal{S}^{(d)}) \to \mathbb{G}$, it reduces to the calculation of the Segre class: $\deg \mathbb{Y}_d = \int_{\mathbb{G}} s(\mathcal{S}^{(d)})$, cf. [7, p. 30, §3.1]. The latter is equal to $\int_{\mathbb{G}} c_9(\mathrm{Sym}_d\, \mathcal{Q})$ in view of the above exact sequence. With the help of Katz & Strømme's Schubert package (see [10]) we get the formula. Here is a script for Macaulay2 [8]:

```
loadPackage "Schubert2"
pt = base d; X = flagBundle({3,3},pt); (S,Q) = bundles X
g=chern(9,symmetricPower_d Q); use QQ[d][H_(2,3)]
g=substitute(g,QQ[d][H_(2,3)]); g=sub(g,H_(2,3)=>1);
toString factor g
binom=(d,m)->product(i=1..m,i->(d-i+1)/i)
f=binom(d+5,8); f=g/oo ; toString factor f
sub(denominator f,ZZ)
toString factor oo
```

□

2.3. **Double point formula.** In view of the formula in 2.2.1, given two general cubic 4-folds f_1, f_2 there are **3402** elements in the pencil of cubics $\alpha_1 f_1 + \alpha_2 f_2$ which contain some 2-plane $\mathbb{P}^2 \subset \mathbb{P}^5$.

Likewise, given a general net of cubic 4-folds, we ask now for the number of its members which contain *two* 2-planes.

Let $\varphi : X^m \to Y^n$ be a map of varieties of dimensions m, n. The double point locus, $\mathbb{D}(\varphi)$, is defined as the closure in X of

$$\{x \in X \mid \exists x' \neq x, \quad \text{with } \varphi(x) = \varphi(x')\}.$$

Under mild conditions, the above is the support of a cycle in the Chow group, expressed by the double point formula,

$$\mathbb{D}(\varphi) = \varphi^\star \varphi_\star [X] - c_{n-m} T_\varphi \cap [X].$$

Here $T_\varphi = TX - \varphi^\star TY$, the virtual normal bundle, cf. [7, p. 166, 9.3], [13].

With notation as in 2.2.1, try and apply the double point formula to

$$X := \mathbb{P}(\mathcal{S}^{(3)}) = \{(p, F_3) \in \mathbb{G} \times \mathbb{P}^{55} \mid p \subset F_3\} \xrightarrow{\varphi} Y := \mathbb{P}^{55}.$$

The map $\mathbb{P}(\mathcal{S}^{(3)}) \xrightarrow{\varphi} \mathbb{P}^{55}$ is generically injective and its image is the hypersurface $\mathbb{Y}_3 \subset \mathbb{P}^{55}$ of cubic 4-folds containing some $p = \mathbb{P}^2 \subset \mathbb{P}^5$. Look at the double point locus $\mathbb{D}(\varphi)$. We have $\dim \mathbb{D}(\varphi) = 53$. Its image $\overline{\mathbb{D}}(\varphi) \subset \mathbb{P}^{55}$ has the expected dimension 53, and degree **5 752 908**; below is a script.

```
loadPackage "Schubert2"
X = flagBundle({3,3}); TX=tangentBundle X; (S,Q)=bundles X
S3=symmetricPower_3(Q);     R=symmetricPower_3(6*OO_X)-S3
```

```
Y = projectiveBundle'(dual R)     ---(this takes too long)
TY=tangentBundle Y;  TP55=56*OO_Y(1)-OO_Y
N=TP55-TY                         ---virtual normal bundle
H=chern(1,OO_Y(1)); toString oo
d=integral (H^54)  ---degree of image in P55
c1=chern(1,N); toString oo
dblpt=d*H-c1;  integral(H^53*dblpt)/2      ---get 5752908
```

But...It turns out that $\overline{\mathbb{D}}(\varphi)$ is reducible. The point is that, when the two planes are in special position, they impose less conditions on the linear system of cubics. We recall from [5] the following regularity bound.

2.3.1. **Lemma.** *The Castelnuovo-Mumford regularity of the saturated homogeneous ideal of a union of $r \geq 2$ subspaces is at most r.* □

2.3.2. Consider the correspondence

$$Z := \{(q_1, q_2, F_3) \in \mathbb{G} \times \mathbb{G} \times \mathbb{P}^{55} \,|\, q_1 \neq q_2,\, F_3 \supset q_1 \cup q_2\}.$$

Let $q_{1,2} = q_1 \cup q_2$. Now Z decomposes into pieces corresponding to the relative position of the plane-pair as we now describe.

• 2 general planes, *e.g.,*

$$\begin{cases} q_1 : x_0 = x_1 = x_2 = 0, \\ q_2 : x_3 = x_4 = x_5 = 0. \end{cases}$$

The homogeneous ideal is generated by the nine quadrics

$$\langle x_i x_j,\, 0 \leq i \leq 2,\, 3 \leq j \leq 5\rangle.$$

Recall the Hilbert polynomial, $P(t) = 2 + 3t + t^2 = 2\binom{t+2}{2}$ measures, for all t beyond the regularity, the number of independent conditions imposed on hypersurfaces of degree t to lie in the homogeneous ideal of $q_{1,2}$. Thus, the dimension of the fiber of Z over such plane pair is $55 - 2\binom{5}{2} = 35$. This yields a component of Z of dimension $35 + 18 = 53$.

• 2 planes meeting at a point; these form a hypersurface in $\mathbb{G}{\times}\mathbb{G}$. Now the homogeneous ideal is of the form

$$\langle x_0,\, x_1x_3, x_1x_4, x_2x_3, x_2x_4\rangle.$$

Its Hilbert polynomial is $P(t) = 1 + 3t + t^2$; hence the fiber of Z has dimension 36. We get again a component of dimension 53.

• 2 planes through a line in $\mathbb{P}^5$. The homogeneous ideal is of the form $\langle x_0, x_1, x_2x_3\rangle$. These plane-pairs vary in a subvariety of $\mathbb{G}{\times}\mathbb{G}$ of codimension 4. The Hilbert polynomial is $P(t) = 1{+}2t{+}t^2$. Thus we find a component of Z of dimension $= 14 + 55 - 16 = 53$.

So, that's a situation where diminishing the dimension of the base is compensated by an equal increase in the dimension of the fiber.

It follows that the locus in $\mathbb{P}^{55}$ consisting of cubic 4-folds which contain a plane-pair in $\mathbb{P}^5$ is of dimension 53. Hence we may conclude that the map $(p, F_3) \overset{\varphi}{\mapsto} F_3$ is generically injective and its double point locus receives contribution from the three configurations as described above.

In order to find the degree of the closure of the "good" locus corresponding to two general planes, we shall pursue below a different route. It amounts to building a smooth 2:1 cover of the component of the Hilbert scheme of unions of 2-planes in $\mathbb{P}^5$.

3. Parameter space for unions of 2-planes in $\mathbb{P}^5$

Start with $\mathbb{G} \times \mathbb{G}$, the variety of ordered plane-pairs. Define

$$\mathbb{S} = \{ \text{ plane-pairs with a common line } \}.$$

Let $\widetilde{\mathbb{G}}$ be the blowup of the diagonal $\mathbb{D}$ in $\mathbb{G} \times \mathbb{G}$.

One checks that, though $\mathbb{S}$ is singular along $\mathbb{D}$, its strict transform $\widetilde{\mathbb{S}} \subset \widetilde{\mathbb{G}}$ is smooth. It is isomorphic to a natural $\mathbb{P}^3 \times \mathbb{P}^3$–bundle over $\mathbb{G}(2,6) = \mathbb{G}r[1,5]$, the grassmannian of lines in $\mathbb{P}^5$, blown-up along its diagonal.

We have $\dim \mathbb{S} = \dim \widetilde{\mathbb{S}} = 14$.

Let $\widehat{\mathbb{G}}$ be the blowup of $\widetilde{\mathbb{G}}$ along $\widetilde{\mathbb{S}}$. Then $\widehat{\mathbb{G}}$ parameterizes a flat family with general member the union of two general planes in $\mathbb{P}^5$.

In other words, there is a surjective map $\widehat{\mathbb{G}} \longrightarrow \mathbb{H}$, where $\mathbb{H}$ denotes the Hilbert scheme component of unions of two planes in $\mathbb{P}^5$. See [3] for the case of pairs of subspaces of codimension 2.

The verification of the assertions above can be done using local coordinates. Instead of working with $\mathbb{G} \times \mathbb{G}$, we may fix the 2-plane

$$p_0 := \langle x_0, x_1, x_2 \rangle$$

and consider the variable 2-plane p_a as in (1), where the $a_{i,j}$ stand for affine coordinates in $\mathbb{G}$ around p_0. Equations for the fiber of $\mathbb{S}$ over p_0 are given by the 5×5 minors of the 6×6 matrix of the system $p_0 = p_a = 0$. It's the same as the ideal $\mathcal{J}$ of the 2×2 minors of $p_a = 0$ alone. One sees at once that this is singular precisely at the origin, *i.e.*, $p_0 : a_{i,j} = 0, i, j = 1, 2, 3$. Blowing up the diagonal means now blowing up $\mathbb{G}$ at p_0. We choose $a_{3,3}$ as the local generator of the exceptional ideal and write

$$a_{1,1} = b_{1,1} a_{3,3}, \ldots, a_{3,2} = b_{3,2} a_{3,3},$$

the 8 relations for the blowup $\widetilde{\mathbb{A}^9} \to \mathbb{A}^9 \subset \mathbb{G}$. Presently

$$a_{3,3}, b_{i,j}, i, j = 1, 2, 3, (i, j) \neq (3, 3)$$

are affine coordinates up in the blowup $\widetilde{\mathbb{A}^9}$. Let $\mathcal{J}'$ be the ideal generated by $\mathcal{J}$ upstairs, *i.e.*, upon plugging in the relations $a_{i,j} = a_{3,3} b_{i,j}$. We find that

$$\mathcal{J}' = (a_{3,3})^2 \widetilde{\mathcal{J}},$$

with

$$\widetilde{\mathcal{J}} = \langle b_{2,2} - b_{2,3} b_{3,2}, b_{2,1} - b_{2,3} b_{3,1}, b_{1,2} - b_{1,3} b_{3,2}, b_{1,1} - b_{1,3} b_{3,1} \rangle.$$

This is the ideal of the (fiber over p_0 of the) strict transform $\widetilde{\mathbb{S}}$ up in the blowup of $\mathbb{G} \times \mathbb{G}$ along the diagonal. Given a plane-pair $(q_1, q_2) \in \mathbb{S}$, the intersection $q_1 \cap q_2$ is a line provided $q_1 \neq q_2$. Thus the rational map

$$\begin{array}{ccc} \mathbb{S} & \dashrightarrow & \mathbb{G}r[1,5] \\ (q_1, q_2) & \longmapsto & q_1 \cap q_2 \end{array}$$

is a morphism off the diagonal $\mathbb{D}$. We claim that it induces a morphism

$$\lambda : \widetilde{\mathbb{S}} \longrightarrow \mathbb{G}r[1,5].$$

Indeed, the lifted linear system $p_0 = \widetilde{p}_a = 0$ restricted to $\widetilde{\mathbb{S}}$ yields the system

$$p_0 : x_0 = x_1 = x_2 = b_{3,1} x_3 + b_{3,2} x_4 + x_5 = 0$$

from which we infer there is always a well defined line lying on both planes. The morphism λ lifts to yield a map $\widetilde{\lambda}$ to the natural $\mathbb{P}^3 \times \mathbb{P}^3$-bundle $\mathbb{U}$ over $\mathbb{G}$r[1,5] defined by picking a pair of 2-planes through a line $\ell \in \mathbb{G}r[1,5]$,

$$\mathbb{U} = \{(q_1, q_2, \ell) \in \mathbb{G} \times \mathbb{G} \times \mathbb{G}r[1,5] \,|\, \ell \subseteq q_1 \cap q_2\}.$$

One also checks that $\widetilde{\lambda}$ actually induces an isomorphism $\lambda : \widetilde{\mathbb{S}} \longrightarrow \widetilde{\mathbb{U}}$ where $\widetilde{\mathbb{U}}$ denotes the blowup of the relative diagonal $(q_1 = q_2)$ in $\mathbb{U}$.

Set for short

$$\mathcal{F}_d := \operatorname{Sym}_d \mathcal{F} = H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(d)),$$

the space of homogeneous polynomials of degree d. There is a natural rational map

$$\begin{array}{ccc} \mu : \mathbb{G} \times \mathbb{G} & \dashrightarrow & \mathbb{G}r(9, \mathcal{F}_2) \\ (q_1, q_2) & \longmapsto & q_1 \cdot q_2 \end{array}$$

defined by multiplying the 3-dimensional subspaces $q_i \subset \mathcal{F}$ of linear polynomials. The scheme of indeterminacy of μ is equal to $\mathbb{S}$. It lifts to a rational map

$$\widetilde{\mathbb{G}} \dashrightarrow \mathbb{G}r(9, \mathcal{F}_2)$$

with scheme of indeterminacy equal to $\widetilde{\mathbb{S}}$. Let $\widehat{\mathbb{G}}$ be the blowup of $\widetilde{\mathbb{G}}$ along $\widetilde{\mathbb{S}}$. We obtain a morphism

$$\widehat{\mu} : \widehat{\mathbb{G}} \longrightarrow \mathbb{G}r(9, \mathcal{F}_2)$$

Over $\widehat{\mathbb{G}}$, for each degree $d \geq 2$, there is a vector subbundle $\mathcal{V}_d \subset \mathcal{F}_d$ of the trivial bundle of homogeneous polynomials of degree d such that:

- The fiber of $\mathcal{V}_d$ over a plane-pair $p_{1,2} \in \widehat{\mathbb{G}}$ is the space of equations of hypersurfaces of degree d containing $p_{1,2}$;
- $\mathrm{rank}\mathcal{V}_d = \binom{d+5}{5} - 2\binom{d+2}{2}$;
- The image $\mathbb{W}_d$ in $\mathbb{P}^N = \mathbb{P}\mathcal{F}_d$ of the projectivization $\mathbb{P}\mathcal{V}_d \subset \widehat{\mathbb{G}} \times \mathbb{P}^N$ is the variety of hypersurfaces containing a (flat specialization of a) plane-pair.

The variety $\widehat{\mathbb{G}}$ inherits a $\mathbb{C}^\star$-action, with (a lot:-) of isolated fix points. The vector bundles $\mathcal{V}_d \to \widehat{\mathbb{G}}$ are equivariant. Bott localization (cf. [6, §2], [15], [1], [2]) applies, enabling us to find the degree of $\mathbb{W}_d$ in $\mathbb{P}\mathcal{F}_d$, $d \geq 3$, to wit,

$$\deg \mathbb{W}_d = \int_{\widehat{\mathbb{G}}} c_{18}(-\mathcal{V}_d) = \sum_{\text{fixpts}} \frac{c_{18}^T(-\mathcal{V}_d)}{c_{18}^T \tau} .$$

Using Maple, in the flavor explained by Meurer [15] we are able to integrate and find, *e.g.,* for $d = 3$, the value $\deg \mathbb{W}_3 = \mathbf{3\,371\,760}$, not in agreement with the double point formula.

An argument employing Grothendieck-Riemann-Roch (cf. [4]) shows that there is a formula for the degree of $\mathbb{W}_d \subset \mathbb{P}\mathcal{F}_d$ as a polynomial in d. Actually we got by interpolation the following polynomial of degree 54 (cf. [16] for a script):

$$\begin{gathered}
\deg \mathbb{W}_d = \tfrac{1}{2^{32} \cdot 3^{23} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17} \tbinom{d}{3} \Big(297797500d^{51} + 16974457500d^{50} + \\
438953515000d^{49} + 6750473730000d^{48} + 67557745255000d^{47} + 446469328305000d^{46} \\
+1821546306580000d^{45} + 3261093465630000d^{44} - 5452213497731000d^{43} \\
-26658904130859000d^{42} - 1792499938229000d^{41} - 807392033197659000d^{40} \\
-6904527757469587700d^{39} + 34775461914517695 00d^{38} + \\
168293105176596569800d^{37} - 832050550503900264 00d^{36} - \\
418358516692370972 5625d^{35} + 4729797968873046725475d^{34} + \\
93623512083339602708675d^{33} - 210025261579623597041475d^{32} - \\
1497759082084784912756740d^{31} + 6691368991089621694295820d^{30} + \\
10512834434651356253342780d^{29} - 127045484364059052592597740d^{28} + \\
173715078834280290838756586d^{27} + 1128680664343084906757160738d^{26} - \\
4994152749025875809985069838d^{25} + 2356774599575513190792679230d^{24} + \\
37766401805433040109235274520d^{23} - 120118775223192214665021263640d^{22} + \\
64160131759384538259802479140d^{21} + 507092558093142767480135015700d^{20} - \\
1451146056063090731464859692765d^{19} + 1272606825133942111965200965455d^{18} + \\
388312586377531571922451794995d^{17} + 3580712277013841049646053016725d^{16} - \\
20845497262217658319851150940560d^{15} + 12031904188478235409221442162320d^{14} + \\
169428261347272908281967678701280d^{13} - 687428225963718953591666450858400d^{12} + \\
134001321234109896458655459015 5520d^{11} - 1087468822379100789481710030842880d^{10} - \\
1847260530393109277960386571454720d^{9} + 8564365120882865993680747841936640d^{8} - \\
18503733474733545031760180202663936d^{7} + 28014319703719681406965987557875712d^{6} - \\
35797411551433954908141875545178112d^{5} + 39893149299289869979218029094174720d^{4} - \\
42655130536947988709557871758540800d^{3} + 40106073268823932733976960565248000d^{2} - \\
24647062713098039616382917672960000d + 9496912828923697566983808614400000 \Big).
\end{gathered} \tag{2}$$

There are three other known families of rational cubic 4 folds: impose F_3 to contain either a Del Pezzo of degree 5 or a scroll of degree 4 (2 types) in $\mathbb{P}^5$ (cf. [11]). It would be nice to find their degrees. The difficulty lies in describing appropriate parameter spaces for those families of surfaces.

Acknowledgement. We would like to thank the anonymous referee for very valuable and detailed comments and corrections.

References

[1] R. Bott, *A Residue Formula for Holomorphic Vector Fields*, J. Differential Geom. 1, 311–330, 1967.
[2] M. Brion, *Equivariant cohomology and equivariant intersection theory.* Notes by Alvaro Rittatore. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Representation theories and algebraic geometry (Montreal, PQ, 1997), 1–37, Kluwer Acad. Publ., Dordrecht, 1998.
[3] D. Chen, I. Coskun & S. Nollet, *Hilbert scheme of a pair of codimension two linear subspaces.* Comm. Algebra 39, no. 8, 3021–3043, 2011. arXiv:0909.5170
[4] F. Cukierman, A. F. Lopez & I. Vainsencher, *Enumeration of surfaces containing an elliptic quartic curve*, (to appear in Proc. Amer. Math. Soc.), 2012. arXiv:1209.3335
[5] H. Derksen & J. Sidman, *A sharp bound for the Castelnuovo-Mumford regularity of subspace arrangements*, Adv. Math., 172 , no. 2, 151–157, 2002. arXiv:math/0109035 http://arxiv.org/abs/math/0109035
[6] G. Ellingsrud & S. A. Strømme, *Bott's formula and enumerative geometry*, J. Amer. Math. Soc. 9, 175–193, 1996. (arXiv:alg-geom/9411005v1) DOI: 10.1090/S0894-0347-96-00189-0
[7] W. Fulton, Intersection Theory, Second Edition, Ergeb. Math. Grenzgeb. Springer-Verlag, Berlin, 1998.
[8] D. R. Grayson & M. E. Stillman, Macaulay2, a software system for research in algebraic geometry, (`http://www.math.uiuc.edu/Macaulay2/`), 2012.
[9] J. Harris, conference ELGA, La Cumbre, Argentina, 2011. `http://mate.dm.uba.ar/~visita16/ELGA-2011/version/v1/research_workshop-en.shtml`
[10] S. Katz & S. A. Strømme, schubert, *A Maple package for Intersection Theory.* `http://linus.mi.uib.no`, 2001.
[11] J. Kollár, K. E. Smith & A. Corti, Rational and Nearly Rational Varieties, Cambridge Univ. Press, 2004.
[12] M. Kontsevich, *Enumeration of rational curves via torus actions*, in The moduli space of curves (Texel Island, 1994), 335–368, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995. arXiv:hep-th/9405035
[13] D. Laksov, *Residual intersections and Todd's formula for the double locus of a morphism*, Acta. Math. 140, 75-92, 1978. DOI: 10.1007/BF02392304
[14] J. A. D. Maia, A. R. Silva, I. Vainsencher & F. Xavier, *Enumeration of surfaces containing a curve of low degree*, J. Pure Appl. Algebra, v. 217, p. 1379-1394, 2013. DOI: 10.1016/j.jpaa.2012.10.016
[15] P. Meurer, *The number of rational quartics on Calabi-Yau hypersurfaces in weighted projective space $P(2,1^4)$*, Math. Scand., 78, 63-83 1996. arXiv:alg-geom/9409001
[16] I. Vainsencher, *Scripts*, `http://www.mat.ufmg.br/~israel/Projetos/GomezMontfest`.

Departamento de Matemática, UFMG, Belo Horizonte, MG, Brazil
E-mail address: `israel@mat.ufmg.br`